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Research Article

An Efficient Technique to Study Time Fractional Extended Fisher-Kolmogorov Equation

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Abstract: Reaction-diffusion partial differential equations are among the most widely used equations in applied mathematical modelling. In this study, we examine the solutions of one such equation, namely, the time-fractional extended Fisher-Kolmogorov equation. This equation is widely used in the study of population growth and wave propagation dynamics. The technique we consider is a combination of the natural transform and the Adomian decomposition method. Fractional derivatives are considered with singular and non-singular kernels. The existence and uniqueness of the solutions are presented. We analyse two different cases of the proposed problem to determine the validity and efficacy of the proposed scheme. Additionally, numerical simulation is shown, and the nature of the achieved solution is captured in terms of plots for various fractional orders. The outcome demonstrates that the method is straightforward, efficient, and dependable. The proposed method does not require any predetermined assumptions, linearization, perturbation, or discretization, and it prevents rounding errors. Therefore, the technique is ready to be implemented for a variety of nonlinear time fractional partial differential equations.

***Keywords*:** time-fractional extended fisher-kolmogorov equation, natural transform, adomian decomposition method

MSC: 26A33, 35A22, 35R11

1. Introduction

Ronald Fisher proposed the reaction-diffusion equation known as Fisher's equation in 1937. It is a partial differential equation in inhomogeneous form. In population growth dynamics and wave propagation, this equation is utilized. The benefits of population dynamics of wave spatial distribution of a beneficial allele were proposed by Fisher [1]. In 1937, Fisher proposed a broader reaction-diffusion model through the contributions of Kolmogorov, Petrovsky, and Piskunov and introduced a new model known as the Kolmogorov-Petrovsky-Piskunov equation, which is employed in population genetics. A branch of genetics called population genetics, which is related to evolutionary biology, examines genetic variations within and between populations. Plasma physics, ecology, phase transition, and physiology issues are all generating more and more of these equations.

The Fisher-Kolmogorov (FK) equation has the following standard form:

$$\phi_t - \Delta\phi + \phi^3 - \phi = 0,\ \Omega \times [0, T]. \tag{1}$$

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The classical FK equation is shown in [2-4]. There has been significant scholarly interest in fractional subdiffusion due to its prevalence in biological systems. This phenomenon is characterised by the mean square displacement of a population exhibiting a sublinear power law relationship with respect to time. The prevailing consensus in the scientific community is that the time fractional diffusion equation is the suitable mathematical model for subdiffusion, assuming that it originates from particles being trapped for indefinite durations. In this study, we have considered one such model, namely, the time fractional extended Fisher-Kolmogorov (TFEFK) equation for the real-valued function P defined on $\Omega \times [0, T]$ as [5-9] was obtained in the aforementioned (1) by adding a stabilising fourth-order derivative term.

$$D_t^{\delta}\phi(\zeta, y, t) + \alpha\Delta^2\phi - \Delta\phi + \phi^3 - \phi = 0, \Omega \times [0, T], \tag{2}$$

with the initial condition,

$$\phi(\zeta, y, 0) = \sin(\zeta)\sin(y). \tag{3}$$

Here, α is a positive constant, Ω is bounded in domain R^2, the Laplace operator is $\Delta = \frac{\partial^2}{\partial\zeta^2} + \frac{\partial^2}{\partial y^2}$, and the biharmonic operator is $\Delta^2 = \frac{\partial^4}{\partial\zeta^4} + 2\frac{\partial^4}{\partial\zeta^2\partial y^2} + \frac{\partial^4}{\partial y^4}$. With the use of the above operators, Equation (2) takes the form

$$D_t^{\delta}\phi(\zeta, y, t) + \alpha\frac{\partial^4\phi}{\partial\zeta^4} + 2\alpha\frac{\partial^4\phi}{\partial\zeta^2\partial y^2} + \alpha\frac{\partial^4\phi}{\partial y^4} - \frac{\partial^2\phi}{\partial\zeta^2} - \frac{\partial^2\phi}{\partial y^2} + \phi^3 - \phi = 0. \tag{4}$$

In the aforementioned Equation (4), the fourth-order term represents the phase transitions that occur close to critical points, sometimes referred to as Lipschitz points. This term plays a crucial role in phase changes or transitions, which are physical processes of transitioning from one state to the next medium at various parameter values that are close to the critical value. Under exceptional circumstances, the expression is widely applied to refer to the changes in the basic matter of states such as gaseous, liquid, solid, and plasma [10].

The primary objective of examining the concept of fractional calculus (FC) is to gain comprehension of the phenomenon of heterogeneity that is linked to complexity. Furthermore, it has been demonstrated that FC is the most effective tool for elucidating the intricacies of diffusion processes. This is due to the limitations of integer order calculus, which fails to capture the intricate dynamics exhibited by complex and nonlinear models that incorporate temporal factors, historical context, and their subsequent ramifications. FC has been extensively used in numerous research and application fields, such as signal processing, physics, electrochemistry, biology, rheology, mechanics, ecology, neural networking, optics, and image processing [11-21]. Recently, the primary goal of many researchers and scientists has been to develop an exact or approximate solution to the linear or nonlinear fractional partial differential equations (FPDEs). This has made substantial advances in a variety of different mathematical fields.

One of the most notable features of fractional derivatives is their lack of classical asymptotic behaviour. Furthermore, the effectiveness of these derivatives is currently under evaluation through the utilisation of real-world procedures, and it is imperative to enhance dynamic numerical methods in order to facilitate their practical implementation. Nevertheless, because the nonlinear operator more accurately explains these occurrences, nonlinearity has drawn significant interest. There have been a lot of fractional derivative definitions during the past few centuries. In the literature, there are several well-known fractional derivative definitions, such as Atangana-Baleanu-Caputo (*ABC*), Riesz, Riemann-Liouville (R-L), Caputo (*C*), Caputo-Fabrizio (*CF*), and Grunwald-Letnikov. Many eminent academics have proposed a separate definition for both integral and differential operators with fractional order. Each fundamental idea, though, has limitations of its own. The R-L derivative does not adequately capture the significance of the beginning circumstances, and the singular kernel is not related to Caputo's notion of FC. The operator was designed by Caputo et al. [22] in 2015 to go beyond the aforementioned restrictions. Subsequent writers used the operator to research and show some interesting behaviour in nonlinear, complicated situations.

Many researchers have recently brought up certain concerns regarding fundamental characteristics, such as the non-local and non-singular kernels, that describe the behaviour of nonlinear problems. Atangana et al. [23] created the new fractional derivative known as the *ABC* derivative in 2016 to overcome these restrictions using Mittag-Leffler functions.

The aforementioned problems were all buried by this derivative. In this regard, numerous numerical techniques have been broadly adopted and developed to solve a wide range of linear and nonlinear problems in FPDEs, such as the Adomian decomposition method [24], residual power series method [25], iterative Laplace transform method [26], Laplace homotopy analysis method [27], homotopy analysis method [28], variational iteration technique [29], homotopy perturbation technique [30], reduced differential transform method [31], modified variational iteration method [32], L1-predictor-corrector method [33], and Daftardar-Gejji and Jafari's iterative method [34].

The goal of this study is to use the natural transform decomposition method (NTDM) to solve the TFEFK equation. For a class of nonlinear partial differential equations (PDEs), Rawashdeh et al. [35] proposed this technique. Round-off errors are eliminated by the NTDM without the need for imposing assumptions, discretization, perturbation, or linearization. Two powerful methods were used to develop the NTDM: natural transform (NT) and Adomian decomposition. This new approach is thought to be a superb tool for quickly and easily resolving particular classes of nonlinear PDEs. The solution provided by this method might be precise or approximative and is based on a quick convergence series. NTDM was recently used to study various time fractional differential equations such as the Zakharov-Kuznetsov equation [36], Klein-Gordon equation [37], Fisher's fractional-order equation [38], Kawahara and modified Kawahara equation [39], and Burgers-Huxely equation [40].

The paper formation is described in the paragraphs that follow. In Section 2, the NT of important definitions is presented, along with some other findings that are helpful in the research. In Section 3, the fundamental ideal of NTDM, along with three fractional derivatives, is presented. Section 4 focuses on convergence and uniqueness solutions. In Section 5, numerical results and discussion, as well as the TFEFK equation approximate solutions, are presented. In Section 6, we have presented the conclusions.

2. Preliminaries

Here, we introduce the essential ideas of singular and nonsingular derivatives and NT.

Definition 2.1 [41] The fractional order derivative of $h(t) \in C_{-1}^{q}$ in C derivative is given by

$$D_t^{\delta} h(t) = \begin{cases} \dfrac{d^q h(t)}{dt^q}, \ \delta = q \in \mathbb{N}, \\ \dfrac{1}{\Gamma(q-\delta)} \displaystyle\int_0^t (t-\xi)^{q-\delta-1} h^q(\xi) d\xi, \ q-1 < \delta \le q, \ q \in \mathbb{N}. \end{cases} \tag{5}$$

Definition 2.2 [42] The *CF* derivative of the function $h(t)$ of order δ is represented by the notation with the property $0 < \delta \le 1$ is defined as

$${}^{CF}D_t^{\delta} h(t) = \frac{1}{1-\delta} \int_0^t h'(\zeta) \exp\left(\frac{-\delta(t-\zeta)}{1-\delta} \right) d\zeta, \ t \ge 0. \tag{6}$$

Definition 2.3 [43] Let $h \in H'(a, e)$, $e > a$, $\delta \in [0, 1]$ be the case. The *ABC* derivative of $h(t)$, is defined by

$${}^{ABC}D_t^{\delta} h(t) = \frac{M[\delta]}{1-\delta} \int_a^t h'(\zeta) E_{\delta} \left(\frac{-\delta(t-\zeta)^{\delta}}{1-\delta} \right) d\zeta. \tag{7}$$

In this case, $M[\delta]$ is a normalisation function or constant with the property $0 < \delta \le 1$.

Definition 2.4 [44] NT for the function $h(t)$ is given by

$$N^{+}[h(t)] = R(s, \phi) = \frac{1}{\phi} \int_0^{\infty} e^{(\frac{-st}{\phi})} h(t) dt, \quad \phi, s > 0. \tag{8}$$

Definition 2.5. [45] The definition of the NT for *C* derivative is

$$N^{+}[{}_{0}^{C}D_{t}^{\delta}\phi(t)] = \left(\frac{s}{v}\right)^{\delta}\left(N^{+}[\phi(t)] - \frac{1}{s}\phi(0)\right). \tag{9}$$

Definition 2.6 [46] The definition of the NT for *CF* derivative is

$$N^{+}[{}_{0}^{CF}D_{t}^{\delta}\phi(t)] = \frac{1}{1-\delta+\delta\left(\frac{v}{s}\right)}\left(N^{+}[\phi(t)] - \frac{1}{s}\phi(0)\right). \tag{10}$$

Definition 2.7 [45] The definition of the NT for *ABC* derivative is

$$N^{+}[{}_{0}^{ABC}D_{t}^{\delta}\phi(t)] = \frac{M[\delta]}{1-\delta+\delta\left(\frac{v}{s}\right)^{\delta}}\left(N^{+}[\phi(t)] - \frac{1}{s}\phi(0)\right). \tag{11}$$

In this case, $M[\delta]$ is a normalisation function that follows the condition $M(0) = M(1) = 1$.

3. NTDM

In this section, the method of NT decomposition is applied to the study of the TFEFK equation. Now, we apply the NT to the nonhomogeneous form of Equation (2) by considering the three fractional derivatives such as *C*, *CF*, and *ABC*.

NTDM$_C$: On taking the NT of Equation (2) and also using the *C* derivative, we have

$$\left(\frac{s}{v}\right)^{\delta}\left[N^{+}[\phi(\zeta, y, t)] - \frac{\sin(\zeta)\sin(y)}{s}\right] = N^{+}[g(\zeta, y, t) - \alpha\Delta^{2}\phi + \Delta\phi + \phi - \phi^{3}]. \tag{12}$$

By applying inverse NT on (12), we get

$$\phi(\zeta, y, t) = N^{-1}\left[\frac{\sin(\zeta)\sin(y)}{s} + \left(\frac{v}{s}\right)^{\delta} N^{+}[g(\zeta, y, t) - \alpha\Delta^{2}\phi + \Delta\phi + \phi - \phi^{3}]\right]. \tag{13}$$

The nonlinear terms can be expressed as

$$\phi^{3} = \sum_{k=0}^{\infty} A_{k}, \tag{14}$$

where A_k denotes the Adomian polynomials. The unknown functions $\phi(\zeta, y, t)$ is an infinite series solution denoted by

$$\phi(\zeta, y, t) = \sum_{k=0}^{\infty} \phi_{k}(\zeta, y, t). \tag{15}$$

By making use of the Equations (14) and (15) into (13), we obtain

$$\sum_{k=0}^{\infty}\phi_k(\zeta, y, t) = N^{-1}\left[\frac{\sin(\zeta)\sin(y)}{s}\right]$$
$$+N^{-1}\left[\left(\frac{v}{s}\right)^{\delta} N^{+}\left[g(\zeta, y, t) - \alpha\sum_{k=0}^{\infty}(\Delta^2\phi)_k + \sum_{k=0}^{\infty}(\Delta\phi)_k + \sum_{k=0}^{\infty}\phi_k - \sum_{k=0}^{\infty}A_k\right]\right]. \tag{16}$$

From Equation (16), we get

$$\begin{aligned}
{}^{C}\phi_0(\zeta, y, t) &= g(\zeta, y, t) + \sin(\zeta)\sin(y),\\
{}^{C}\phi_1(\zeta, y, t) &= N^{-1}\left[\left(\frac{v}{s}\right)^{\delta} N^{+}[\alpha(\Delta^2\phi)_0 + (\Delta\phi)_0 + \phi_0 - A_0]\right],\\
{}^{C}\phi_2(\zeta, y, t) &= N^{-1}\left[\left(\frac{v}{s}\right)^{\delta} N^{+}[\alpha(\Delta^2\phi)_1 + (\Delta\phi)_1 + \phi_1 - A_1]\right],\\
&\vdots\\
{}^{C}\phi_{k+1}(\zeta, y, t) &= N^{-1}\left[\left(\frac{v}{s}\right)^{\delta} N^{+}[\alpha(\Delta^2\phi)_k + (\Delta\phi)_k + \phi_k - A_k]\right],\ k \geq 0.
\end{aligned} \tag{17}$$

By substituting (17) into (15), we obtain the solution as

$${}^{C}\phi(\zeta, y, t) = {}^{C}\phi_0(\zeta, y, t) + {}^{C}\phi_1(\zeta, y, t) + {}^{C}\phi_2(\zeta, y, t) + \ldots. \tag{18}$$

NTDM$_{CF}$: On taking the NT of Equation (2) with the help of *CF* derivative, we have

$$\frac{1}{1-\delta+\delta\left(\frac{v}{s}\right)}\left[N^{+}[\phi(\zeta, y, t)] - \frac{\sin(\zeta)\sin(y)}{s}\right] = N^{+}[g(\zeta, y, t) - \alpha\Delta^2\phi + \Delta\phi + \phi - \phi^3]. \tag{19}$$

Using inverse NT on Equation (19), we obtain

$$\phi(\zeta, y, t) = N^{-1}\left[\frac{\sin(\zeta)\sin(y)}{s} + \left(1-\delta+\delta\left(\frac{v}{s}\right)\right)N^{+}[g(\zeta, y, t) - \alpha\Delta^2\phi + \Delta\phi + \phi - \phi^3]\right]. \tag{20}$$

Now, we substitute Equations (14) and (15) into (20)

$$\sum_{k=0}^{\infty}\phi_k(\zeta, y, t) = N^{-1}\left[\frac{\sin(\zeta)\sin(y)}{s}\right]$$
$$+N^{-1}\left[\left(1-\delta+\delta\left(\frac{v}{s}\right)\right)N^{+}\left[g(\zeta, y, t) - \alpha\sum_{k=0}^{\infty}(\Delta^2\phi)_k + \sum_{k=0}^{\infty}(\Delta\phi)_k + \sum_{k=0}^{\infty}\phi_k - \sum_{k=0}^{\infty}A_k\right]\right]. \tag{21}$$

From Equation (21), we obtain

$$
\begin{aligned}
&{}^{CF}\phi_0(\zeta, y, t) = g(\zeta, y, t) + \sin(\zeta)\sin(y), \\
&{}^{CF}\phi_1(\zeta, y, t) = N^{-1}\left[\left(1-\delta+\delta\left(\frac{v}{s}\right)\right)N^{+}[\alpha(\Delta^2\phi)_0 + (\Delta\phi)_0 + \phi_0 - A_0]\right], \\
&{}^{CF}\phi_2(\zeta, y, t) = N^{-1}\left[\left(1-\delta+\delta\left(\frac{v}{s}\right)\right)N^{+}[\alpha(\Delta^2\phi)_1 + (\Delta\phi)_1 + \phi_1 - A_1]\right], \\
&\quad\vdots \\
&{}^{CF}\phi_{k+1}(\zeta, y, t) = N^{-1}\left[\left(1-\delta+\delta\left(\frac{v}{s}\right)\right)N^{+}[\alpha(\Delta^2\phi)_k + (\Delta\phi)_k + \phi_k - A_k]\right], k \geq 0.
\end{aligned} \tag{22}
$$

By substituting (22) into (15), we obtain the solution as

$$
{}^{CF}\phi(\zeta, y, t) = {}^{CF}\phi_0(\zeta, y, t) + {}^{CF}\phi_1(\zeta, y, t) + {}^{CF}\phi_2(\zeta, y, t) + \dots. \tag{23}
$$

NTDM$_{ABC}$: On taking the NT of Equation (2) with the help of *ABC* derivative, we have

$$
\frac{M[\delta]}{1-\delta+\delta\left(\frac{v}{s}\right)^{\delta}}\left[N^{+}[\phi(\zeta, y, t)] - \frac{\sin(\zeta)\sin(y)}{s}\right] = N^{+}[g(\zeta, y, t) - \alpha\Delta^2\phi + \Delta\phi - \phi^3 + \phi]. \tag{24}
$$

By taking inverse NT on Equation (24), we get

$$
\phi(\zeta, y, t) = N^{-1}\left[\frac{\sin(\zeta)\sin(y)}{s} + \frac{1-\delta+\delta\left(\frac{v}{s}\right)^{\delta}}{M[\delta]}N^{+}[g(\zeta, y, t) - \alpha\Delta^2\phi + \Delta\phi + \phi - \phi^3]\right]. \tag{25}
$$

Now, we substitute Equations (14) and (15) into (25)

$$
\begin{aligned}
\sum_{k=0}^{\infty}\phi_k(\zeta, y, t) = {} & N^{-1}\left[\frac{\sin(\zeta)\sin(y)}{s}\right] \\
& + N^{-1}\left[\frac{1-\delta+\delta\left(\frac{v}{s}\right)^{\delta}}{M[\delta]}N^{+}\left[g(\zeta, y, t) - \alpha\sum_{k=0}^{\infty}(\Delta^2\phi)_k + \sum_{k=0}^{\infty}(\Delta\phi)_k + \sum_{k=0}^{\infty}\phi_k - \sum_{k=0}^{\infty}A_k\right]\right].
\end{aligned} \tag{26}
$$

From Equation (26), we obtain

$$^{ABC}\phi_0(\zeta, y, t) = g(\zeta, y, t) + \sin(\zeta)\sin(y),$$

$$^{ABC}\phi_1(\zeta, y, t) = N^{-1}\left[\frac{1-\delta+\delta\left(\frac{v}{s}\right)^{\delta}}{M[\delta]} N^{+}[\alpha(\Delta^2\phi)_0 + (\Delta\phi)_0 + \phi_0 - A_0]\right],$$

$$^{ABC}\phi_2(\zeta, y, t) = N^{-1}\left[\frac{1-\delta+\delta\left(\frac{v}{s}\right)^{\delta}}{M[\delta]} N^{+}[\alpha(\Delta^2\phi)_1 + (\Delta\phi)_1 + \phi_1 - A_1]\right], \quad (27)$$

$$\vdots$$

$$^{ABC}\phi_{k+1}(\zeta, y, t) = N^{-1}\left[\frac{1-\delta+\delta\left(\frac{v}{s}\right)^{\delta}}{M[\delta]} N^{+}\left[\alpha(\Delta^2\phi)_k + (\Delta\phi)_k + \phi_k - A_k\right]\right], k \geq 0.$$

By substituting (27) into (15), we obtain the solution as

$$^{ABC}\phi(\zeta, y, t) = {}^{ABC}\phi_0(\zeta, y, t) + {}^{ABC}\phi_1(\zeta, y, t) + {}^{ABC}\phi_2(\zeta, y, t) + \dots. \quad (28)$$

4. Convergence analysis

We state the convergence and uniqueness statements of the approximate solutions in a similar manner of [45] in this section.

Theorem 1 The proposed method solution $^{C}\phi$ of (2) is unique when $0 < (\xi_1 + \xi_2)\frac{t^{\delta}}{\Gamma(1+\delta)} < 1$.

Theorem 2 The proposed method solution $^{C}\phi$ of (2) is convergent.

Similarly, projected method solutions $^{CF}\phi$ and $^{ABC}\phi$ of (2) will be unique and convergent.

5. Numerical illustrations

In this section, we have presented the proposed technique solutions of Equation (2) with the initial condition (3).

5.1 *Case 1*

By making use of the procedure presented in Section 3, we will find successive solutions with respect to three fractional derivatives.

NTDM$_C$:

$$^{C}\phi_0(\zeta, y, t) = \sin(\zeta)\sin(y),$$

$$^{C}\phi_1(\zeta, y, t) = -\frac{\sin(\zeta)t^{\delta}\sin(y)(4\alpha + \sin^2(\zeta)\sin^2(y) + 1)}{\Gamma(\delta+1)},$$

$$^{C}\phi_2(\zeta, y, t) = \Big(\big(3\sin^4(\zeta)\sin^4(y) + 80\sin^2(\zeta)\sin^2(y)\big)$$

$$+\big(3(29\cos(2\zeta) - 5)\cos^2(y) - 102\cos^2(\zeta)\sin^2(y) + 25\big)\Big)\frac{\sin(\zeta)t^{2\delta}\sin(y)}{\Gamma(2\delta+1)},$$

$$\vdots$$

By making use of these successive values in (18), we get the approximate solution.

NTDM$_{CF}$:

$$^{CF}\phi_0(\zeta, y, t) = \sin(\zeta)\sin(y),$$

$$^{CF}\phi_1(\zeta, y, t) = \sin(\zeta)(\delta - \delta t - 1)\sin(y)(4\alpha + \sin^2(\zeta)\sin^2(y) + 1),$$

$$^{CF}\phi_2(\zeta, y, t) = \big(256\alpha(4\alpha - 1) - 164\cos(2\zeta) + 4\cos(2y)(4(336\alpha + 23)\cos(2\zeta) - 41) + (9 - 12\cos(2\zeta))\cos(4y)$$

$$+24\cos(4\zeta)\sin^4(y) + 27\big)\frac{1}{128}\big(\delta^2((t-4)t + 2) + 4\delta(t-1) + 2\big)\sin(y)\sin(\zeta),$$

$$\vdots$$

By making use of these successive values in (23), we get the approximate solution.

NTDM$_{ABC}$:

$$^{ABC}\phi_0(\zeta, y, t) = \sin(\zeta)\sin(y),$$

$$^{ABC}\phi_1(\zeta, y, t) = \sin(\zeta)\sin(y)\left(\delta - \frac{\delta t^{\delta}}{\Gamma(\delta+1)} - 1\right)\big(4\alpha + \sin^2(\zeta)\sin^2(y) + 1\big),$$

$$^{ABC}\phi_2(\zeta, y, t) = \big(256\alpha(4\alpha - 1) - 164\cos(2\zeta) + 4\cos(2y)(4(336\alpha + 23)\cos(2\zeta) - 41 + 24\cos(4\zeta)\sin^4(y) + 27)$$

$$+(9 - 12\cos(2\zeta))\cos(4y)\big)\frac{1}{64}\sin(\zeta)\sin(y)\left((\delta-1)^2 + \frac{\delta^2 t^{2\delta}}{\Gamma(2\delta+1)} - \frac{2t^{\delta}}{\Gamma(\delta-1)}\right),$$

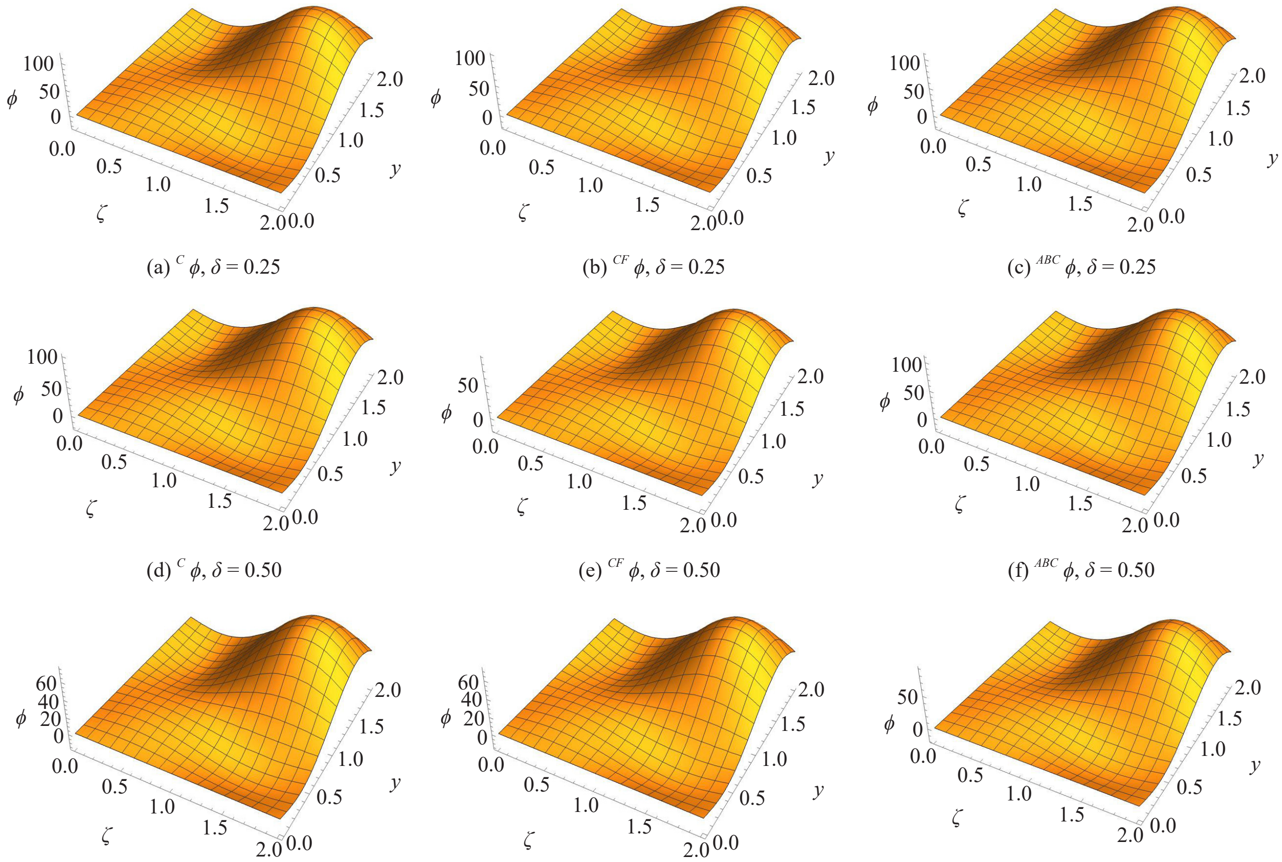
$$\vdots$$

By making use of these successive values in (28), we get the approximate solution.

The approximate solution of Case 1 for different values of ζ, y, and fixed t at different δ orders is shown in Table 1. Figure 1 illustrates the characteristics of the solution for the homogeneous case of the TFEFK equation, which is defined in Case 1 with distinct fractional orders. The approximate solutions of two-dimensional graphical representations for various fractional order values are shown in Figure 2.

Table 1. Approximate solution of Case 1 for different values of ζ, y and δ with $t = 1$

ζ	y	$\delta = 0.7$			$\delta = 0.8$		
		$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$	$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$
0.25	0.25	3.52	3.31484	4.19717	3.03051	2.9612	3.65629
0.25	0.5	4.36881	4.12372	5.2498	3.74168	3.66678	4.55319
0.25	0.75	1.35744	1.3108	1.75643	1.10199	1.1131	1.4618
0.5	0.25	4.36881	4.12372	5.2498	3.74168	3.66678	4.55319
0.5	0.5	5.58135	5.28601	6.78296	4.74327	4.66838	5.84545
0.5	0.75	2.22951	2.16486	2.93784	1.78405	1.81649	2.42066
0.75	0.25	1.35744	1.3108	1.75643	1.10199	1.1131	1.4618
0.75	0.5	2.22951	2.16486	2.93784	1.78405	21.81649	2.42066
0.75	0.75	2.39911	2.35674	3.28193	1.86087	1.92804	2.64972
ζ	y	$\delta = 0.9$			$\delta = 1$		
		$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$	$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$
0.25	0.25	2.55524	2.56042	2.9517	2.11249	2.11249	2.11249
0.25	0.5	3.13432	3.14892	3.64738	2.57013	2.57013	2.57013
0.25	0.75	0.859243	0.889042	1.08357	0.638626	0.638626	0.638626
0.5	0.25	3.13432	3.14892	3.64738	2.57013	2.57013	2.57013
0.5	0.5	3.93444	3.96839	4.62936	3.18605	3.18605	3.18605
0.5	0.75	1.36222	1.42167	1.75821	0.980398	0.980398	0.980398
0.75	0.25	0.859243	0.889042	1.08357	0.638626	0.638626	0.638626
0.75	0.5	1.36222	1.42167	1.75821	0.980398	0.980398	0.980398
0.75	0.75	1.35432	1.44218	1.84309	0.899163	0.899163	0.899163

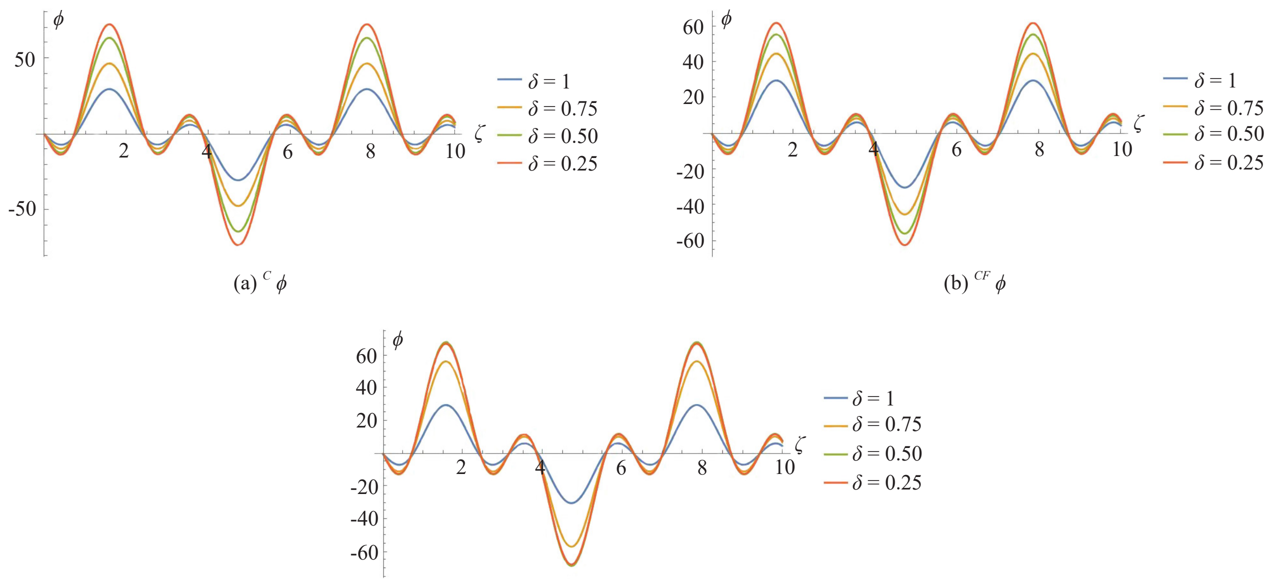


(a) $^{C}\phi$, $\delta = 0.25$ (b) $^{CF}\phi$, $\delta = 0.25$ (c) $^{ABC}\phi$, $\delta = 0.25$

(d) $^{C}\phi$, $\delta = 0.50$ (e) $^{CF}\phi$, $\delta = 0.50$ (f) $^{ABC}\phi$, $\delta = 0.50$

(g) $^{C}\phi$, $\delta = 0.75$ (h) $^{CF}\phi$, $\delta = 0.75$ (i) $^{ABC}\phi$, $\delta = 0.75$

Figure 1. Approximate solution of Case 1 with $t = 1$, different values of δ



(a) $^{C}\phi$ (b) $^{CF}\phi$

(c) $^{ABC}\phi$

Figure 2. Approximate solution of Case 1 with $t = 1$, $y = 2$ for different values of δ

5.2 *Case 2*

In this case, we have considered the nonhomogeneous form of Equation (2) by taking the $g(\zeta, y, t) = 4\alpha e^{-t}\sin(\zeta)\sin(y) + e^{-3t}\sin^3(\zeta)\sin^3(y)$ with the initial condition (3).

NTDM$_C$:

$$^{C}\phi_0(\zeta, y, t) = \sin(\zeta)\sin(y),$$

$$^{C}\phi_1(\zeta, y, t) = -\frac{\sin(\zeta)t^{\delta}\sin(y)}{\Gamma(\delta+1)},$$

$$^{C}\phi_2(\zeta, y, t) = \sin(\zeta)t^{\delta}\sin(y)\left(\frac{t^{\delta}\left(4\alpha+3\sin^2(\zeta)\sin^2(y)+1\right)}{\Gamma(2\delta+1)} - \frac{t\left(4\alpha+3\sin^2(\zeta)\sin^2(y)\right)}{\Gamma(\delta+2)}\right),$$

$$\vdots$$

By making use of these successive values in (18), we obtain the approximate solution.

NTDM$_{CF}$:

$$^{CF}\phi_0(\zeta, y, t) = \sin(\zeta)\sin(y),$$

$$^{CF}\phi_1(\zeta, y, t) = \sin(\zeta)(\delta+\delta(-t)-1)\sin(y),$$

$$^{CF}\phi_2(\zeta, y, t) = \frac{1}{2}\sin(\zeta)\sin(y)\Big(4\alpha(\delta-1)(\delta((t-4)t+2)+2(t-1))+\delta^2((t-4)t+2)$$

$$+4\delta(t-1)+3(\delta-1)\sin^2(\zeta)(\delta((t-4)t+2)+2(t-1))\sin^2(y)+2\Big),$$

$$\vdots$$

By making use of these successive values in (23), we obtain the approximate solution.

NTDM$_{ABC}$:

$$^{ABC}\phi_0(\zeta, y, t) = \sin(\zeta)\sin(y),$$

$$^{ABC}\phi_1(\zeta, y, t) = \sin(\zeta)\sin(y)\left(\delta - \frac{t^{\delta}}{\Gamma(\delta)} - 1\right),$$

$$^{ABC}\phi_2(\zeta, y, t) = \sin(\zeta)\sin(y)\left(\frac{\delta^2 t^{2\delta}\left(4\alpha+3\sin^2(\zeta)\sin^2(y)+1\right)}{\Gamma(2\delta+1)} - \right.$$

$$\left.\frac{t^{\delta}\left(2\delta^2+4\alpha\left(2\delta^2+t-2\right)+3\sin^2(\zeta)\left(2\delta^2+t-2\right)\sin^2(y)-2\right)}{(\delta+1)\Gamma(\delta)}\right)$$

$$+(\delta-1)\left(\delta+4\alpha(\delta+t-1)+3\sin^2(\zeta)(\delta+t-1)\sin^2(y)-1\right),$$

$$\vdots$$

By making use of these successive values in (28), we obtain the approximate solution. Approximate solutions with respect to C, CF, and ABC converge to the exact solution

Table 2 shows the absolute error of Case 2 for various values of t and ζ when $y = 0.1$ and $\delta = 1$. Table 3 displays the approximate solution for Case 2 with varying values of δ, t, and ζ when $y = 0.1$. The 2D and 3D simulations for a range of δ values are shown in Figures 3 and 4.

In this section, we have obtained the approximate solution of the TFEFK equation with the use of NTDM. Three fractional derivatives were applied to the proposed model to investigate the fractional effects. The findings of this study are presented in tabular and graphical simulations.

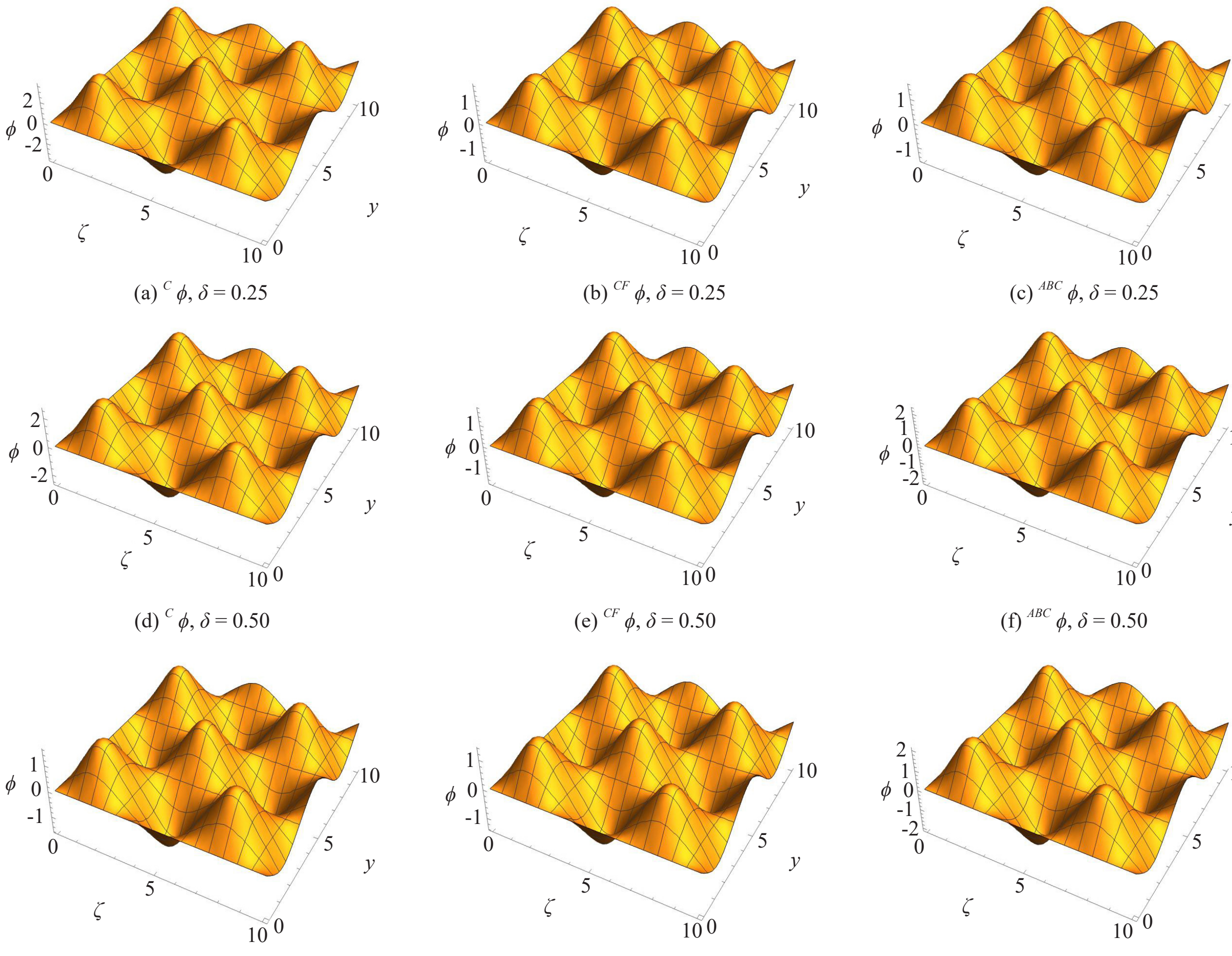
As indicated in figures and tables, the numerical investigation was conducted on the TFEFK equation using the NTDM with distinct time and space variables at different fractional order values. The nature of the solution for the homogeneous case of the TFEFK equation defined in Case 1 with distinct fractional order is depicted in Figure 1. Figure 2 presents the 2D graphical representations of the approximate solution for different fractional order values. Figures 3 and 4 display the 3D and 2D simulations for various values of δ. By choosing various fractional orders δ, the solitary wave solutions of Cases 1 and 2 are physically characterised in the figures. It is clear from the graphical simulations that the solutions can be utilised to investigate the physical process of the transition from one state medium to another, as well as the population growth dynamics and wave propagation. Table 1 represents the approximate solution of Case 1 for various values of ζ, y, and fixed t at various δ orders. Table 2 displays the absolute error of Case 2 for different values of t and ζ with $y = 0.1$ and $\delta = 1$. The approximate solution of Case 2 with different values of δ, t, and ζ with $y = 0.1$ is shown in Table 3. It is clear from Table 2 that the approximate solutions converge to the exact solution. From the reported tables and figures, it is observed that all three derivatives show good agreement. The presented method demonstrates that, as the order approaches the classical case, the obtained solution approaches the analytical solution and confirms the accuracy of the employed scheme. Consequently, the physical representation of our results may serve as a beneficial tool for investigating further findings for nonlinear wave problems in applications of science.

Table 2. Absolute error of Case 2 for different values of t and ζ with $y = 0.1$ and $\delta = 1$

t	ζ	$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$
0.025	0.25	6.39E-08	6.39E-08	6.39E-08
0.05	0.25	5.08E-07	5.08E-07	5.08E-07
0.075	0.25	1.70E-06	1.70E-06	1.70E-06
0.1	0.25	4.02E-06	4.02E-06	4.02E-06
0.025	0.5	1.24E-07	1.24E-07	1.24E-07
0.05	0.5	9.85E-07	9.85E-07	9.85E-07
0.075	0.5	3.30E-06	3.30E-06	3.30E-06
0.1	0.5	7.78E-06	7.78E-06	7.78E-06
0.025	0.75	1.76E-07	1.76E-07	1.76E-07
0.05	0.75	1.40E-06	1.40E-06	1.40E-06
0.075	0.75	4.70E-06	4.70E-06	4.70E-06
0.1	0.75	1.11E-05	1.11E-05	1.11E-05

Table 3. Approximate solution of Case 2 with different values of δ, t and ζ with $y = 0.1$

δ	t	ζ	$NTDM_C$	$NTDM_{CF}$	$NTDM_{ABC}$
0.7	0.025	0.25	2.30916E-02	2.85291E-02	3.07396E-02
	0.05	0.25	2.24679E-02	2.86446E-02	3.20623E-02
	0.075	0.25	2.21287E-02	2.87547E-02	3.31446E-02
	0.1	0.25	2.19582E-02	2.88594E-02	3.40865E-02
	0.025	0.5	4.47482E-02	5.53067E-02	5.95966E-02
	0.05	0.5	4.35408E-02	5.55313E-02	6.21636E-02
	0.075	0.5	4.28847E-02	5.57453E-02	6.42642E-02
	0.1	0.5	4.25558E-02	5.59488E-02	6.60921E-02
	0.025	0.75	6.36239E-02	7.86785E-02	8.47901E-02
	0.05	0.75	6.19095E-02	7.89991E-02	8.84476E-02
	0.075	0.75	6.09792E-02	7.93047E-02	9.14403E-02
	0.1	0.75	6.05142E-02	7.95952E-02	9.40445E-02
0.8	0.025	0.25	2.34719E-02	2.47011E-02	2.54063E-02
	0.05	0.25	2.27329E-02	2.47013E-02	2.58888E-02
	0.075	0.25	2.21734E-02	2.47014E-02	2.63228E-02
	0.1	0.25	2.17325E-02	2.47015E-02	2.67307E-02
	0.025	0.5	4.54847E-02	4.78768E-02	4.92459E-02
	0.05	0.5	4.40532E-02	4.78778E-02	5.01830E-02
	0.075	0.5	4.29696E-02	4.78787E-02	5.10260E-02
	0.1	0.5	4.21157E-02	4.78795E-02	5.18182E-02
	0.025	0.75	6.46698E-02	6.80909E-02	7.00425E-02
	0.05	0.75	6.26355E-02	6.80936E-02	7.13793E-02
	0.075	0.75	6.10960E-02	6.80962E-02	7.25818E-02
	0.1	0.75	5.98832E-02	6.80987E-02	7.37115E-02
0.9	0.025	0.25	2.38182E-02	2.32212E-02	2.32519E-02
	0.05	0.25	2.31196E-02	2.29848E-02	2.30785E-02
	0.075	0.25	2.25053E-02	2.27552E-02	2.29330E-02
	0.1	0.25	2.19531E-02	2.25326E-02	2.28105E-02
	0.025	0.5	4.61555E-02	4.50016E-02	4.50616E-02
	0.05	0.5	4.48019E-02	4.45438E-02	4.47265E-02
	0.075	0.5	4.36117E-02	4.40994E-02	4.44454E-02
	0.1	0.5	4.25419E-02	4.36685E-02	4.42087E-02
	0.025	0.75	6.56232E-02	6.39882E-02	6.40747E-02
	0.05	0.75	6.36989E-02	6.33382E-02	6.36000E-02
	0.075	0.75	6.20071E-02	6.27073E-02	6.32021E-02
	0.1	0.75	6.04865E-02	6.20955E-02	6.28672E-02

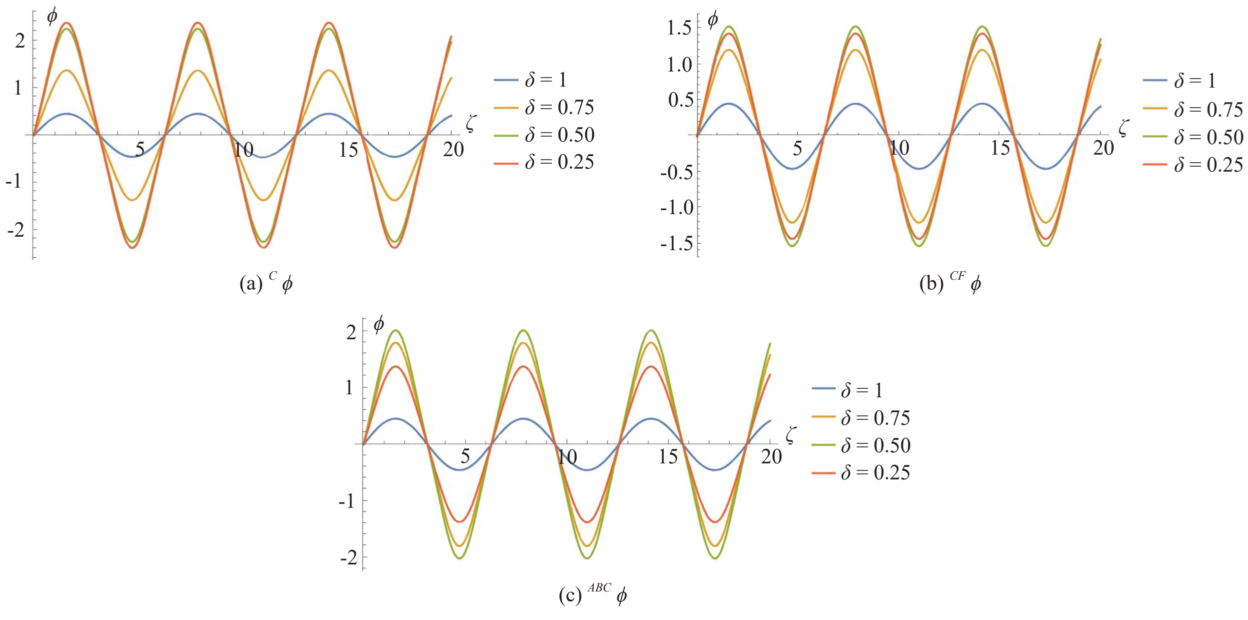


(a) ${}^{C}\phi$, $\delta = 0.25$ (b) ${}^{CF}\phi$, $\delta = 0.25$ (c) ${}^{ABC}\phi$, $\delta = 0.25$

(d) ${}^{C}\phi$, $\delta = 0.50$ (e) ${}^{CF}\phi$, $\delta = 0.50$ (f) ${}^{ABC}\phi$, $\delta = 0.50$

(g) ${}^{C}\phi$, $\delta = 0.75$ (h) ${}^{CF}\phi$, $\delta = 0.75$ (i) ${}^{ABC}\phi$, $\delta = 0.50$

Figure 3. Approximate solution of Case 2 with $t = 1$, different values of δ



(a) ${}^{C}\phi$ (b) ${}^{CF}\phi$

(c) ${}^{ABC}\phi$

Figure 4. Approximate solution of Case 2 with $t = 1$, $y = 2$ for different values of δ

6. Conclusions

In this work, we used NTDM to estimate the solutions of the TFEFK equation for the fractional derivatives of *C*, *CF*, and *ABC*. Results for different fractional orders at different ζ and t were obtained. All three fractional derivative approaches have very strong agreement with one another, according to the solution analysis. For different orders of the fractional derivative, the results of numerical computations are reported. The graphs exhibit the effectiveness and viability of the suggested strategy. Additionally, because it is straightforward in concept but effective in studying nonlinear time fractional differential equations, this method can be used to identify potential solutions for a wide variety of similar problems that arise in mathematical physics.

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Conflict of interest

The authors declare no competing financial interest.

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