


## Research Article

# Certain Integral Transforms and Their Applications in Propagation of Laguerre-Gaussian Schell-model Beams

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**Abstract:** The principal aim of the present work is to investigate a new class of integral transforms involving the product of Bessel function of the first kind of arbitrary order with generalized Laguerre polynomials and logarithmic functions which deduce some new results in terms of the digamma function and Kampé de Fériet functions. A novel expression is found for the Kampé de Fériet function  $F_{2;1}^{1;2,1}$  in terms of hypergeometric functions  ${}_1F_1$  and  ${}_2F_2$ . Finally, the results obtained are applied in the problem of propagation of Laguerre-Bessel-Gaussian Schell-model beams as an application.

**Keywords:** integral transform, Bessel function, generalized Laguerre polynomials, generalized hypergeometric function, digamma function

**MSC:** 33B15, 33C10, 33C15

## 1. Introduction

A remarkably large number of integral transforms involving a variety of special functions have been presented (see [1-11]). Among those special functions (see [12-16]), due mainly to the greater abstruseness of their properties, Bessel functions have helped resolve many applications in various problems of mathematical physics [17-18], thereby, several integral formulas involving Bessel functions and their various extensions (or generalizations) have been investigated [19], see e.g., p. 40, Equation (8).

The  $\delta$  expansion is investigated by Wandzura [20] to study the structure functions of refractive-index fluctuation for optical propagation in turbulent. Recently, based on a comparison between quadratic approximation and  $\delta$  expansion by the study of some laser fields in turbulent atmosphere, Chu and Liu [18] have introduced the first integral to evaluate the Huygens-Fresnel diffraction integral. Earlier, Cang et al. [17] have studied the average intensity of Laguerre-Gaussian and Bessel-Gaussian Schell-model beams propagation through paraxial optical system in turbulent atmosphere. Afterwards, the authors have derived a closed-form of the integral  $I_{n,\alpha}^{\gamma,\lambda}$  and the problem has become very important for the researchers interested in propagation of Laguerre-Bessel-Gaussian beams.

Throughout this paper, let  $\mathbb{C}$ ,  $\mathbb{R}^+$ ,  $\mathbb{R}$  and  $\mathbb{Z} \setminus \{0\}$  be the sets of complex numbers, positive real numbers, positive and

non-positive integers respectively. For the present investigation, we recall the following definitions.

The natural generalization of Gauss's hypergeometric function  ${}_2F_1$  is called the generalized hypergeometric series  ${}_pF_q$  which is defined by (see [21],[22, p. 71-75])

$${}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n z^n}{(\beta_1)_n \dots (\beta_q)_n n!}$$

$$(\beta_j \neq 0, -1, -2, \dots, j = 1, \dots, q), \quad (1)$$

where,  $(\alpha)_n$  is the Pochhammer symbol defined by (see [22, p. 2 and p. 5])

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}; \quad \alpha \neq 1, -1, -2, \dots$$

In 1921, the four Appell functions were unified and generalized by Kampé de Fériet, who defined a general hypergeometric function of two variables. The notation introduced by Kampé de Fériet for his double hypergeometric function of superior order has been subsequently abbreviated by Burchnall and Chaundy. Here, the definition of a more general double hypergeometric function in a slightly modified notation (see [23-24]).

$$F_{q;m;n}^{p;l;k} \left[ \begin{matrix} (a)_p : (b)_l; (c)_k; \\ (\alpha)_q : (\beta)_m; (\gamma)_n; \end{matrix} x, y \right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{r+s} \prod_{j=1}^l (b_j)_r \prod_{j=1}^k (c_j)_s}{\prod_{j=1}^q (\alpha_j)_{r+s} \prod_{j=1}^m (\beta_j)_r \prod_{j=1}^n (\gamma_j)_s} \frac{x^r y^s}{r! s!}, \quad (2)$$

where, for convergence,

- (i)  $p + l < q + m + 1, p + k < q + n + 1, |x| < \infty, |y| < \infty$ , or
- (ii)  $p + l = q + m + 1, p + k = q + n + 1$ , and

$$\begin{cases} |x|^{l/(p-q)} + |y|^{l/(p-q)} < 1, & \text{if } p > q, \\ \max\{|x|, |y|\} < 1, & \text{if } p \leq q. \end{cases}$$

The Bessel function  $J_\lambda(x)$  of the first kind and order  $\lambda$  (see [19]) is defined by

$$J_\lambda(x) = \sum_{l=0}^{\infty} \frac{(-1)^l (x/2)^{\lambda+2l}}{\Gamma(\lambda+l+1)l!}. \quad (3)$$

The generalized Laguerre polynomial (see [21]) (or the Sonine polynomial [25]) is given as

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{m=0}^n \frac{(-n)_m}{(\alpha+1)_m} \frac{x^m}{m!}. \quad (4)$$

The polygamma functions  $\psi^{(n)}(x)$  ( $n \in \mathbb{N}$ ) are defined as

$$\psi^{(n)}(x) = \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(x) = \frac{d^n}{dx^n} \psi(x) \quad (x \in \mathbb{C} \setminus \mathbb{Z}_0^-), \quad (5)$$

where  $\Gamma(x)$  is the familiar gamma function and the psi-function  $\psi$  is defined by  $\psi(x) = \frac{d}{dx} \log \Gamma(x) = \psi^{(0)}(x)$  (see, [22], [26, Chapter 1]).

## 2. Evaluation of $I_{n,\alpha}^{\gamma,\lambda}$

Here, we present the integral transforms involving Bessel function, generalized Laguerre polynomial and logarithmic function  $I_{n,\alpha}^{\gamma,\lambda}$ , which are asserted by the following theorem.

**Theorem 2.1.** The following transformation holds true.

$$\int_0^\infty x^{2\gamma+1} e^{-\sigma x^2} L_n^{(\alpha)}(\beta x^2) J_\lambda(\mu x) \ln(bx) dx = \frac{\mu^\lambda}{2^{\lambda+2} \lambda! \sigma^{\gamma+1}} \sum_{m=0}^n \frac{\Gamma(n+\alpha+1)(-\beta/\sigma)^m}{\Gamma(n-m+1)\Gamma(\alpha+m+1)m!} \Gamma(\chi_m) A_{m,\lambda}, \quad (6)$$

where

$$A_{m,\lambda} = \left[ \ln\left(\frac{b^2}{\sigma}\right) {}_1F_1(\chi_m; \lambda+1; -\mu^2/4\sigma) + \psi(\chi_m-1) {}_2F_2(1, \chi_m; \lambda+1, 1; -\mu^2/4\sigma) \right] - \frac{1}{(1-\chi_m)} F_{2;1}^{1;2;0} \left[ \begin{matrix} \chi_m : & \chi_m-1, 1; 1; \\ \lambda+1, 1 : & \chi_m; & - \end{matrix} ; -\mu^2/4\sigma, -\mu^2/4\sigma \right], \quad (7)$$

and  $Re(2\gamma + \lambda + 1) > -1$ ,  $Re(\sigma) > 0$  and  $b > 0$ .

*Proof.* For that, by using the following identities (see [19])

$$L_n^{(\alpha)}(\beta x^2) = \sum_{m=0}^n \frac{\Gamma(n+\alpha+1)(-\beta x^2)^m}{\Gamma(n-m+1)\Gamma(\alpha+m+1)m!}, \quad (8)$$

and

$$J_\lambda(\mu x) = \sum_{l=0}^{\infty} \frac{(-1)^l (\mu x/2)^{\lambda+2l}}{\Gamma(\lambda+l+1)l!}, \quad (9)$$

the integral  $I_{n,\alpha}^{\gamma,\lambda}$  becomes

$$I_{n,\alpha}^{\gamma,\lambda} = \sum_{m=0}^n \frac{\Gamma(n+\alpha+1)(-\beta)^m}{\Gamma(n-m+1)\Gamma(\alpha+m+1)m!} \sum_{l=0}^{\infty} \frac{(-1)^l (\mu/2)^{\lambda+2l}}{\Gamma(\lambda+l+1)l!} I_{m,l}^{\gamma}, \quad (10)$$

where

$$I_{m,l}^{\gamma} = \int_0^\infty x^{2\gamma+2m+2l+\lambda+1} e^{-\sigma x^2} \ln(bx) dx, \quad (11)$$

with  $Re(2\gamma + 2m + 2l + \lambda + 1) > -1$ ,  $Re(\sigma) > 0$  and  $b > 0$ .

This last integral can be evaluated by using the following identity (see [27])

$$\int_0^\infty x^{2\xi+1} e^{-\sigma x^2} \ln(bx) dx = \frac{\Gamma(\xi+1)}{4\sigma^{\xi+1}} \left[ \ln\left(\frac{b^2}{\sigma}\right) + \psi(\xi+1) \right], \quad (12)$$

By taking  $\xi = \gamma + m + \lambda/2 + l$ , Equation (11) becomes

$$I_{m,l}^{\gamma} = \frac{\Gamma(\gamma+m+l+\lambda/2+1)}{4\sigma^{(\gamma+m+l+\lambda/2+1)}} \left[ \ln\left(\frac{b^2}{\sigma}\right) + \psi(\gamma+m+l+\lambda/2+1) \right]. \quad (13)$$

By substituting this last expression in Equation (10), the integral  $I_{n,\alpha}^{\gamma,\lambda}$  becomes

$$I_{n,\alpha}^{\gamma,\lambda} = \frac{\mu^\lambda}{2^{\lambda+2}\sigma^{\gamma+\lambda/2+1}} \sum_{m=0}^n \frac{\Gamma(n+\alpha+1)(-\beta/\sigma)^m}{\Gamma(n-m+1)\Gamma(\alpha+m+1)m!} I_\psi^{(m)}, \quad (14)$$

where

$$I_\psi^{(m)} = \sum_{l=0}^{\infty} \frac{\Gamma(\chi_m+l)(-\mu^2/4\sigma)^l}{\Gamma(\lambda+l+1)l!} \left[ \ln\left(\frac{b^2}{\sigma}\right) + \psi(\chi_m+l) \right], \quad (15)$$

with  $\chi_m = \gamma + m + \lambda/2 + 1$ .

With the help of the following identities (see [28])

$$\sum_{m=0}^{\infty} \psi(b+m) \frac{(\mu_p)_m}{(\varrho_q)_m} z^m = \psi(b-1) {}_{p+1}F_q(1, (\mu_p); (\varrho_q); z) - \frac{1}{(1-b)} F_{q;1;0}^{p;2;1} \left[ \begin{matrix} (\mu_p) : & b-1, 1; 1; \\ (\varrho_q) : & b; \quad -; \end{matrix} \quad z, z \right], \quad (16)$$

where  $F_{q;m;n}^{p;l;k}$  is the Kampé de Fériet function see Equation (2).

It is easy to show the following expression

$$\begin{aligned} & \sum_{m=0}^{\infty} \psi(\chi_m+l) \frac{(\chi)_l}{(1)_l(\lambda+1)_l} (-z)^l \\ &= \psi(\chi_m-1) {}_2F_2(1, \chi_m; \lambda+1, 1; -z) - \frac{1}{(1-\chi_m)} F_{2;1;0}^{1;2;1} \left[ \begin{matrix} \chi_m : & \chi_m-1, 1; 1; \\ \lambda+1, 1 : & \chi_m; \quad -; \end{matrix} \quad -z, -z \right], \end{aligned} \quad (17)$$

and consequently Equation (15) becomes

$$\begin{aligned} I_\psi^{(m)} &= \frac{\Gamma(\chi_m)}{\lambda!} \ln\left(\frac{b^2}{\sigma}\right) {}_1F_1(\chi_m; \lambda+1; -\mu^2/4\sigma) + \frac{\Gamma(\chi_m)}{\lambda!} \psi(\chi_m-1) {}_2F_2(1, \chi_m; \lambda+1, 1; -\mu^2/4\sigma) \\ &\quad - \frac{\Gamma(\chi_m)}{\lambda!(1-\chi_m)} F_{2;1;0}^{1;2;1} \left[ \begin{matrix} \chi_m : & \chi_m-1, 1; 1; \\ \lambda+1, 1 : & \chi_m; \quad -; \end{matrix} \quad -\mu^2/4\sigma, -\mu^2/4\sigma \right]. \end{aligned} \quad (18)$$

By substituting Equation (18) in Equation (14), one finds Equation (6). This completes the proof.

### 3. Special cases

Among a large number of possible special cases and the other versions of the main result in Theorem 2.1, we obtain another version of the formula in Equation (7) in the following corollaries.

**Corollary 3.1.** If we take  $\lambda = 0$  in Equation (6), we arrive at the following integral transform

$$\int_0^\infty x^{2\gamma+1} e^{-\sigma x^2} L_n^{(\alpha)}(\beta x^2) J_0(\mu x) \ln(bx) dx = \frac{1}{4\sigma^{\gamma+1}} \sum_{m=0}^n \frac{\Gamma(n+\alpha+1)(-\beta/\sigma)^m}{\Gamma(n-m+1)\Gamma(\alpha+m+1)m!} \Gamma(\gamma+m+1) A_{m,0}, \quad (19)$$

where

$$A_{m,0} = \left[ \ln \left( \frac{b^2}{\sigma} \right) + \psi(\gamma + m) \right] {}_1F_1(\gamma + m + 1; 1; -\mu^2 / 4\sigma) + \frac{1}{(\gamma + m)} F_{2;1;0}^{1;2;1} \left[ \begin{matrix} \gamma + m + 1: & \gamma + m, 1; & 1; \\ 1, 1: & \gamma + m + 1; & -; \end{matrix} \middle| -\mu^2 / 4\sigma, -\mu^2 / 4\sigma \right], \quad (20)$$

and  $Re(2\gamma + 1) > -1$ ,  $Re(\sigma) > 0$  and  $b > 0$ .

**Corollary 3.2.** For  $\alpha = 0$  in Equation (19), we get the following integral transform

$$\int_0^\infty x^{2\gamma+1} e^{-\sigma x^2} L_n(\beta x^2) J_0(\mu x) \ln(bx) dx = \frac{n!}{4\sigma^{\gamma+1}} \sum_{m=0}^n \frac{(-\beta/\sigma)^m}{\Gamma(n-m+1)(m!)^2} \Gamma(\gamma + m + 1) A_{m,0}, \quad (21)$$

where  $A_{m,0}$  is given by Equation (20) and  $Re(2\gamma + 1) > -1$ ,  $Re(\sigma) > 0$  and  $b > 0$ .

**Corollary 3.3.** On setting  $\alpha = \gamma = 0$  in Equation (19), we obtain the following integral transform

$$\int_0^\infty x e^{-\sigma x^2} L_n(\beta x^2) J_0(\mu x) \ln(bx) dx = \frac{n!}{4\sigma} \sum_{m=0}^n \frac{(-\beta/\sigma)^m}{\Gamma(n-m+1)m!} A_{m,0}, \quad (22)$$

where

$$A_{m,0} = \left[ \ln \left( \frac{b^2}{\sigma} \right) + \psi(m) \right] {}_1F_1(m + 1; 1; -\mu^2 / 4\sigma) + \frac{1}{m} F_{2;1;0}^{1;2;1} \left[ \begin{matrix} m + 1: & m, 1; & 1; \\ 1, 1: & m + 1; & -; \end{matrix} \middle| -\mu^2 / 4\sigma, -\mu^2 / 4\sigma \right], \quad (23)$$

and  $Re(\sigma) > 0$  and  $b > 0$ .

#### 4. Evaluation of $I_{n,\alpha}^{\gamma,\lambda,\mu}$

In this section, we present the integral transforms which involves two Bessel functions, generalized Laguerre polynomial and logarithmic function  $I_{n,\alpha}^{\gamma,\lambda,\mu}$ , which are asserted by the following theorem.

**Theorem 4.1.** The following transformation holds true.

$$\int_0^\infty x^{2\gamma+1} e^{-\sigma x^2} I_n^{(\alpha)}(\beta x^2) J_\lambda(\mu x) J_\nu(\chi x) \ln(bx) dx = \frac{(\chi/2\sqrt{\sigma})^\nu}{2^{\lambda+2} \lambda! \sigma^{\gamma+\lambda/2+1}} \sum_{k=0}^\infty \frac{(-\chi^2/4\sigma)^k}{k! \Gamma(\nu+k+1)} \sum_{m=0}^\infty \frac{\Gamma(n+\alpha+1)(-\beta/\sigma)^m}{\Gamma(n-m+1)\Gamma(\alpha+m+1)m!} \Gamma(\chi'_m) A_{m,\lambda}, \quad (24)$$

where

$$A_{m,\lambda} = \left[ \ln \left( \frac{b^2}{\sigma} \right) {}_1F_1(\chi'_m; \lambda + 1; -\mu^2 / 4\sigma) + \psi(\chi'_m - 1) {}_2F_2(1, \chi'_m; \lambda + 1, 1; -\mu^2 / 4\sigma) \right] + \frac{1}{(1-\chi'_m)} F_{2;1;0}^{1;2;1} \left[ \begin{matrix} \chi'_m: & \chi'_m - 1, 1; & 1; \\ \lambda + 1, 1: & \chi'_m; & -; \end{matrix} \middle| -\mu^2 / 4\sigma, -\mu^2 / 4\sigma \right], \quad (25)$$

and  $Re(2\gamma + \lambda + \nu + 1) > -1$ ,  $Re(\sigma) > 0$  and  $b > 0$ .

*Proof.* For that, we use the expression of Bessel function of the first kind given as Equation (9), to write the expansion of  $I_{n,\alpha}^{\gamma,\lambda,\mu}$  as follow

$$I_{n,\alpha}^{\gamma,\lambda,\mu} = \sum_{k=0}^{\infty} \frac{(-1)^k (\chi/2)^{\nu+2k}}{k! \Gamma(\nu+k+1)} \int_0^{\infty} x^{2\gamma+2k+\nu+1} e^{-\sigma x^2} L_n^{(\alpha)}(\beta x^2) J_{\lambda}(\mu x) \ln(bx) dx. \quad (26)$$

This theorem can be established by replacing  $\chi'_m = \gamma + k + m + \nu/2 + \lambda/2 + 1$  in Equation (26), and using the result obtained in Equation (6). This completes the proof.

## 5. Special cases

Among a large number of possible special cases and other versions of the result in Theorem 4.1. We obtain another version of the formula (24) in the following corollaries.

**Corollary 5.1.** If we put  $\nu = \lambda = 0$  in Equation (24), we obtain

$$\begin{aligned} & \int_0^{\infty} x^{2\gamma+1} e^{-\sigma x^2} L_n^{(\alpha)}(\beta x^2) J_0(\mu x) J_0(\chi x) \ln(bx) dx \\ &= \frac{1}{4\sigma^{\gamma+1}} \sum_{k=0}^{\infty} \frac{(-\chi^2/4\sigma)^k}{(k!)^2} \sum_{m=0}^n \frac{\Gamma(n+\alpha+1)(-\beta/\sigma)^m}{\Gamma(n-m+1)\Gamma(\alpha+m+1)m!} \Gamma(\gamma+k+m+1) A_{m,0}, \end{aligned} \quad (27)$$

where

$$\begin{aligned} A_{m,0} = & \left[ \ln\left(\frac{b^2}{\sigma}\right) + \psi(\gamma+k+m) \right] {}_1F_1(\gamma+k+m+1; 1; -\mu^2/4\sigma) \\ & + \frac{1}{(\gamma+k+m)} F_{2;1;0}^{1;2;1} \left[ \begin{matrix} \gamma+k+m+1: & \gamma+k+m, 1; & 1; \\ 1, 1: & \gamma+k+m+1; & -; \end{matrix} \right. \left. -\mu^2/4\sigma, -\mu^2/4\sigma \right], \end{aligned} \quad (28)$$

and  $Re(2\gamma+1) > -1$ ,  $Re(\sigma) > 0$  and  $b > 0$ .

**Corollary 5.2.** If we take  $\alpha = 0$  in Equation (27), we find

$$\begin{aligned} & \int_0^{\infty} x^{2\gamma+1} e^{-\sigma x^2} L_n(\beta x^2) J_0(\mu x) J_0(\chi x) \ln(bx) dx \\ &= \frac{1}{4\sigma^{\gamma+1}} \sum_{k=0}^{\infty} \frac{(-\chi^2/4\sigma)^k}{(k!)^2} \sum_{m=0}^n \frac{n!(-\beta/\sigma)^m}{\Gamma(n-m+1)(m!)^2} \Gamma(\gamma+k+m+1) A_{m,0}, \end{aligned} \quad (29)$$

where  $A_{m,0}$  is given in Equation (28) and  $Re(2\gamma+1) > -1$ ,  $Re(\sigma) > 0$  and  $b > 0$ .

**Corollary 5.3.** Applying  $\alpha = \gamma = 0$  in Equation (27), we obtain the following formula.

$$\begin{aligned} & \int_0^{\infty} x e^{-\sigma x^2} L_n(\beta x^2) J_0(\mu x) J_0(\chi x) \ln(bx) dx \\ &= \frac{1}{4\sigma} \sum_{k=0}^{\infty} \frac{(-\chi^2/4\sigma)^k}{(k!)^2} \sum_{m=0}^n \frac{n!(-\beta/\sigma)^m}{\Gamma(n-m+1)(m!)^2} \Gamma(k+m+1) A_{m,0}, \end{aligned} \quad (30)$$

where

$$\begin{aligned} A_{m,0} = & \left[ \ln\left(\frac{b^2}{\sigma}\right) + \psi(k+m) \right] {}_1F_1(k+m+1; 1; -\mu^2/4\sigma) \\ & + \frac{1}{(k+m)} F_{2;1;0}^{1;2;1} \left[ \begin{matrix} k+m+1: & k+m, 1; & 1; \\ 1, 1: & k+m+1; & -; \end{matrix} \right. \left. -\mu^2/4\sigma, -\mu^2/4\sigma \right], \end{aligned} \quad (31)$$

and  $Re(\sigma) > 0$  and  $b > 0$ .

## 6. Applications

### 6.1 Propagation of Laguerre-Gaussian Schell-model beams

As mentioned in [17], the spectral density function of a focused beam propagating through an ABCD paraxial optical system in turbulent atmosphere is given as

$$S(p, z) = f(p, z) + \Delta S(p, z), \quad (32)$$

where

$$\Delta S(p, z) = A(z) \int_0^\infty x^3 e^{-\sigma x^2} \ln(bx) L_n(\beta x^2) J_0(\mu x) dx, \quad (33)$$

$z$  is the propagation distance and  $\rho$  is the radial coordinate. In Equation (32),  $f$  is the average intensity of the considered beam propagation without turbulence. By using our main result obtained in Equation (6) with  $\gamma = 1$ ,  $\lambda = 0$  and  $\alpha = 0$ , one finds

$$\Delta S(p, z) = \frac{A(z)}{4\sigma^2} \sum_{m=0}^n \frac{n!(-\beta/\sigma)^m}{\Gamma(n-m+1)(m!)^2} \Gamma(\chi_m) A_{m,0} \quad (34)$$

where

$$A_{m,0} = \left[ \ln\left(\frac{b^2}{\sigma}\right) {}_1F_1(\chi_m; 1; -\mu^2/4\sigma) + \psi(\chi_m - 1) {}_2F_2(1, \chi_m; 1, 1; -\mu^2/4\sigma) \right] - \frac{1}{(1-\chi_m)} F_{2:1:0}^{1:2:1} \left[ \begin{matrix} \chi_m & \chi_m - 1, 1, 1; & -\mu^2/4\sigma, -\mu^2/4\sigma \\ 1, 1 & \chi_m; & - \end{matrix} \right]. \quad (35)$$

We can also investigate other characteristics of this beams family by applying our main result, especially the evaluation of the Huygens-Fresnel diffraction integral.

As an example of application, we study the evolution of the spectral density function of a focused beam propagating through an ABCD paraxial optical system in a turbulent atmosphere. In all the following simulations, we take the parameters as follows:  $\sigma = 3.6$ ,  $n = 2$  and  $b = 2.5$ . Figure 1 shows the evolution of the spectral density of the propagation of Laguerre-Gaussian beam in turbulent atmosphere for two  $\beta$  values:  $\beta = 2$  and  $\beta = 5$ . From the illustration of this figure, we can find that with the increase of  $\beta$ , some lobes of the spectral density function of a focused beam propagating in a turbulent medium appears.

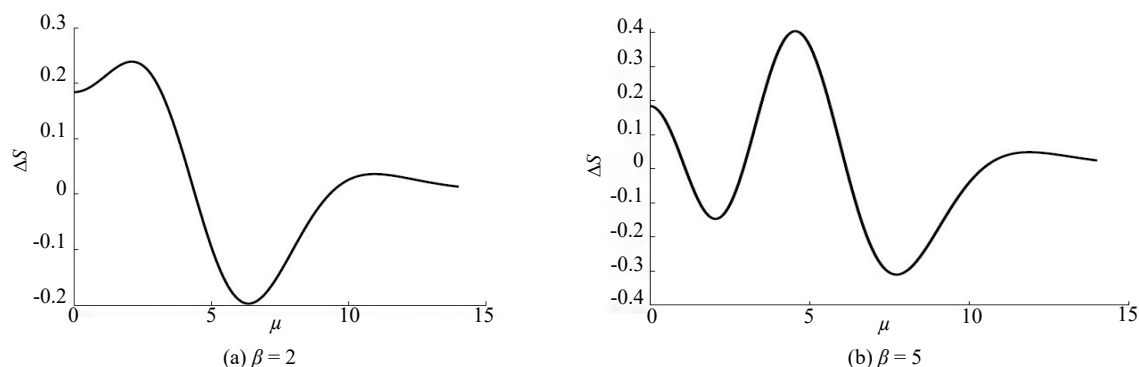


Figure 1. Illustration of propagation of Laguerre-Gaussian Schell-model beams with  $\sigma = 3.6$ ,  $n = 2$  and  $b = 2.5$

## 6.2 Propagation of Laguerre-Bessel-Gaussian Schell-model beams

In this case, Equation (33) becomes

$$\Delta S(p, z) = A(z) \int_0^\infty x^3 e^{-\sigma x^2} \ln(bx) L_n(\beta x^2) J_0(\mu x) J_0(\chi x) dx, \quad (36)$$

this quantity can be evaluated by replacing  $\gamma = 1$  and  $\nu = \lambda = \alpha = 0$  in Equation (24). The result for this case is given by

$$\Delta S(p, z) = \frac{A(z)}{4\sigma^2} \sum_{k=0}^{\infty} \frac{(-\chi^2 / 4\sigma)^k}{(k!)^2} \sum_{m=0}^n \frac{n!(-\beta / \sigma)^m}{\Gamma(n-m+1)(m!)^2} \Gamma(\chi'_m) A_{m,0}, \quad (37)$$

where

$$A_{m,0} = \left[ \ln\left(\frac{b^2}{\sigma}\right) {}_1F_1(\chi'_m; 1; -\mu^2 / 4\sigma) + \psi(\chi'_m - 1) {}_2F_2(1, \chi'_m; 1, 1; -\mu^2 / 4\sigma) \right] - \frac{1}{(1 - \chi'_m)} F_{2;1}^{1;2;1} \left[ \begin{matrix} \chi'_m : \chi'_m - 1, 1; 1; \\ 1, 1 : \chi'_m; -; \end{matrix} \begin{matrix} -\mu^2 / 4\sigma, -\mu^2 / 4\sigma \end{matrix} \right], \quad (38)$$

and  $\chi'_m = \gamma + k + m + 1$ .

Figure 2 illustrates also the behaviour of the spectral density function of a Laguerre-Bessel-Gaussian Schell-model beam in terms of the parameter  $\mu$  with two values of  $\beta = 2$  and  $\beta = 5$  and propagating in a turbulent environment.

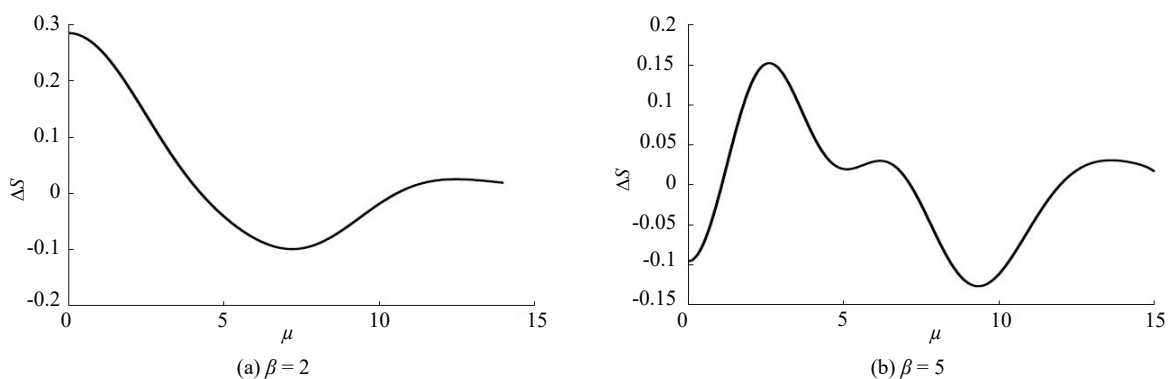


Figure 2. Illustration of propagation of Laguerre-Bessel-Gaussian Schell-model beams with  $\sigma = 3.6$ ,  $n = 2$ ,  $b = 2.5$  and  $x = 2$

## 7. Conclusion

We have evaluated two integrals transforms involving orthogonal polynomials and Bessel functions, which are interesting to study the behaviour of light and matter laser beams. Our main results are expressed in terms of digamma function and Kampé de Fériet functions. We have also given a novel expression of this later function in terms of some hypergeometric functions. Two applications are given to show the importance of our main results.

## Conflict of interest

The author declare no conflict of interest.



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