Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Dynamics and Bifurcations in Fractional Lozi Map

Sumit S Pakhare1[1*], Prashant M Gade[2]

[1]Department of Physics, Priyadarshini J. L. College of Engineering, Nagpur, India
[2]Department of Physics, Rashtrasant Tukadoji Maharaj Nagpur University, MJP Campus, Nagpur, India
E-mail: sumitspakhare@gmail.com

Received: 6 February 2023; **Revised:** 27 October 2023; **Accepted:** 27 October 2023

Abstract: We propose a fractional version of the Lozi map. The Lozi map is quite similar to the Henon map except that the square term of the Henon map is replaced by a linear term in the Lozi map. This map is known to be analytically tractable. The fractional version of the Lozi map displays rich dynamical behaviour. In this work, we carry out a time-series analysis of the map and present the bifurcation diagrams along with attractors for different parametric values of the map for various values of fractional order parameter (α).

***Keywords*:** Lozi map, henon map, fractional calculus, chaotic maps, coupled map lattice

MSC: 37E99, 26A33

1. Introduction

Nonlinear maps manifest various phenomena that occur in flows as well. We can observe chaos even in one-dimensional nonlinear maps while the chaos is not possible in one or two-dimensional differential equations. In some cases, it is possible to derive maps from flows using the Poincaré map. Nonetheless, higher dimensional maps show dynamically rich behaviour.

One of the earliest and most studied chaotic maps is the Henon map [1]. Later on, a more analytically tractable version of the map known as the Lozi map was introduced and has been studied in great detail [2]. The Lozi map is piecewise linear. Piecewise linear maps have been studied in great detail recently due to their application in power electronics [3]. Bifurcations and the onset of chaos in nonlinear maps have been an object of immense interest to both mathematicians and physicists. We study a fractional version of the Lozi map in this work. Of late, fractional versions of nonlinear chaotic maps have attracted much attention [4–9]. Pioneering work has been done by Miller and Ross [10], Atici and Eloe [11] and notable others [12, 13]. The complex nature of fractional nonlinear models has also been investigated in [14]. A mechanical system can also be modelled using a fractional nonlinear map like the chaotic water wheel model in [15]. Another notable work investigating nonlinear maps of fractional order is [16]. Due to the possible applications of these systems in varied engineering and control processes, they are studied with interest. A nonlinear map or difference equation as it is studied often does not have any long-term memory. This is because the current state of the system depends upon the previous state of the system only. If we want to model a system with memory, the evolution should explicitly depend upon all the previous states of the system. This explicit dependence of the current state on the previous states introduces

DOI: https://doi.org/10.37256/cm.5420242423

long-term memory in the system. All definitions of the fractional difference equations include this explicit dependence on the previous states. Thus fractional difference equations are suitable mathematical models for systems having long-term memory. This makes the study of fractional nonlinear systems very interesting and useful from the point of view of many engineering and scientific applications. Chaos of the fractional difference maps has been the focus of a recent study by Xiao et al. [17]. Notable work has been done by Gejji [18], Baleanu [19], Atici [20], Edelman [21], etc on fractional difference equations.

Here the analysis done is mainly numerical. However, the extension of the theory of Lyapunov functions to fractional maps will be of great interest. It has been carried out for fractional differential equations [22].

We also note that unstable periodic orbits play an important role in the dynamics of Lozi map. However, for fractional order maps, we have results only for fixed point and period-2 orbit [23]. The Lozi map is similar to the Henon map. The difference is the presence of the piecewise linear term in place of the quadratic term in the Henon map. Lozi introduced a two-dimensional map in 1978 [24]. It resembles the Henon map [1] with the quadratic term in the Henon map replaced with a linear relation. The integer order Lozi map is given as,

$$x(t) = 1 + y(t-1) - a|x(t-1)|, \tag{1}$$

$$y(t) = \beta x(t-1), \tag{2}$$

where a and β are the map parameters that could be varied. The Lozi map is a modified version of the Henon map where the quadratic term in the Henon map is replaced by a linear term. For $\beta = 0.05$ and $\beta = 0.2$, we have shown a bifurcation diagram in Figure 1. Some studies on fractional versions of the Henon map (although with slightly different definitions) and Henon-Lozi map have been carried out previously [25, 26].

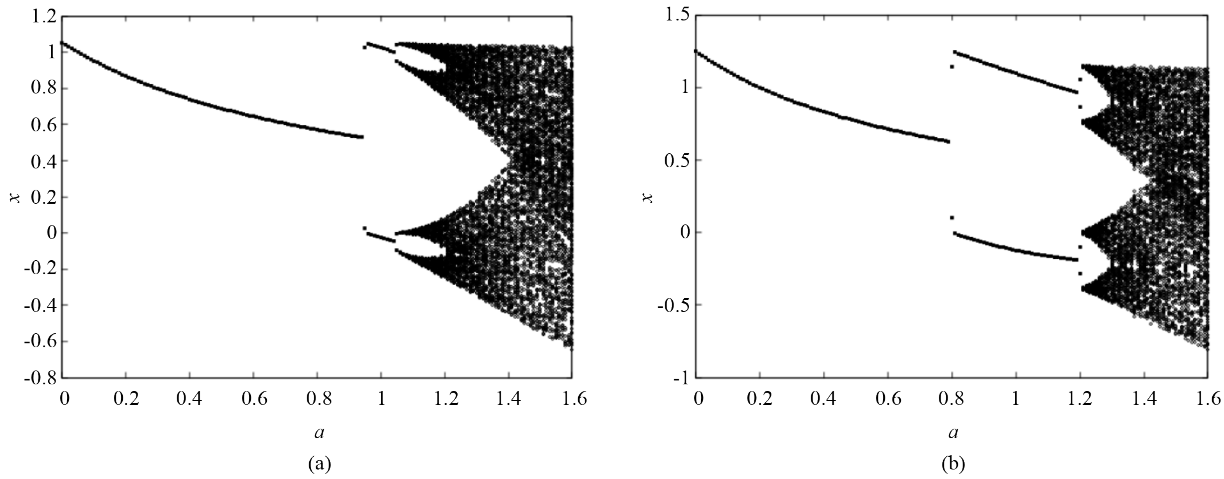

Figure 1. Bifurcation diagram for (a) $\beta = 0.05$ and (b) $\beta = 0.2$

We expect the dynamics to be different for the Lozi map than the Henon map. The dynamical behaviour of integer order Lozi map has been investigated extensively. The fractional version of the Lozi map displays rich dynamical behaviour like abrupt order-to-order or order-to-chaos transitions, infinite sets of period-3 cycles embedded in the sea of chaos [27]. We also note that analytical studies are possible in the Lozi map due to its piecewise linear nature. For example, Misiurewicz proved the conjecture of Lozi about the existence of a chaotic attractor for the Lozi map [28].

We define the Lozi map and show its bifurcation diagram for some parameters so that we can compare these dynamics to a fractional version of the Lozi map. This would help us to understand the impact of long-term memory on the dynamics of these systems.

2. Fractional Lozi map

We use the definition of fractional difference operator by Deshpande and Daftardar-Gejji [18]. According to their definition, the evolution of a 1-d fractional nonlinear map is given as,

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} \frac{\Gamma(t-j+\alpha)}{\Gamma(t-j+1)} [f(x(j-1)) - x(j-1)], \tag{3}$$

where $f(x)$ is a nonlinear map and α is the fractional order parameter which can have any real number values between 0 and 1. We extend this definition to 2-d by replacing x with vector $(x,\ y)$. A similar definition can be observed in [29]. The fractional version of the Lozi map is then given as,

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} \frac{\Gamma(t-j+\alpha)}{\Gamma(t-j+1)} [1 + y(j-1) - a|x(j-1)| - x(j-1)], \tag{4}$$

$$y(t) = y(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} \frac{\Gamma(t-j+\alpha)}{\Gamma(t-j+1)} [\beta x(j-1) - y(j-1)]. \tag{5}$$

As is evident from the above definition, the evolution of a current state explicitly depends upon all the previous states of the system. This is the reason the fractional nonlinear system is said to have memory. There are two map parameters, namely a and β.

We evolve the system defined by eq. (4) and (5) in time for various values of fractional order parameter α and map parameter β. We discard 10^4 initial iterates of the system and use the last 10 values of variable x to plot the bifurcation diagram as a function of map parameter a. Even if we discard the first 10^5 transients, the bifurcation diagram is unchanged. We observe that, for $\alpha = 0.4$, the system tends to go to infinity above $\beta = 0.1$. For $\alpha = 0.6$, the system goes to infinity above $\beta = 0.26$ and for $\alpha = 0.8$, it goes to infinity above $\beta = 0.54$. Thus we can explore a larger range of β for a larger value of α. We take $x = y = 0$ as the initial conditions. For $\beta = 0.05$, we observe a period-1, followed by period-2 and then period-4. Period-4 attractor develops into 4-band attractor which continuously merges and leads to 2-band attractor. The 2-bands continuously grow and merge to form a single attractor. The same dynamics is seen for larger β except that the fixed point is not observed. We observe that for larger dissipation i.e. for a large value of β the size of the chaotic attractor shrinks. The bifurcation diagrams for β = 0.05, 0.08 and 0.1 are presented in Figure 2. We can infer that for a larger value of fractional order parameter α, the system is more stable and doesn't go to infinity for a wider range of β values. For the lesser value of α, the system tends to go to infinity for lesser values of β. Thus for the higher values of the fractional order parameter α, the chaos is much robust. Increase in fractional order parameter α generally makes the system much more stable.

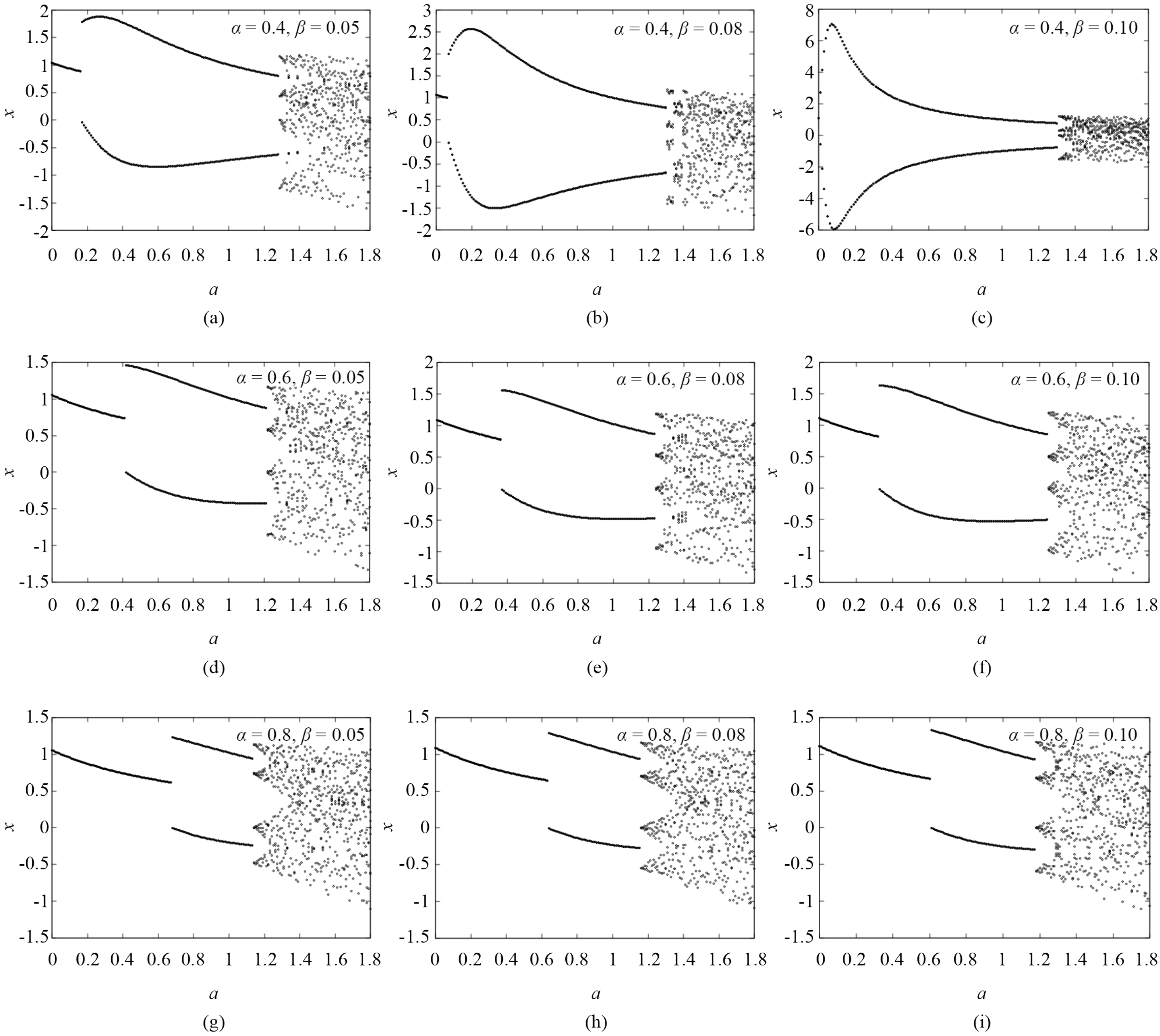

Figure 2. Bifurcation diagrams for fractional Lozi map for various values of fractional-order parameter α and map parameter β

The attractors can be visualized by the time-series plots. We iterate the system for 1.5×10^4 time steps and plot the variable values as a function of time. We observe that, as the map parameter a is increased, the system goes through a transition from a periodic regime to a chaotic regime in the manner described above. The period-1, period-2, period-4, 4-band, 2-band and single chaotic attractor can be observed from the time series given in Figure 3.

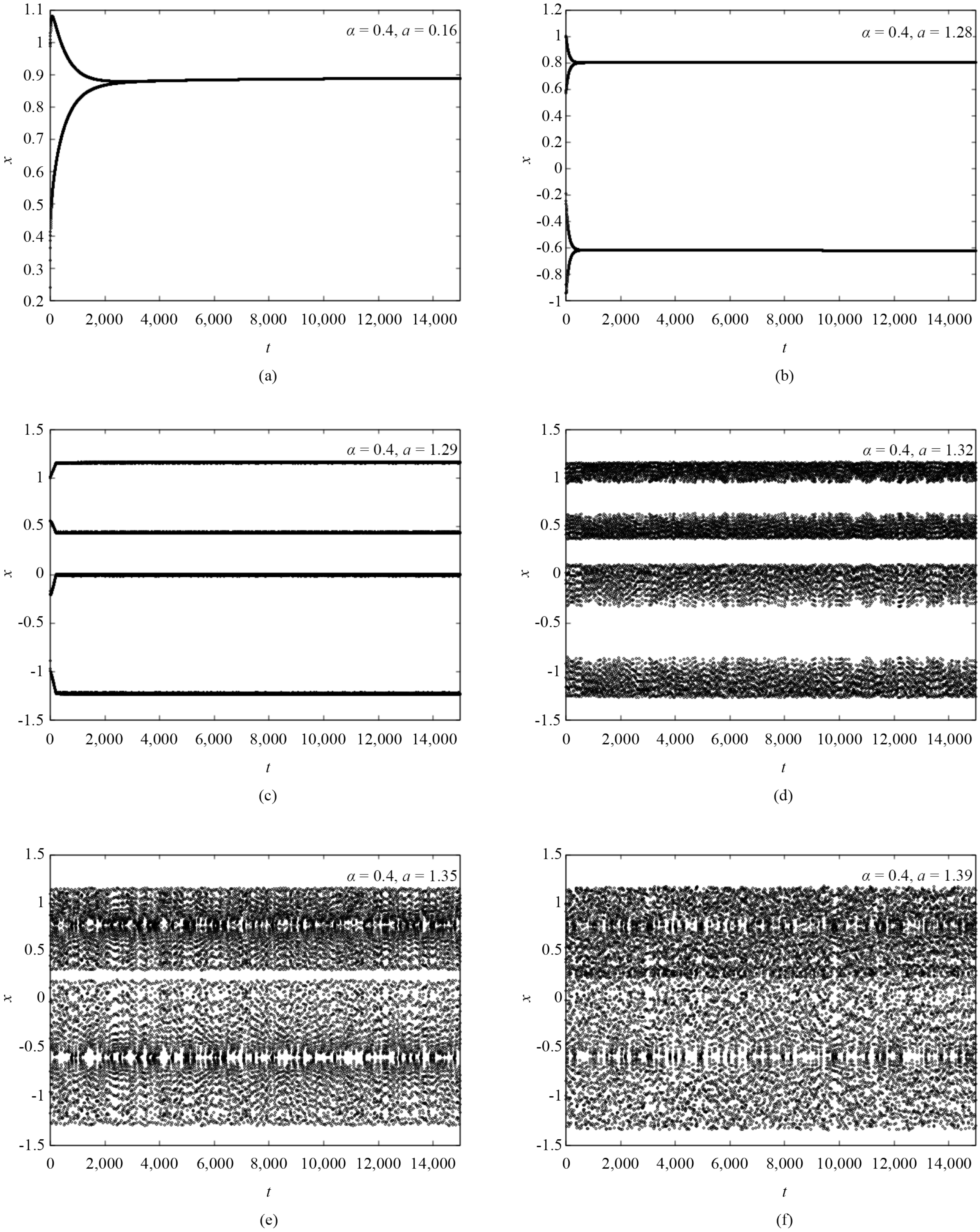

Figure 3. Time series plots for the fractional Lozi map showing (a) period-1 (b) period-2 (c) period-4 (d) 4-band (e) 2-band and (f) single band

We study the attractors for the system for fractional order parameter $\alpha = 0.4$ and map parameter $\beta = 0.05$ in further detail. The attractors shown in the figures below clearly show the transition as the value of a is changed from 1.29 for period-4 bifurcation to a single band for $a = 1.39$. We have shown period-4, 4-band, 2-band, and single band attractor in Figure 4 for $\alpha = 0.4$ as representative figures. Similar dynamics is observed for $\alpha = 0.6$ and $\alpha = 0.8$.

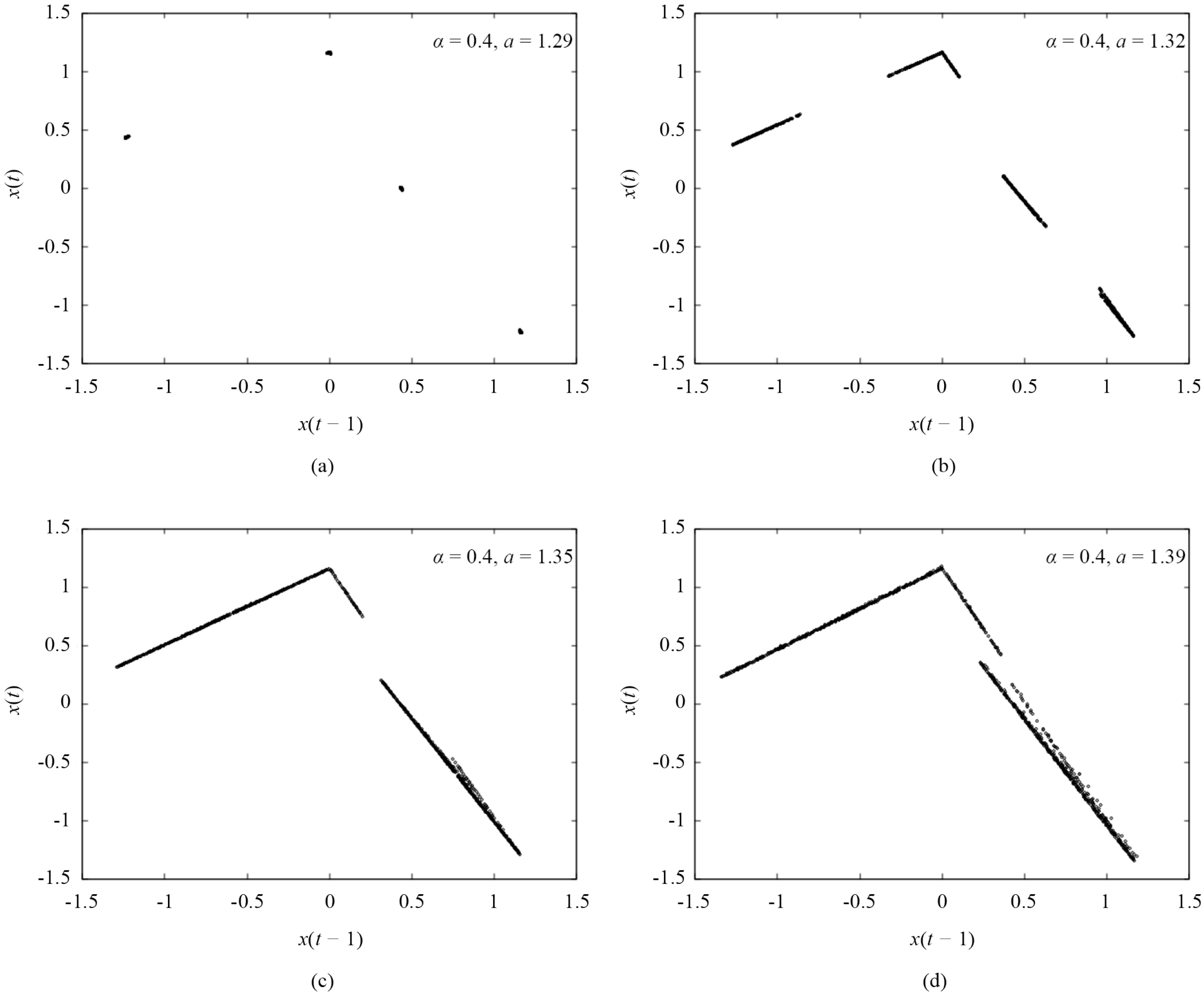

Figure 4. We plot $x(t-1)$ versus $x(t)$ for Lozi map after discarding several transients for fractional order ($\alpha = 0.4$) Lozi map for few values of β

3. Results and discussion

From the bifurcation diagrams, it is clear that the fractional-order parameter α affects the evolution of the fractional Lozi map system and the system can become unstable in the range the original map was stable. However, the overall bifurcation structure does not change much even for small values of α. The map parameter β also affects the evolution of the system. An increase in β however makes the system more unstable for the same value of α. We can explore larger values of β for larger values of α. From the control viewpoint, we find that the stable fixed point is observed over a larger range of parameters for smaller α. However, there is no significant change in the parameter regime where chaos is observed as we change α. Thus in this particular case, the fractional version of the map can be very useful for stabilizing the fixed point. The robustness of chaos is not lost and there are no periodic orbits inside the chaotic domain. In the chaotic region, the fractional order map is effectively higher dimensional compared to its integer order version. Hence it can have applications in cryptography and secure communication [30].

There have been several analytical results in the Lozi map. The genuine Lozi system does not show a period-doubling cascade. The smoother versions of this map show a period-doubling cascade. They also show abrupt transitions. We may think that the fractional version of the map is smoother and qualitatively different behaviour is obtained. This is not the case. Our results indicate that the dynamics deforms continuously on the introduction of the memory. It would be interesting to study if similar analytic studies can be carried out for the fractional Lozi map.

Acknowledgement

PMG thanks DST-SERB (CRG/2020/003993) for financial assistance.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Henon M. A two-dimensonal mapping with a strange attractor. *Communications in Mathematical Physics*. 1976; 50: 376-392.
[2] Elhadj Z. *Lozi Mappings: Theory and Applications*. CRC Press; 2013.
[3] Banerjee S, Ranjan P, Grebogi C. Bifurcations in two-dimensional piecewise smooth maps-theory and applications in switching circuits. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*. 2000; 47(5): 633-643.
[4] Hartley TT, Lorenzo CF. Dynamics and control of initialized fractional-order systems. *Nonlinear Dynamics*. 2002; 29(1): 201-233.
[5] Hwang C, Leu JF, Tsay SY. A note on time-domain simulation of feedback fractional-order systems. *IEEE Transactions on Automatic Control*. 2002; 47(4): 625-631.
[6] Petráš I, Podlubnỳ I, O'Leary P, Dorčák L, Vinagre BM. *Analogue Realizations of Fractional Order Controllers*. Technická univerzita; 2002.
[7] Babakhani A, Baleanu D, Khanbabaie R. Hopf bifurcation for a class of fractional differential equations with delay. *Nonlinear Dynamics*. 2012; 69(3): 721-729.
[8] Baleanu D, Diethelm K, Scalas E, Trujillo JJ. *Fractional Calculus: Models and Numerical Methods*. vol. 3. World Scientific; 2012.
[9] Li C, Peng G. Chaos in Chen's system with a fractional order. *Chaos, Solitons & Fractals*. 2004; 22(2): 443-450.
[10] Miller KS, Ross B. Fractional difference calculus. In: *Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and Their Applications*. Chichester: Horwood; 1989. p.139-152.
[11] Atıcı FM, Eloe PW. Two-point boundary value problems for finite fractional difference equations. *Journal of Difference Equations and Applications*. 2011; 17(04): 445-456.
[12] Bhalekar S, Gade PM, Joshi D. Stability and dynamics of complex order fractional difference equations. *Chaos, Solitons & Fractals*. 2022; 158: 112063.
[13] Peterson AC, Goodrich C. *Discrete Fractional Calculus*. Springer; 2015.
[14] Veeresha P. The efficient fractional order based approach to analyze chemical reaction associated with pattern formation. *Chaos, Solitons & Fractals*. 2022; 165: 112862.
[15] Deepika S, Veeresha P. Dynamics of chaotic waterwheel model with the asymmetric flow within the frame of Caputo fractional operator. *Chaos, Solitons & Fractals*. 2023; 169: 113298.
[16] Veeresha P, Baskonus HM, Gao W. Strong interacting internal waves in rotating ocean: novel fractional approach. *Axioms*. 2021; 10(2): 123.

[17] Xiao H, Ma Y, Li C. Chaotic vibration in fractional maps. *Journal of Vibration and Control*. 2014; 20(7): 964-972.
[18] Deshpande A, Daftardar-Gejji V. Chaos in discrete fractional difference equations. *Pramana*. 2016; 87(4): 1-10.
[19] Wu GC, Baleanu D. Discrete fractional logistic map and its chaos. *Nonlinear Dynamics*. 2014; 75(1): 283-287.
[20] Atici F, Eloe P. Initial value problems in discrete fractional calculus. *Proceedings of the American Mathematical Society*. 2009; 137(3): 981-989.
[21] Edelman M. Caputo standard α-family of maps: Fractional difference vs. fractional. *Chaos: An Interdisciplinary Journal of Nonlinear Science*. 2014; 24(2): 023137.
[22] Aguila-Camacho N, Duarte-Mermoud MA, Gallegos JA. Lyapunov functions for fractional order systems. *Communications in Nonlinear Science and Numerical Simulation*. 2014; 19(9): 2951-2957.
[23] Bhalekar S, Gade PM. Fractional-order periodic maps: Stability analysis and application to the periodic-2 limit cycles in the nonlinear systems. *Journal of Nonlinear Science*. 2023; 33: 119.
[24] Lozi R. Un attracteur étrange (?) du type attracteur de Hénon. *Le Journal de Physique Colloques*. 1978; 39(C5): C5-9.
[25] Hu T. Discrete chaos in fractional Hénon map. *Applied Mathematics*. 2014; 5(15): 2243-2248.
[26] Ouannas A, Khennaoui AA, Wang X, Pham VT, Boulaaras S, Momani S. Bifurcation and chaos in the fractional form of Hénon-Lozi type map. *The European Physical Journal Special Topics*. 2020; 229(12): 2261-2273.
[27] Botella-Soler V, Castelo J, Oteo J, Ros J. Bifurcations in the Lozi map. *Journal of Physics A: Mathematical and Theoretical*. 2011; 44(30): 305101.
[28] Misiurewicz M, Štimac S. Lozi-like maps. *arXiv*. 2017. Available from: https://doi.org/10.48550/arXiv.1705.08511.
[29] Wang L, Sun K, Peng Y, He S. Chaos and complexity in a fractional-order higher-dimensional multicavity chaotic map. *Chaos, Solitons & Fractals*. 2020; 131: 109488.
[30] Cao W, Cai H, Hua Z. n-Dimensional Chaotic Map with application in secure communication. *Chaos, Solitons & Fractals*. 2022; 163: 112519.