

# Exact solutions of Benjamin-Bona-Mahoney-Burgers equation with dual power-law nonlinearity by modified exp-function method

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## Abstract

In this work, the Benjamin-Bona-Mahoney-Burgers equation has examined which includes the dual-power law nonlinearity and diffraction term. By using the modified exp-function method, the exact solutions of governing equation have been obtained. The resulting travelling wave solutions have been found to exhibit various characteristics such as being dark solitons, periodic and singular depending on the values of certain constants. To further illustrate these solutions, 3-D, 2-D, and contour graphs have been displayed. To the best of our knowledge, it is the first time in literature that the dark solitons, periodic soliton and singular soliton solutions of considered equations have been obtained by utilizing the modified exp-function method.

## Keywords

Benjamin-Bona-Mahoney-Burgers equation, Modified exp-function method, Travelling wave solution.

## 1 Introduction

It is completely obvious that nonlinear partial differential equations (NLPDEs) have been used to explain a great deal of the mathematical and physical models that arise in the many disciplines of sciences including quantum mechanics, classical mechanics, optical fibres, plasma physics and biological science. Finding solutions to many distinct types of NLPDEs has caught the interest of numerous researchers. Since NLPDEs have such a broad variety of applications in applied mathematics, it is essential to look for different sorts of solutions to the NLPDEs. There are different methods which have been utilized to derive the analytic solutions of NLPDEs like tanh method [1, 2], Jacobi elliptic function method [3, 4],  $G'/G$  method [5–7], F-expansion method [8, 9], modified simple equation method [10, 11] etc. The Lie classical method [12–14], nonclassical method [15–20] and direct method [21] can also be used to find the exact solutions of NLPDEs. Also, fractional natural decomposition method [22] is strong approach to find iterative solution of PDEs. The q-homotopy analysis transform technique [23] is used to analyze the nature and capture the corresponding consequences of the solutions obtained for PDEs. The modified exp-function method [24] is also an efficient method that can be used to find sigular, periodic, dark and bright solitary wave solutions.

In this manuscript, we consider Benjamin-Bona-Mahoney-Burgers(BBMB) equation with dual power-law nonlinearity,

$$v_t + \epsilon_1 v_x + (\epsilon_2 v^{2n} + \epsilon_3 v) v_x + \epsilon_4 v_{xx} + \epsilon_5 v_{xxt} = 0, \quad (1)$$

This equation is a variant of the more general BBM equation [25], which is itself a modification of the Korteweg-de Vries (KdV) equation [26, 27]. The BBM equation and its variants are used to model a wide range of physical phenomena, including fluid dynamics, plasma physics, and nonlinear optics. This equation explains the unidirectional propagation of long waves with small amplitudes in water. The Benjamin equation [28] explains the one-directional movement of

low-amplitude, long internal waves that occur at the interface between two layers of fluid, subject to the combined influences of surface tension and gravity. The Khater et. al. [29] obtained the solitary wave solutions of BBMB equation with dual power law nonlinearity by using extended tanh function method. Wang et.al. [30] derived the shock wave solutions of the BBMB equation with the Lie symmetry analysis.

The modified exp-function method is powerful mathematical tool used to obtain exact solutions for NLPDEs that arise in various fields of physics, engineering and applied mathematics. The main idea behind this method is to transform the original NLPDE into an ordinary differential equation (ODE) by introducing a wave transformation. This transformation leads to the reduction of the number of independent variables, which simplifies the problem considerably. The travelling wave solutions of reduced equation are assumed in the form of an exponential function  $exp(-\phi)$  solution, where  $\phi$  is a solution of the first-order differential equation. The obtained solutions are then represented graphically, enabling a better understanding of the physical properties exhibited by the considered equation. By studying the graphical representations, various physical parameters can be extracted, such as the speed of propagation of the wave, the amplitude, and the shape of the wave. The computational software MAPLE is used to solve the system of nonlinear equations and to depict the graphical representation of obtained exact solutions.

The structure of this manuscript is as follows. In §1, introduction provides a brief overview of the paper's objectives and motivation. It includes a description of the problem under consideration and its significance in the field of mathematics. In §2, description of modified exp-function method to find the exact solutions of NLPDE is given. In §3, the modified exp-function method is successfully applied to the Eq.(1). The exact solutions obtained using the method are presented and discussed. In §4, graphical representations of the derived analytic solutions for some specific values of constants are given. The figures help to illustrate the behaviour of the solutions and provide insight into the physical interpretation of the results. In §5, the paper's key findings and conclusions are summarized. The authors provide their final remarks on the modified exp-function method and its usefulness in solving NLPDEs. They may also suggest avenues for further research in the field.

## 2 Modified exp-function method [31]

Consider the following NLPDE

$$F(V, V_x, V_t, V_{xt}, V_{tt}, V_{xx}, \dots) = 0 \quad (2)$$

in which  $V = V(x, t)$  is function of  $x$  and  $t$ ,  $F$  is polynomial of  $V(x, t)$ , and its derivatives. The following are the key concepts behind this method:

**Step 1:** Applying the travelling wave transformation to Eq.(2), which is clearly described as

$$V(x, t) = V(\xi), \xi = (x - c_1 t), \quad (3)$$

where  $c_1$  describes the wave velocity. Eq.(2) is changed into the nonlinear ODE using the above wave transformation

$$G(V, V', V'', V''', \dots) = 0, \quad (4)$$

where prime denotes the derivatives with respect to  $\xi$ .

**Step 2::** Suppose that Eq.(4) has the following form of solution

$$V(\xi) = \sum_{i=0}^N A_i [\exp(-\phi(\xi))]^i = A_0 + \epsilon_1 \exp(-\phi) + \dots + A_N \exp(N(-\phi)) \quad (5)$$

where  $A_i$ 's ( $0 \leq i \leq N$ ) are the constants to be determined later such that,  $A_N \neq 0$  and  $\phi = \phi(\xi)$  is the solution of first-order ODE

$$\phi'(\xi) = \exp(-\phi(\xi)) + \delta \exp(\phi(\xi)) + \theta, \quad (6)$$

where  $\delta$  and  $\theta$  are real constants.

Eq. (6) has five exact solutions which are give below:

- When  $\delta \neq 0, \theta^2 - 4\delta > 0$ ,

$$\phi(\xi) = \ln \left( \frac{-\sqrt{\theta^2 - 4\delta}}{2\delta} \tanh \left( \frac{\sqrt{\theta^2 - 4\delta}}{2} (\xi + E) \right) - \frac{\theta}{2\delta} \right). \quad (7)$$

- When  $\delta \neq 0, \theta^2 - 4\delta < 0$ ,

$$\phi(\xi) = \ln \left( \frac{\sqrt{-\theta^2 + 4\delta}}{2\delta} \tan \left( \frac{\sqrt{-\theta^2 + 4\delta}}{2} (\xi + E) \right) - \frac{\theta}{2\delta} \right). \quad (8)$$

- $\delta = 0, \theta \neq 0, \theta^2 - 4\delta > 0$ ,

$$\phi(\xi) = -\ln \left( -\frac{\theta}{\exp(\theta(\xi + E)) - 1} \right). \quad (9)$$

- $\delta \neq 0, \theta \neq 0, \theta^2 - 4\delta = 0$ ,

$$\phi(\xi) = -\ln \left( -\frac{2\theta(\xi + E) + 4}{\theta^2(\xi + E)} \right). \quad (10)$$

- When  $\delta = 0, \theta = 0, \theta^2 - 4\delta = 0$ ,

$$\phi(\xi) = \ln(\xi + E). \quad (11)$$

The constants to be obtained subsequently are the coefficients  $A_0, \epsilon_1, A_2, \dots, A_N, \theta$  and  $\delta$ . Comparing the nonlinear term with the highest order derivative term, the value of  $N$  is obtained.

**Step 3:** Substituting the Eq.(5) with the obtained value of  $N$  into Eq.(4) and using Eq.(6), a system of nonlinear equations is obtained by comparing the coefficients of various powers of  $\exp(-\phi(\xi))$  to zero.

**Step 4:** The obtained system is solved for the constants  $A_0, \epsilon_1, A_2, \dots, A_N, \theta$  and  $\delta$  to obtain the travelling wave solutions of Eq.(2).

### 3 Exact solutions of the BBMB equation with dual power-law nonlinearity

Consider the BBMB equation with dual power-law nonlinearity

$$v_t + \epsilon_1 v_x + (\epsilon_2 v^{2n} + \epsilon_3 v) v_x + \epsilon_4 v_{xx} + \epsilon_5 v_{xxt} = 0, \quad (12)$$

Where  $\epsilon_1$  indicates the intensity of defection or drifting, the parameters  $\epsilon_4, \epsilon_5$  are the dissipative diffraction coefficients, the exponent  $n$  is the power law nonlinearity parameter and  $\epsilon_2, \epsilon_3$  quantify the strength of the two nonlinear terms.

Substituting  $n = 1$  in above equation, we get the following form of Eq.(12)

$$v_t + \epsilon_1 v_x + (\epsilon_2 v^2 + \epsilon_3 v) v_x + \epsilon_4 v_{xx} + \epsilon_5 v_{xxt} = 0, \quad (13)$$

Using  $v(x, t) = u(\xi)$  where  $(\xi = x - c_1 t)$ , above equation becomes

$$(\epsilon_1 - c_1) u' + (\epsilon_2 u^2 + \epsilon_3 u) u' + \epsilon_4 u'' - \epsilon_5 c_1 u''' = 0. \quad (14)$$

Suppose that Eq.(14) has the following form of travelling wave solution

$$u(\xi) = \sum_{i=0}^N A_i [\exp(-\phi(\xi))]^i = A_0 + A_1 \exp(-\phi) + \dots + A_N \exp(N(-\phi)) \quad (15)$$

Balancing  $u^2 u'$  and  $u'''$  in Eq.(14), we obtained  $(2N + N + 1 = N + 3) \Rightarrow (N = 1)$ , so the Eq.(14) can be rewritten as

$$v(\xi) = A_0 + A_1 \exp(-\phi(\xi)) \quad (16)$$

where  $A_1 \neq 0$ . Substituting Eq.(16) along with the Eq.(6), we obtained a system of nonlinear equations after comparing the various powers of  $\exp(-\phi(\xi))$ , solving the obtained system of nonlinear equations, the following values of the constants are obtained

**Set 1**

$$\theta = \theta, \delta = \frac{\theta^2 A_1^4 \epsilon_2^2 \epsilon_5 - 2A_1^4 \epsilon_2^2 + 12A_1^2 \epsilon_1 \epsilon_2 \epsilon_5 - 3A_1^2 \epsilon_3^2 \epsilon_5 + 12\epsilon_4^2 \epsilon_5}{4A_1^4 \epsilon_2^2 \epsilon_5}, A_0 = \frac{\theta A_1^2 \epsilon_2 - A_1 \epsilon_3 + 2\epsilon_4}{2A_1 \epsilon_2}, \quad (17)$$

$$A_1 = A_1, c_1 = \frac{A_1^2 \epsilon_2}{6\epsilon_5}$$

Taking the values of constant from set(1) and using Eqs.(7-11), we obtained the following solutions of the considered equation

- When  $\delta \neq 0, \theta^2 - 4\delta > 0$ ,

$$u_1(x, t) = \frac{\theta A_1^2 \epsilon_2 - A_1 \epsilon_3 + 2\epsilon_4}{2A_1 \epsilon_2} - \frac{2A_1 \delta}{\sqrt{\theta^2 - 4\delta} \tanh\left(\frac{\sqrt{\theta^2 - 4\delta}\left(x - \frac{A_1^2 \epsilon_2}{6\epsilon_5} t + E\right)}{2}\right)} + \theta \quad (18)$$

where  $\delta = \frac{\theta^2 A_1^4 \epsilon_2^2 \epsilon_5 - 2A_1^4 \epsilon_2^2 + 12A_1^2 \epsilon_1 \epsilon_2 \epsilon_5 - 3A_1^2 \epsilon_3^2 \epsilon_5 + 12\epsilon_4^2 \epsilon_5}{4A_1^4 \epsilon_2^2 \epsilon_5}$

- When  $\delta \neq 0, \theta^2 - 4\delta < 0$

$$u_2(x, t) = \frac{\theta A_1^2 \epsilon_2 - A_1 \epsilon_3 + 2\epsilon_4}{2A_1 \epsilon_2} - \frac{2A_1 \delta}{-\sqrt{-\theta^2 + 4\delta} \tan\left(\frac{\sqrt{-\theta^2 + 4\delta}\left(-\frac{A_1^2 \epsilon_2}{6\epsilon_5} t + E + x\right)}{2}\right)} + \theta \quad (19)$$

where  $\delta = \frac{\theta^2 A_1^4 \epsilon_2^2 \epsilon_5 - 2A_1^4 \epsilon_2^2 + 12A_1^2 \epsilon_1 \epsilon_2 \epsilon_5 - 3A_1^2 \epsilon_3^2 \epsilon_5 + 12\epsilon_4^2 \epsilon_5}{4A_1^4 \epsilon_2^2 \epsilon_5}$

- $\delta = 0, \theta \neq 0, \theta^2 - 4\delta > 0$ ,

If system of nonlinear equations is solved with  $\delta = 0$ , we obtained the following solution of the considered equation.

$$u_3(x, t) = \frac{\sqrt{\frac{-2A_1^4 \epsilon_2^2 + 12A_1^2 \epsilon_1 \epsilon_2 \epsilon_5 - 3A_1^2 \epsilon_3^2 \epsilon_5 + 12\epsilon_4^2 \epsilon_5}{\epsilon_5}} - A_1 \epsilon_3 + 2\epsilon_4}{2A_1 \epsilon_2} + \frac{A_1 \theta}{e^{\left(-\frac{A_1^2 \epsilon_2}{6\epsilon_5} t + E + x\right)}} \quad (20)$$

where

$$\theta = \frac{\sqrt{\frac{-2A_1^4 \epsilon_2^2 + 12A_1^2 \epsilon_1 \epsilon_2 \epsilon_5 - 3A_1^2 \epsilon_3^2 \epsilon_5 + 12\epsilon_4^2 \epsilon_5}{\epsilon_5}}}{\epsilon_2 A_1^2},$$

- $\delta \neq 0, \theta \neq 0, \theta^2 - 4\delta = 0$ , solving system of nonlinear equation with using the condition  $\theta^2 - 4\delta = 0$ , the following solution of original equation is obtained

$$u_4(x, t) = A_0 - \frac{A_1 \theta^2 \left( -\frac{(12\epsilon_1 \epsilon_5 \epsilon_2 - 3\epsilon_5 \epsilon_3^2 + \sqrt{144\epsilon_1^2 \epsilon_2^2 \epsilon_5^2 - 72\epsilon_1 \epsilon_2 \epsilon_3^2 \epsilon_5^2 + 9\epsilon_3^4 \epsilon_5^2 + 96\epsilon_2^2 \epsilon_4^2 \epsilon_5})t}{24\epsilon_2 \epsilon_5} + E + x \right)}{4 + 2\theta \left( -\frac{(12\epsilon_1 \epsilon_5 \epsilon_2 - 3\epsilon_5 \epsilon_3^2 + \sqrt{144\epsilon_1^2 \epsilon_2^2 \epsilon_5^2 - 72\epsilon_1 \epsilon_2 \epsilon_3^2 \epsilon_5^2 + 9\epsilon_3^4 \epsilon_5^2 + 96\epsilon_2^2 \epsilon_4^2 \epsilon_5})t}{24\epsilon_2 \epsilon_5} + E + x \right)} \quad (21)$$

where  $A_1$  is given by

$$A_1 = \frac{\sqrt{12\epsilon_1 \epsilon_5 \epsilon_2 - 3\epsilon_5 \epsilon_3^2 + \sqrt{144\epsilon_1^2 \epsilon_2^2 \epsilon_5^2 - 72\epsilon_1 \epsilon_2 \epsilon_3^2 \epsilon_5^2 + 9\epsilon_3^4 \epsilon_5^2 + 96\epsilon_2^2 \epsilon_4^2 \epsilon_5}}}{2\epsilon_2}$$

- When  $\delta = 0, \theta = 0, \theta^2 - 4\delta = 0$ ,

$$u_5(x, t) = \frac{A_0 \left( -\frac{(12\epsilon_1 \epsilon_5 \epsilon_2 - 3\epsilon_5 \epsilon_3^2 + \sqrt{144\epsilon_1^2 \epsilon_2^2 \epsilon_5^2 - 72\epsilon_1 \epsilon_2 \epsilon_3^2 \epsilon_5^2 + 9\epsilon_3^4 \epsilon_5^2 + 96\epsilon_2^2 \epsilon_4^2 \epsilon_5})t}{24\epsilon_2 \epsilon_5} + E + x \right) + A_1}{-\frac{(12\epsilon_1 \epsilon_5 \epsilon_2 - 3\epsilon_5 \epsilon_3^2 + \sqrt{144\epsilon_1^2 \epsilon_2^2 \epsilon_5^2 - 72\epsilon_1 \epsilon_2 \epsilon_3^2 \epsilon_5^2 + 9\epsilon_3^4 \epsilon_5^2 + 96\epsilon_2^2 \epsilon_4^2 \epsilon_5})t}{24\epsilon_2 \epsilon_5} + E + x} \quad (22)$$

where  $A_0$  and  $A_1$  are given by

$$A_0 = \frac{-\sqrt{12\epsilon_1\epsilon_5\epsilon_2 - 3\epsilon_5\epsilon_3^2} + \sqrt{144\epsilon_1^2\epsilon_2^2\epsilon_5^2 - 72\epsilon_1\epsilon_2\epsilon_3^2\epsilon_5^2 + 9\epsilon_3^4\epsilon_5^2 + 96\epsilon_2^2\epsilon_4^2\epsilon_5\epsilon_3} + 2\epsilon_4}{2\epsilon_2}$$

$$A_1 = \frac{\sqrt{12\epsilon_1\epsilon_5\epsilon_2 - 3\epsilon_5\epsilon_3^2} + \sqrt{144\epsilon_1^2\epsilon_2^2\epsilon_5^2 - 72\epsilon_1\epsilon_2\epsilon_3^2\epsilon_5^2 + 9\epsilon_3^4\epsilon_5^2 + 96\epsilon_2^2\epsilon_4^2\epsilon_5}}{2\epsilon_2}$$

**Set 2**

$$\theta = \frac{\sqrt{-\frac{-2A_1^4\epsilon_2^2 + 12A_1^2\epsilon_1\epsilon_2\epsilon_5 - 3A_1^2\epsilon_3^2\epsilon_5 + 12\epsilon_4^2\epsilon_5}{\epsilon_5}}}{\epsilon_2 A_1^2}, \delta = 0, \quad (23)$$

$$A_0 = \frac{\sqrt{-\frac{-2A_1^4\epsilon_2^2 + 12A_1^2\epsilon_1\epsilon_2\epsilon_5 - 3A_1^2\epsilon_3^2\epsilon_5 + 12\epsilon_4^2\epsilon_5}{\epsilon_5}} - A_1\epsilon_3 + 2\epsilon_4}{2A_1\epsilon_2}, A_1 = A_1, c_1 = \frac{A_1^2\epsilon_2}{6\epsilon_5}$$

when  $\bullet \delta = 0, \theta \neq 0, \theta^2 - 4\delta > 0$ , the following solution of original equation is obtained

$$u_6(x, t) = \frac{A_0 e^{\theta \left( -\frac{A_1^2 \epsilon_2 t}{6\epsilon_5} + E + x \right)} + A_1 \theta - A_0}{\theta \left( -\frac{A_1^2 \epsilon_2 t}{6\epsilon_5} + E + x \right)} \quad (24)$$

where values of  $\theta$  and  $A_0$  are from set 2.

## 4 Physical Interpretation

One of the most effective means of comprehending the physical dynamics of real-world systems is through visual representation. In this regard, we have illustrated the graphical characteristics of the analytical solutions derived in the previous section. By selecting appropriate parameter values, we have rendered the dark, singular, and periodic solutions in various graphical forms, including 3D, 2D, and contour plots. In order to illustrate various dynamical wave patterns of derived exact solutions to the considered problem, this study extensively use the modified exp-function technique. Fig.1 represents the singular dark soliton solution  $u_1(x, t)$  with values of parameters  $A_1 = 3, \epsilon_1 = 3.5, \epsilon_2 = 9, \epsilon_3 = -7, \epsilon_4 = 5, \epsilon_5 = 4, E = 1, \theta = 1$ . Fig.2 displays the periodic singular solution  $u_2(x, t)$  with values of parameters  $A_1 = 1, \epsilon_1 = 3, \epsilon_2 = -4, \epsilon_3 = 4, \epsilon_4 = 5, \epsilon_5 = 4, E = 1, \theta = 1$ . 2D plot of fig.2 shows the periodic behaviour of obtained travelling wave solution. Fig.3 shows the 3D, 2D and contour plot of exponential singular solution  $u_3(x, t)$  with the values of constants  $A_1 = 1, \epsilon_1 = 4, \epsilon_2 = 3, \epsilon_3 = 11, \epsilon_4 = 3, \epsilon_5 = 5, E = 1$ . From 2D plot of fig.3, it can be observed that the line  $y = -1$  is asymptotes of the obtained travelling wave solution. Fig.4 shows the singular soliton solution  $u_6(x, t)$  with unknown values  $A_1 = 1, \epsilon_1 = -0.4, \epsilon_2 = 9, \epsilon_3 = -1.5, \epsilon_4 = 3, \epsilon_5 = 2.5, E = 1$ . By exploiting travelling wave solutions obtained in this manuscript, it is possible to predict the shape and speed of waves in water as they propagate through these environments. This information can be useful in designing structures such as breakwaters and seawalls to mitigate the impact of waves on coastal communities.

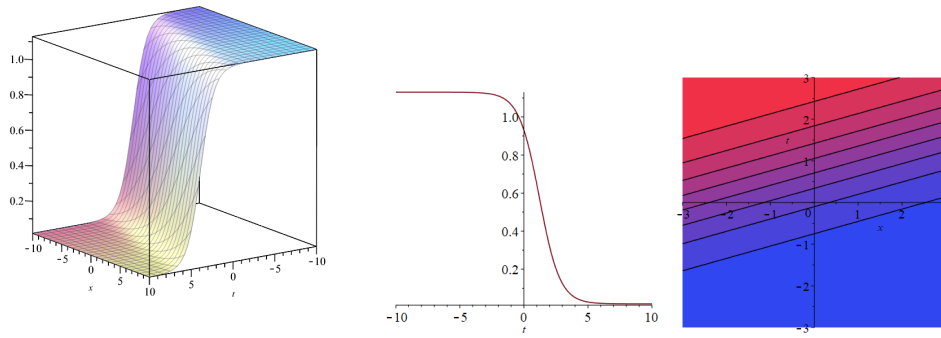


Figure 1: 3D, 2D and Contour plot of  $u_1(x, t)$

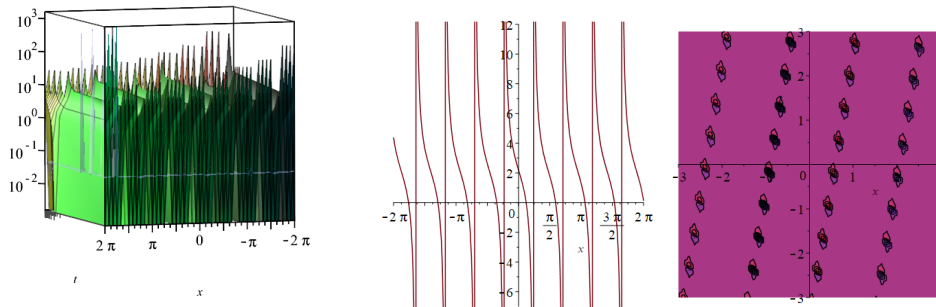


Figure 2: 3D, 2D and Contour plot of  $u_2(x, t)$

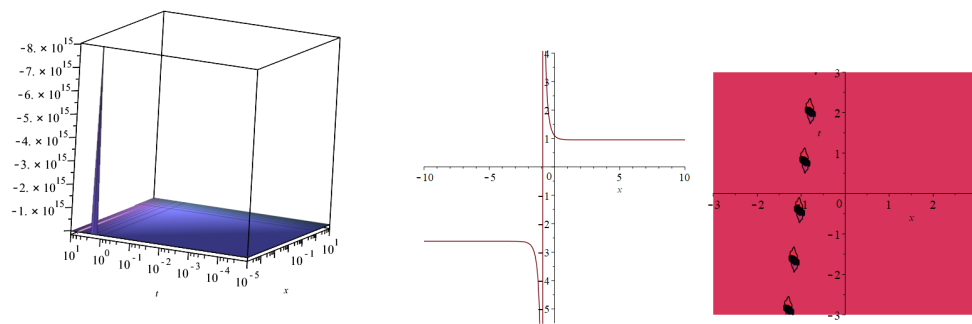


Figure 3: 3D, 2D and Contour plot of  $u_3(x, t)$

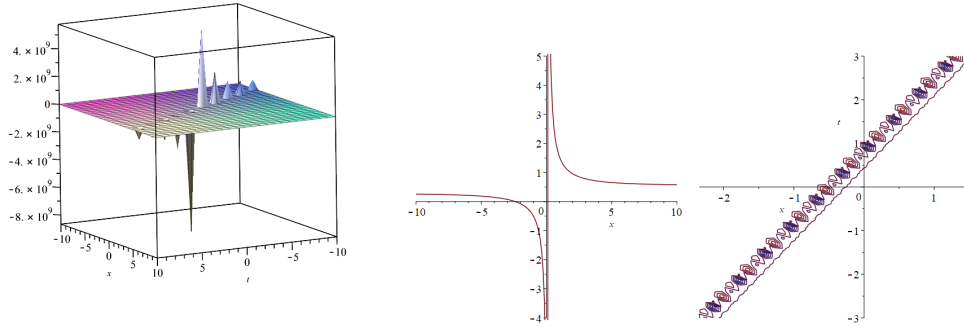


Figure 4: 3D, 2D and Contour plot of  $u_6(x, t)$

## 5 Conclusion

The modified exp-function method has been utilized to obtain the traveling wave solution of Eq.(12). The considered equation has been reduced to an ODE by assuming the wave variable. The periodic, singular, and dark soliton solutions of the considered equation have been obtained by considering the solutions of the reduced ODE in the form of an exponential function. The obtained solutions have been validated by plugging them back into the original equation, and their behavior has been depicted through a graphical representation of the resulting traveling wave solutions. The computational software Maple has been utilized to illustrate the visual depictions of the acquired solutions. By making certain modifications to this method, it can be utilized to tackle other intricate models that arise in the fields of mathematics, physics, and applied mathematics.

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