

## Research Article

# A Caputo-Type Fractional-Order Model for the Transmission of Chlamydia Disease

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**Abstract:** In this article, we study a mathematical model of the chlamydia infection caused by sexual contact. To analyze this model, we combine the homotopy perturbation Laplace transform technique with the fractional order formulation in the Caputo sense. This article illustrates the increased degree of freedom that fractional derivative models allow to investigate disease dynamics for a given data set and to highlight memory effects. The existence, singularity, and consistency of the problem are also examined in the research. The unique parameter estimation for each value of the noninteger order makes this study more useful.

**Keywords:** Caputo fractional derivative, homotopy perturbation Laplace transform method, chlamydia model, stability analysis

**MSC:** 26A33, 34A08

## 1. Introduction

These days, the study of infectious diseases and their cures is widely practiced in many different fields [1, 2]. We can better anticipate future disease outbreaks with the help of mathematical models that consider existing data. Fractional differential equation (FDE) models were extensively used to study the spread of viral infections [3-7]. To date, several models have been developed to investigate chlamydia transmission. The authors defined an optimal control derivation for chlamydia modeling in [8]. In [9], an optimal control is used. A model of chronic *Chlamydia trachomatis* disease was developed, taking into account a combination treatment with tryptophan and antibiotics. Because of their memory effect, FDEs have gained importance in modeling many scientific and engineering fields. Vellappandi et al. [10] also studied the chlamydia disease model in fractional order. There is no precise method for dealing with fractional-order differential equations. Several numerical and analytical methods have been used to obtain the approximate solution of FDEs, viz., the homotopy perturbation method (HPM) [11], the homotopy perturbation general transform method [12], the residual power series method [13], the homotopy analysis method [14], the L1-Predictor-Corrector method [15], the Caputo-Fabrizio derivatives [16], etc.

In this work, we examine a nonlinear chlamydia model discussed by Shah et al. [17]. The prime parameters utilized

in the described model are the susceptible class as  $S(t)$ , the exposed class as  $E(t)$ , infected individuals due to sexual activity as  $I_s(t)$ , infected individuals due to the unhygienic environment as  $I_u(t)$ , and the recovered class as  $R(t)$ . This model is specified using a set of differential equations, as mentioned below

$$\begin{aligned} \frac{dS}{dt} &= \beta - a_1 S(t)E(t) + a_6 R(t) - \mu S(t), \\ \frac{dE}{dt} &= a_1 S(t)E(t) - a_2 E(t) - a_3 E(t) - \mu E(t), \\ \frac{dI_s}{dt} &= a_2 E(t) - a_4 I_s(t) - \mu I_s(t), \\ \frac{dI_u}{dt} &= a_3 E(t) - a_5 I_u(t) - \mu I_u(t), \\ \frac{dR}{dt} &= a_4 I_s(t) + a_5 I_u(t) - a_6 R(t) - \mu R(t), \end{aligned} \tag{1}$$

where initially we take the values as

$$S(0) = S_0, E(0) = E_0, I_s(0) = I_{s_0}, I_u(0) = I_{u_0}, R(0) = R_0. \tag{2}$$

We consider the total population of the system as  $N(t)$ , where

$$N(t) = S(t) + E(t) + I_s(t) + I_u(t) + R(t). \tag{3}$$

**Table 1.** Chlamydia model parameters

Notation	Value	Description
$\beta$	0.018	The average global birth rate
$a_1$	0.8	The rate of transmission from $S$ to $E$
$a_2$	0.67	The rate of transmission from $E$ to $I_s$
$a_3$	0.32	The rate of transmission from $E$ to $I_u$
$a_4$	0.92	Recovery rate from $I_s$
$a_5$	0.95	Recovery rate from $I_u$
$a_6$	0.05	The rate of transmission from $R$ to $S$
$\mu$	0.01	Escape rate

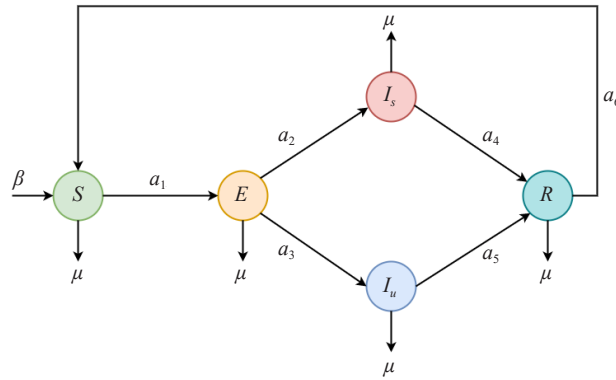


Figure 1. Diagram of chlamydia model

## 2. Preliminaries

**Definition 1** The Caputo time-fractional derivative [18] of  $u(x, t)$  with order  $\alpha > 0$  is

$${}_0^C D_t^\alpha \psi(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\sigma)^{n-\alpha-1} \frac{\partial^n \psi(\sigma)}{\partial \sigma^n} d\sigma, & n-1 < \alpha < n \\ \frac{\partial^n \psi(\sigma)}{\partial \sigma^n}, & \alpha = n \in N. \end{cases} \quad (4)$$

**Definition 2** The Laplace transform of (4) is given by [18],

$$L\{\psi(t)\} = s^\alpha L\{\psi(t)\} - \sum_{k=0}^{n-1} s^{\alpha-1-k} \psi^{(k)}(0), n-1 < \alpha \leq n. \quad (5)$$

The solution of FDE [19]

$${}_0^C D_t^\alpha \psi(t) = f(t); \text{ where } f(t) \in C([0, T]), \alpha \in (0, 1], \quad (6)$$

with  $\psi(0) = \psi_0$ , is given by

$$\psi(t) = \psi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\sigma)^{\alpha-1} f(\sigma) d\sigma. \quad (7)$$

**Theorem 1** [20, 21] Let  $V = C[0, T]$  be the Banach space of continuous real valued functions defined on  $[0, T]$ . Define a Banach space  $B = V \times V \times V \times V \times V$ , with norm,

$$\|\psi\| = \|S, E, I_s, I_u, R\| = \sup_{t \in [0, T]} S(t) + E(t) + I_s(t) + I_u(t) + R(t), \quad (8)$$

where  $\psi \in B$  and  $S, E, I_s, I_u, R \in V$ . Moreover, consider a convex subset  $A$  of Banach space  $B$ . If the operator  $X$  on  $A$  are such that they satisfy the following three conditions, then there exists at least one fixed point  $w \in A$  for  $X$ ; i.e.,

1.  $Xw \in A, Aw \in A$ .
2.  $X$  is a contraction.
3.  $X$  is continuous and compact.

### 3. Qualitative analysis

For system (1), we construct a function

$$\begin{aligned}
 f_1(t, S(t), E(t), I_s(t), I_u(t), R(t)) &= \beta - a_1 S(t)E(t) + a_6 R(t) - \mu S(t), \\
 f_2(t, S(t), E(t), I_s(t), I_u(t), R(t)) &= a_1 S(t)E(t) - a_2 E(t) - a_3 E(t) - \mu E(t), \\
 f_3(t, S(t), E(t), I_s(t), I_u(t), R(t)) &= a_2 E(t) - a_4 I_s(t) - \mu I_s(t), \\
 f_4(t, S(t), E(t), I_s(t), I_u(t), R(t)) &= a_3 E(t) - a_5 I_u(t) - \mu I_u(t), \\
 f_5(t, S(t), E(t), I_s(t), I_u(t), R(t)) &= a_4 I_s(t) + a_5 I_u(t) - a_6 R(t) - \mu R(t).
 \end{aligned} \tag{9}$$

Also, we generalize (6) as

$${}_0^C D_t^\alpha \psi(t) = \omega(t, \psi(t)), \tag{10}$$

where  $0 < \alpha \leq 1, t \in [0, T], \psi(0) = \psi_0$ .

As mentioned in Lemma 1, (10) yields

$$\psi(t) = \psi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \sigma)^{\alpha-1} \omega(\sigma, \psi(\sigma)) d\sigma, \tag{11}$$

where

$$\psi(t) = \begin{bmatrix} S(t) \\ E(t) \\ I_s(t) \\ I_u(t) \\ R(t) \end{bmatrix}, \quad \psi_0 = \begin{bmatrix} S_0 \\ E_0 \\ (I_s)_0 \\ (I_u)_0 \\ R_0 \end{bmatrix}, \quad \omega(t, \psi(t)) = \begin{bmatrix} f_1(t, S(t), E(t), I_s(t), I_u(t), R(t)) \\ f_2(t, S(t), E(t), I_s(t), I_u(t), R(t)) \\ f_3(t, S(t), E(t), I_s(t), I_u(t), R(t)) \\ f_4(t, S(t), E(t), I_s(t), I_u(t), R(t)) \\ f_5(t, S(t), E(t), I_s(t), I_u(t), R(t)) \end{bmatrix},$$

and

$$\omega_0 = \begin{bmatrix} f_1(0, S(0), E(0), I_s(0), I_u(0), R(0)) \\ f_2(0, S(0), E(0), I_s(0), I_u(0), R(0)) \\ f_3(0, S(0), E(0), I_s(0), I_u(0), R(0)) \\ f_4(0, S(0), E(0), I_s(0), I_u(0), R(0)) \\ f_5(0, S(0), E(0), I_s(0), I_u(0), R(0)) \end{bmatrix}. \tag{12}$$

### 4. Existence and uniqueness

Let us consider the following axioms and Lipschitz conditions to prove existence and uniqueness:

**Hypothesis 1** There exist  $C_\omega$  and  $D_\omega$  such that

$$\omega(t, \psi(t)) \leq C_\omega \|\psi\| + D_\omega. \quad (13)$$

**Hypothesis 2** There exist  $L_\omega > 0$  such that

$$\forall \psi, \bar{\psi} \in B, \omega(t, \psi) - \omega(t, \bar{\psi}) \leq L_\omega \|\psi - \bar{\psi}\|. \quad (14)$$

**Theorem 2** Considering Hypotheses 1 and 2, system (10) has at least one solution to (11) if  $L_\omega < 1$ .

**Proof.** Proof of theorem is given by following two steps:

1. Consider  $\bar{\psi} \in A$ , where  $A = \{\psi \in B, \|\psi\| \leq \rho, \rho > 0\}$  is a closed and convex set. For  $X(\psi)$  from equations (13) and (14),

$$\|X(\psi) - X(\bar{\psi})\| = \sup_{t \in [0, T]} |\omega(t, \psi(t)) - \omega(t, \bar{\psi}(t))| \leq L_\omega \|\psi - \bar{\psi}\|. \quad (15)$$

Hence,  $X$  is a contraction.

2. Here, we show that the operator  $X$  is relatively compact, i.e.,  $X$  should be equicontinuous and bounded. It follows that if  $\omega$  is continuous, then  $X$  is also continuous for all  $\psi \in A$ ,

$$\begin{aligned} \|X(\psi)\| &= \sup_{t \in [0, T]} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(\tau, \psi(\tau)) d\tau, \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^T (T-\tau)^{\alpha-1} \omega(\tau, \psi(\tau)) d\tau, \\ &\leq \frac{T^\alpha}{\Gamma(\alpha)} [C_\omega \|\psi\| + D_\omega], \\ &\leq \frac{T^\alpha}{\Gamma(\alpha)} [C_\omega \rho + D_\omega]. \end{aligned} \quad (16)$$

Thus,  $X$  is bounded.

For equicontinuity, we consider  $t_1 > t_2 \forall t_1, t_2 \in [0, T]$ , such that

$$\begin{aligned} X(\psi(t_1)) - X(\psi(t_2)) &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} (t_1 - \tau)^{\alpha-1} \omega(\tau, \psi(\tau)) d\tau \right. \\ &\quad \left. - \int_0^{t_2} (t_2 - \tau)^{\alpha-1} \omega(\tau, \psi(\tau)) d\tau \right|, \\ &\leq \frac{|t_1^\alpha - t_2^\alpha|}{\Gamma(\alpha)} [C_\omega \rho + D_\omega]. \end{aligned} \quad (17)$$

As  $t_1 \rightarrow t_2$ , we have  $|t_1^\alpha - t_2^\alpha| \rightarrow 0$ , and from (17),

$$|X(\psi(t_1)) - X(\psi(t_2))| \rightarrow 0.$$

Therefore,  $X$  is uniformly continuous and bounded. From the Arzela-Ascoli theorem, we say that  $X$  is relatively

compact and entirely continuous. Hence, from Theorem 1, we conclude that (11) has at least one solution.

**Theorem 3** Let us consider Hypotheses 1 and 2, system (10) holds a solution as (11) if  $\frac{T^\alpha L_\omega}{\Gamma(\alpha+1)} < 1$ .

**Proof.** For Banach space  $B$ , the operator  $X : B \rightarrow B$  defined as

$$X[\psi(t)] = \psi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \omega(\tau, \psi(\tau)) d\tau, \quad (18)$$

For  $\psi, \bar{\psi} \in B$ , we get

$$\begin{aligned} \|X(\psi) - X(\bar{\psi})\| &\leq \frac{1}{\alpha} \sup_{t \in [0, T]} \left| \int_0^t (t-\tau)^{\alpha-1} \omega(\tau, \psi(\tau)) d\tau - \int_0^t (t-\tau)^{\alpha-1} \omega(\tau, \bar{\psi}(\tau)) d\tau \right|, \\ &\leq \frac{1}{\alpha} \sup_{t \in [0, T]} \int_0^t (t-\tau)^{\alpha-1} \|\omega(\tau, \psi(\tau)) - \omega(\tau, \bar{\psi}(\tau))\| d\tau, \\ &\leq \frac{T^\alpha L_\omega}{\Gamma(\alpha+1)} \|\psi - \bar{\psi}\|. \end{aligned} \quad (19)$$

From (19),  $X$  is contraction. Now, using Banach fixed point theorem, operator  $X$  has a unique fixed point,  $\psi(t)$ , i.e.,  $X(\psi(t)) = \psi(t)$ , which gives the unique solution of generalized FDE (10).

## 5. Stability analysis

### 5.1 Local stability

We will find equilibrium points to discuss the local stability of our system.

### 5.2 Equilibrium points

There are two equilibrium points in our chlamydia system (1).

i) Disease-free equilibrium point  $E_0$ :

$$E_0 = \left( \frac{\beta}{\mu}, 0, 0, 0, 0 \right). \quad (20)$$

It means when there is no infection  $E = I_s = I_u = R = 0$ . Thus, the model (1) has a unique equilibrium point  $E_0$ .

ii) Endemic equilibrium point  $E_{end}^*$ :

$$E_{end}^* = (S^*, E^*, I_s^*, I_u^*, R^*), \quad (21)$$

where

$$S^* = \frac{a_2 + a_3 + \mu}{a_1},$$

$$E^* = \frac{(a_5 + \mu)(a_6 + \mu)(a_4 + \mu)(-\mu^2 + (-a_2 - a_3)\mu + \beta a_1)}{a_1 \mu \left[ \mu^3 + (a_2 + a_3 + a_4 + a_5 + a_6)\mu^2 + \{(a_2 + a_3 + a_5 + a_6)a_4 + (a_2 + a_3 + a_6)a_5 + (a_2 + a_3)a_6\}\mu + \{(a_2 + a_3 + a_6)a_5 + a_3 a_6\}a_4 + a_2 a_5 a_6 \right]}$$

$$I_s^* = \frac{(a_5 + \mu)a_2(a_6 + \mu)(-\mu^2 + (-a_2 - a_3)\mu + \beta a_1)}{a_1 \mu \left[ \mu^3 + (a_2 + a_3 + a_4 + a_5 + a_6)\mu^2 + \{(a_2 + a_3 + a_4 + a_6)a_5 + (a_2 + a_3 + a_6)a_4 + a_6(a_2 + a_3)\}\mu + \{(a_2 + a_3 + a_6)a_4 + a_6 a_2\}a_5 + a_3 a_4 a_6 \right]}$$

$$I_u^* = \frac{(a_6 + \mu)a_3(a_4 + \mu)(-\mu^2 + (-a_2 - a_3)\mu + \beta a_1)}{a_1 \mu \left[ \mu^3 + (a_2 + a_3 + a_4 + a_5 + a_6)\mu^2 + \{(a_2 + a_3 + a_5 + a_6)a_4 + (a_2 + a_3 + a_6)a_5 + a_6(a_2 + a_3)\}\mu + \{(a_2 + a_3 + a_6)a_5 + a_3 a_6\}a_4 + a_2 a_5 a_6 \right]}$$

$$R^* = \frac{((a_2 a_4 + a_3 a_5)\mu + a_4 a_5(a_2 + a_3))(-\mu^2 + (-a_2 - a_3)\mu + \beta a_1)}{\left( a_1 \mu \left[ \mu^3 + (a_2 + a_3 + a_4 + a_5 + a_6)\mu^2 + \{(a_2 + a_3 + a_5 + a_6)a_4 + (a_2 + a_3 + a_6)a_5 + a_6(a_2 + a_3)\}\mu + \{(a_2 + a_3 + a_6)a_5 + a_3 a_6\}a_4 + a_2 a_5 a_6 \right] \right)}$$

### 5.3 Basic reproduction number

A basic reproduction number  $R_0$  is calculated to get the transmission rate of chlamydia disease, using the next-generation matrix (NGM) algorithm [22, 23].

Let  $X = (S, E, I_s, I_u, R)$ , then the model can be rewrite as

$$X' = F(X) - V(X),$$

$$\text{where } F(X) = \begin{bmatrix} a_1 S E \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } V(X) = \begin{bmatrix} E(a_2 + a_3 + \mu) \\ -a_2 E + I_s(a_2 + \mu) \\ -a_3 E + I_u(a_5 + \mu) \\ -a_4 I_s - a_5 I_u + R(a_6 + \mu) \\ -\beta + a_1 S E - a_6 R + \mu S \end{bmatrix}. \quad (22)$$

Here,  $F(X)$  and  $V(X)$  are the rate of appearance of new infections in the compartment and the rate of transfer individuals respectively. By calculating Jacobian matrices at  $E_0$ , we get

$$D(F(E_0)) = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix} \text{ and } D(V(E_0)) = \begin{bmatrix} v & 0 \\ J_1 & J_2 \end{bmatrix},$$

where

$$f = \frac{\partial F_i(E_0)}{\partial X_j} = \begin{bmatrix} a_1 S & 0 & 0 & 0 & a_1 E \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$v = \frac{\partial V_i(E_0)}{\partial X_j} = \begin{bmatrix} a_2 + a_3 + \mu & 0 & 0 & 0 & a_1 E \\ -a_2 & a_4 + \mu & 0 & 0 & 0 \\ -a_3 & 0 & a_3 + \mu & 0 & 0 \\ 0 & -a_4 & -a_5 & a_6 + \mu & 0 \\ a_1 S & 0 & 0 & -a_6 & a_1 E + \mu \end{bmatrix}. \quad (23)$$

Now, we calculate  $\text{NGM}fv^{-1}$ , and find the largest eigenvalues of  $fv^{-1}$  is

$$R_0 = \frac{a_1 \beta}{\mu(a_2 + a_3 + \mu)}. \quad (24)$$

**Theorem 4** The system (1) with disease-free equilibrium point  $E_0$  is locally asymptotically stable if  $\frac{\beta}{\mu} \leq \frac{a_2 + a_3 + \mu}{a_1}$ .

**Proof.** The Jacobian matrix for the system (1) at  $E_0$  can be evaluated by

$$J(E_0) = \begin{bmatrix} -\mu & -\frac{a_1 \beta}{\mu} & 0 & 0 & a_6 \\ 0 & -a_2 - a_3 - \mu & 0 & 0 & 0 \\ 0 & a_2 & -a_4 - \mu & 0 & 0 \\ 0 & a_3 & 0 & -a_5 - \mu & 0 \\ 0 & 0 & a_4 & a_5 & -a_6 - \mu \end{bmatrix}. \quad (25)$$

Therefore, the eigenvalues of matrix  $J(E_0)$  are

$$\lambda_1 = -\mu$$

$$\lambda_2 = -(a_6 + \mu)$$

$$\lambda_3 = -(a_5 + \mu)$$

$$\lambda_4 = -(a_4 + \mu)$$

$$\lambda_5 = \frac{\beta a_1 - \mu(a_2 + a_3 + \mu)}{\mu}. \quad (26)$$



Here, we can clearly see that  $\lambda_1, \lambda_2, \lambda_3$ , and  $\lambda_4$  are negative. Moreover, if  $\frac{\beta}{\mu} < \frac{a_2 + a_3 + \mu}{a_1}$ , then  $\lambda_5 < 0$ .

As we can see that all eigenvalues are negative. So, the system (1) with  $E_0$  is locally asymptotically stable.

**Theorem 5** The system (1) with endemic equilibrium point  $E_{end}^*$  is locally asymptotically stable if  $S^* \leq \max\left\{\frac{a_5}{a_1}, \frac{\mu^2}{a_1 a_5}\right\}$ .

**Proof.** The Jacobian matrix by linearizing the system (1) at  $E_{end}^*$  can be defined as

$$J(E^*) = \begin{bmatrix} -a_1 E^* - \mu & -a_1 S^* & 0 & 0 & a_6 \\ a_1 E^* & a_1 S^* - a_2 - a_3 - \mu & 0 & 0 & 0 \\ 0 & a_2 & -a_4 - \mu & 0 & 0 \\ 0 & a_3 & 0 & -a_5 - \mu & 0 \\ 0 & 0 & a_4 & a_5 & -a_6 - \mu \end{bmatrix}. \quad (27)$$

By basic matrix calculation, we get

$$\begin{aligned} S^* a_1 < a_5 \text{ and } S^* a_1 a_5 < \mu^2 \\ \Rightarrow S^* < \frac{a_5}{a_1} \text{ and } S^* < \frac{\mu^2}{a_1 a_5} \\ \Rightarrow S^* \leq \max\left\{\frac{a_5}{a_1}, \frac{\mu^2}{a_1 a_5}\right\} \end{aligned} \quad (28)$$

Hence, theorem is proved.

## 5.4 Global stability

The stability analysis of FDEs is one of the significant factors [24]. There are various forms and types of stability, and Ulam-Hyers (UH) stability represents one of the significant types. This stability was proposed by Ulam in 1940 and further investigated by Hyers [25]. Rassias extended this stability into a more general form known as Ulam-Hyers-Rassias (UHR) stability.

Let us consider  $S = C[0, T]$  as the Banach space of real-valued continuous functions in  $[0, T]$ , and let us define Banach space  $B = S \times S \times S$  having norm  $\phi \in B$ , as a sup norm. Now, taking positive real number  $F_\omega : \Omega \rightarrow R^+$  and for  $\xi > 0$ , and assuming following Ulam's stability postulates

$$\left| {}^C_0 D_t^\alpha \phi(t) - \omega(t, \phi(t)) \right| \leq \xi F_\omega, \quad (29)$$

$$\left| {}^C_0 D_t^\alpha \phi(t) - \omega(t, \phi(t)) \right| \leq \xi, \quad (30)$$

$$\left| {}^C_0 D_t^\alpha \phi(t) - \omega(t, \phi(t)) \right| \leq F_\omega, \quad (31)$$

where  $\forall t \in \Omega$  and  $\xi = \max(\xi_i)$ , for  $i = 1, 2, 3$ .

**Definition 3** The fractional chlamydia model (4) is UH stable, provided for each  $\xi > 0$  and  $\phi \in B$  of (29). For a positive real number  $F_\omega > 0$ , the existence of  $\psi \in B$  for model (4) ensures

$$|\phi(t) - \psi(t)| \leq \xi C_\omega, t \in \Omega, \quad (32)$$

where  $\xi = \max(\xi_i)$  and  $C_\omega = \max(C_{\omega_i})$ , for  $i = 1, 2, 3$ .

**Definition 4** The fractional model as given in (4) is said to have the stability of the type generalized Ulam-Hyers (GUH): corresponding to a continuous function  $F_\omega : R \rightarrow R$  having condition  $F_\omega(0) = 0$ , if  $\forall \xi > 0$  and  $\forall \phi \in B$  of (30), then  $\psi \in B$  of model (4) having

$$|\phi(t) - \psi(t)| \leq F_\omega(\xi), t \in \Omega, \quad (33)$$

where  $\xi = \max(\xi_i)$  and  $F_\omega = \max(F_{\omega_i})$ ,  $i = 1, 2, 3$ .

**Remark 2** A mapping  $\phi \in B$  will be the result of (29), provided equivalently, we have a mapping  $\mathcal{G}$  in  $B$  so that the following conditions are satisfied:

- (a)  $|\mathcal{G}(t)| \leq \xi, \mathcal{G} = \max(\mathcal{G})$ ,
- (b)  ${}_0^C D_t^\alpha \phi(t) = \omega(t, \phi(t)) + \mathcal{G}(t), \forall t \in \Omega$ .

**Lemma 2** For  $0 \leq \alpha \leq 1$  if  $\phi$  is a member of the Banach space  $B$ , and is the result of (29), then  $\phi$  satisfies

$$|\phi(t) - X(\psi(t))| \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \xi. \quad (34)$$

**Proof.** Let  $\phi$  be a result of (29) and therefore from Remark 2(b), we get

$$\begin{aligned} {}_0^C D_t^\alpha \phi(t) &= \omega(t, \phi(t)) + \mathcal{G}(t), t \in [0, T], \\ \phi(0) &= \phi_0 \geq 0. \end{aligned} \quad (35)$$

Following that, the solution of (35) can be written as

$$\begin{aligned} \phi(t) &= \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t - \tau)^{\alpha - 1} \omega(t, \phi(\tau)) d\tau, \\ &\leq \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t - \tau)^{\alpha - 1} \mathcal{G}(\tau) d\tau. \end{aligned} \quad (36)$$

Now, using Remark 2, we get

$$\begin{aligned} &\left| \phi(t) - \phi_0 - \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t - \tau)^{\alpha - 1} \omega(t, \phi(\tau)) d\tau \right| \\ &\leq \left| \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t - \tau)^{\alpha - 1} \mathcal{G}(\tau) d\tau \right| \\ &\leq \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t - \tau)^{\alpha - 1} |\mathcal{G}(\tau)| d\tau \\ &\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \xi. \end{aligned} \quad (37)$$

Hence, the proof holds.

**Theorem 4** Considering a real-valued continuous map  $\omega$  on  $[0, T] \times B$  (or  $[0, T] \times B$ ), so that for every  $\psi(t) \in B, \omega \in C([0, T] \times B, R)$ . Therefore, under the assumption of Hypothesis 2 and conclusion of Theorem 3, fractional chlamydia system (4) is UH stable on  $[0, T]$ .

**Proof.** Suppose that  $\xi > 0$  and let  $\phi \in B$  be any response of (29). Considering  $\psi \in B$  to be the only outcome of the system (11), as

$${}_0^C D_t^\alpha \psi(t) = \omega(t, \phi(t)), t \in [0, T], \text{ with } \psi(0) = \psi_0, \quad (38)$$

where

$$\psi(t) = \psi_0 + \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t - \tau)^{\alpha - 1} \omega(t, \psi(\tau)) d\tau. \quad (39)$$

In the light of Lemma 2, and Hypothesis 2, we get

$$\begin{aligned} |\phi(t) - \psi(t)| &\leq \left| \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t - \tau)^{\alpha - 1} \omega(t, \phi(\tau)) d\tau - \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t - \tau)^{\alpha - 1} \omega(t, \psi(\tau)) d\tau \right|, \\ &\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \xi + \frac{\alpha C_\omega}{\Gamma(\alpha + 1)} \int_0^t (t - \tau)^{\alpha - 1} |\phi(\tau) - \psi(\tau)| d\tau, \\ &\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \xi + \frac{T^\alpha C_\omega}{\Gamma(\alpha + 1)} |\phi(\tau) - \psi(\tau)|. \end{aligned} \quad (40)$$

It follows that

$$|\phi(t) - \psi(t)| \leq \frac{\Delta \xi}{1 - \Delta C_\omega}, \text{ where } \Delta = \frac{T^\alpha}{\Gamma(\alpha + 1)}. \quad (41)$$

**Corollary 1** Considering  $F_\omega(\xi) = \xi C_\omega$  in Theorem 4, such that  $F_\omega(0) = 0$ , we have the fractional chlamydia system (4) GUH stable.

**Definition 5** Let  $F_\omega$  be continuous positive real-valued function over  $[0, T] \equiv [0, T]$ , i.e.,  $F_\omega \in C([0, T], R^+)$ , the fractional chlamydia model (1) is UHR stable, if we have real constant  $K_{F_\omega} > 0$ , such that for every  $\xi > 0$  and every result  $\phi$  of (30), there lies the result  $\psi$  in  $B$ , such that

$$\phi(t) - \psi(t) \leq K_{F_\omega} \xi F_\omega(t), t \in [0, T], \quad (42)$$

where  $\xi = \max(\xi_i), F_\omega = \max(F_{\omega_i})$ , and  $K_{F_\omega} = \max(K_{F_{\omega_i}})$  for  $i = 1, 2, 3$ .

**Definition 6** The fractional chlamydia model (4) is generalized Ulam-Hyers-Rassias (GUHR) stable if there exist a real constant  $K_{F_\omega} > 0$  and a mapping  $F_\omega \in C([0, T], R^+)$ , such that for each  $\phi$  of (31), there exists a solution  $\psi \in B$  of (4),

$$\phi(t) - \psi(t) \leq K_{F_\omega} F_\omega(t), t \in [0, T], \quad (43)$$

where  $F_\omega = \max(F_{\omega_i})$  and  $K_{F_\omega} = \max(K_{F_{\omega_i}})$ , for  $i = 1, 2, 3$ .

**Remark 3** A mapping  $\phi \in B$  is solution of (30), provided mapping  $\theta \in B$  such that

- (a)  $|\theta(t)| \leq \xi, \theta = \max(\theta_i)$ ,
- (b)  ${}_0^C D_t^\alpha \phi(t) = \omega(t, \phi(t)) + \theta(t), \forall t \in [0, T]$ .

**Hypothesis 3** For  $F_\omega \in B$  and  $\exists \lambda_{F_\omega} > 0$ , such that for  $\forall t \in [0, T]$  inequality of fractional integral is

$${}^C I_t^\alpha F_\omega \leq \lambda_{F_\omega} F_\omega(t). \quad (44)$$

**Lemma 3** For  $0 \leq \alpha \leq 1$ , if  $\phi \in B$  is a result of (30), then  $\phi$  satisfies

$$\left| \phi(t) - \phi_0 - \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \phi(\tau)) d\tau \right| \leq \xi \lambda_{F_\omega} F_\omega(t). \quad (45)$$

**Proof.** Let  $\phi \in B$  be an outcome of (30) and considering Remark 3(b), we get

$${}^C D_t^\alpha \phi(t) = \omega(t, \phi(t)) + \theta(t), t \in [0, T], \phi(0) = \phi_0. \quad (46)$$

The solution of (46) is given by

$$\begin{aligned} \phi(t) &= \phi_0 + \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \phi(\tau)) d\tau, \\ &\leq \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^{\alpha-1} \theta(\tau) d\tau. \end{aligned} \quad (47)$$

Now, using Remark 3, we get

$$\begin{aligned} &\left| \phi(t) - \phi_0 - \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \phi(\tau)) d\tau \right| \\ &\leq \left| \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^{\alpha-1} \theta(\tau) d\tau \right|, \\ &\leq \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^{\alpha-1} |\theta(\tau)| d\tau, \\ &\leq \xi \lambda_{F_\omega} F_\omega(t). \end{aligned} \quad (48)$$

Hence, proved.

**Theorem 5** Considering the mapping  $\omega \in C([0, T] \times R, R) \forall \psi \in B$ , the fractional chlamydia system (4) is UHR stable on  $[0, T]$  by assumptions of Hypothesis 2 and (11).

**Proof.** In the view of Lemma 3, Hypotheses 2 and 3, we get

$$\begin{aligned}
|\phi(t) - \psi(t)| &\leq \left| \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \phi(\tau)) d\tau - \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^{\alpha-1} \omega(t, \psi(\tau)) d\tau \right| \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)} \xi + \frac{\alpha C_\omega}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^{\alpha-1} |\phi(\tau) - \psi(\tau)| d\tau \\
&\leq \lambda_{F_\omega} F_\omega(t) \xi + \frac{T^\alpha C_\omega}{\Gamma(\alpha+1)} |\phi(\tau) - \psi(\tau)|
\end{aligned} \tag{49}$$

It follows that

$$|\phi(t) - \psi(t)| \leq K_{F_\omega} \xi F_\omega(t) \text{ where } K_{F_\omega} = \frac{\lambda_{F_\omega}}{1 - \frac{T^\alpha C_\omega}{\Gamma(\alpha+1)}}. \tag{50}$$

Hence, the theorem is proved.

**Corollary 2** Considering  $\xi = 1$  in Theorem 5, we have the fractional chlamydia system (4) is GUHR stable.

## 6. Working of homotopy perturbation Laplace transform method

Consider a generalized FDE with Caputo derivative as:

$${}_0^C D_t^\alpha \psi(t) + R\psi(t) + N\psi(t) = f(t), t > 0, n-1 < \alpha \leq n, n \in \mathbb{N}, \tag{51}$$

such that

$$\psi(0) = C_0, \frac{d\psi(0)}{dt} = C_1, \frac{d^2\psi(0)}{dt^2} = C_2, \dots, \frac{d^{n-1}\psi(0)}{dt^{n-1}} = C_{n-1}, \tag{52}$$

where  $\left(D_t^\alpha \equiv \frac{d}{dt}\right)$  is fractional differential operator,  $R$  is linear terms,  $N$  is nonlinear terms of  $\psi(t)$ , and  $f(t)$  is continuous function.

Operating Laplace transform to equation (51),

$$\mathcal{L}\left\{{}_0^C D_t^\alpha \psi(t)\right\} = -\mathcal{L}\{R\psi(t)\} - \mathcal{L}\{N\psi(t)\} + \mathcal{L}\{f(t)\}, \tag{53}$$

and using differentiation properties (5),

$$s^\alpha \mathcal{L}\{u(t)\} - \sum_{k=0}^{n-1} s^{\alpha-1-k} u^k(0) = -\mathcal{L}\{R\psi(t)\} - \mathcal{L}\{N\psi(t)\} + \mathcal{L}\{f(t)\}. \tag{54}$$

From equation (52), we get

$$\begin{aligned} \mathcal{L}\{\psi(t)\} = & \left[ \frac{1}{s}\psi(0) + \frac{1}{s^2} \frac{d\psi(0)}{dt} + \dots + \frac{1}{s^n} \frac{d^{n-1}\psi(0)}{dt^{n-1}} \right] \\ & + \frac{1}{s^\alpha} [-\mathcal{L}\{R\psi(t)\} - \mathcal{L}\{N\psi(t)\} + \mathcal{L}\{f(t)\}], \end{aligned} \quad (55)$$

taking inverse Laplace transform (ILT) to equation (55),

$$\psi(t) = \omega(t) - \mathcal{L}^{-1} \left[ \frac{1}{s^\alpha} \mathcal{L}\{R\psi(t)\} + \frac{1}{s^\alpha} \mathcal{L}\{N\psi(t)\} \right], \quad (56)$$

where  $\omega(t)$  is ILT of first and last terms of equation (55). Applying HPM [11] to equation (56) gives

$$\sum_{i=0}^{\infty} p^i \psi_i(t) = \omega(t) - p \mathcal{L}^{-1} \left[ \frac{1}{s^\alpha} \mathcal{L} \left\{ R \left( \sum_{i=0}^{\infty} p^i \psi_i(t) \right) \right\} + \frac{1}{s^\alpha} \mathcal{L} \left\{ N \left( \sum_{i=0}^{\infty} p^i \psi_i(t) \right) \right\} \right] \quad (57)$$

to determine nonlinear terms of the above equation (57), we use He's [26] polynomial,

$$N\psi(t) = \sum_{m=0}^{\infty} p^m H_m(\psi), \quad (58)$$

where  $H_m(\psi_0, \psi_1, \psi_2, \dots, \psi_m) = \frac{1}{m!} \left[ \frac{d^m}{dp^m} N \left( \sum_{i=0}^{\infty} p^i \psi_i(t) \right) \right]_{p=0}$ ,  $m = 0, 1, 2, \dots$

Substituting equation (58) into (57), and comparing the coefficients of  $p^0, p^1, p^2, \dots$ , we have

$$\begin{aligned} p^0 : \psi_0(t) &= \omega(t), \\ p^1 : \psi_1(t) &= -\mathcal{L}^{-1} \left[ \frac{1}{s^\alpha} \mathcal{L}\{R\psi_0(t)\} + \left( \frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L}\{H_0\} \right], \\ p^2 : \psi_2(t) &= -\mathcal{L}^{-1} \left[ \frac{1}{s^\alpha} \mathcal{L}\{R\psi_1(t)\} + \left( \frac{(1-\alpha)s^\alpha + \alpha}{s^\alpha} \right) \mathcal{L}\{H_1\} \right], \\ &\vdots \end{aligned} \quad (59)$$

The solution of equation (51) can be obtained as

$$\psi(t) = \sum_{i=0}^{\infty} p^i \psi_i(t) = \psi_0(t) + p^1 \psi_1(t) + p^2 \psi_2(t) + \dots \quad (60)$$

as  $p \rightarrow 1$  gives

$$\psi(t) = \psi_0(t) + \psi_1(t) + \psi_2(t) + \dots \quad (61)$$

## 7. Approximate solution of chlamydia model

In this section, we present the analytical approach to system (1) given by

$$\begin{aligned}
 {}_0^C D_t^\alpha S(t) &= \beta - a_1 S(t)E(t) + a_6 R(t) - \mu S(t), \\
 {}_0^C D_t^\alpha E(t) &= a_1 S(t)E(t) - a_2 E(t) - a_3 E(t) - \mu E(t), \\
 {}_0^C D_t^\alpha I_s(t) &= a_2 E(t) - a_4 I_s(t) - \mu I_s(t), \\
 {}_0^C D_t^\alpha I_u(t) &= a_3 E(t) - a_5 I_u(t) - \mu I_u(t), \\
 {}_0^C D_t^\alpha R(t) &= a_4 I_s(t) + a_5 I_u(t) - a_6 R(t) - \mu R(t),
 \end{aligned} \tag{62}$$

Applying Laplace transform to (62) in Caputo sense, we achieve

$$\begin{aligned}
 \mathcal{L}\left\{{}_0^C D_t^\alpha S(t)\right\} &= \mathcal{L}\{\beta - a_1 S(t)E(t) + a_6 R(t) - \mu S(t)\}, \\
 \mathcal{L}\left\{{}_0^C D_t^\alpha E(t)\right\} &= \mathcal{L}\{a_1 S(t)E(t) - a_2 E(t) - a_3 E(t) - \mu E(t)\}, \\
 \mathcal{L}\left\{{}_0^C D_t^\alpha I_s(t)\right\} &= \mathcal{L}\{a_2 E(t) - a_4 I_s(t) - \mu I_s(t)\}, \\
 \mathcal{L}\left\{{}_0^C D_t^\alpha I_u(t)\right\} &= \mathcal{L}\{a_3 E(t) - a_5 I_u(t) - \mu I_u(t)\}, \\
 \mathcal{L}\left\{{}_0^C D_t^\alpha R(t)\right\} &= \mathcal{L}\{a_4 I_s(t) + a_5 I_u(t) - a_6 R(t) - \mu R(t)\}.
 \end{aligned} \tag{63}$$

using differentiation property (5), we get

$$\begin{aligned}
 s^\alpha \mathcal{L}\{S(t)\} - s^{\alpha-1} S(0) &= \mathcal{L}\{\beta - a_1 S(t)E(t) + a_6 R(t) - \mu S(t)\}, \\
 s^\alpha \mathcal{L}\{E(t)\} - s^{\alpha-1} E(0) &= \mathcal{L}\{a_1 S(t)E(t) - a_2 E(t) - a_3 E(t) - \mu E(t)\}, \\
 s^\alpha \mathcal{L}\{I_s(t)\} - s^{\alpha-1} I_s(0) &= \mathcal{L}\{a_2 E(t) - a_4 I_s(t) - \mu I_s(t)\}, \\
 s^\alpha \mathcal{L}\{I_u(t)\} - s^{\alpha-1} I_u(0) &= \mathcal{L}\{a_3 E(t) - a_5 I_u(t) - \mu I_u(t)\}, \\
 s^\alpha \mathcal{L}\{R(t)\} - s^{\alpha-1} R(0) &= \mathcal{L}\{a_4 I_s(t) + a_5 I_u(t) - a_6 R(t) - \mu R(t)\}.
 \end{aligned} \tag{64}$$

Applying ic (4) and taking inverse Laplace, we have

$$\begin{aligned}
S(t) &= S_0 + L^{-1} \left[ \frac{1}{s^\alpha} L \{ \beta - a_1 S(t) E(t) + a_6 R(t) - \mu S(t) \} \right], \\
E(t) &= E_0 + L^{-1} \left[ \frac{1}{s^\alpha} L \{ a_1 S(t) E(t) - a_2 E(t) - a_3 E(t) - \mu E(t) \} \right], \\
I_s(t) &= I_{s_0} + L^{-1} \left[ \frac{1}{s^\alpha} L \{ a_2 E(t) - a_4 I_s(t) - \mu I_s(t) \} \right], \\
I_u(t) &= I_{u_0} + L^{-1} \left[ \frac{1}{s^\alpha} L \{ a_3 E(t) - a_5 I_u(t) - \mu I_u(t) \} \right], \\
R(t) &= R_0 + L^{-1} \left[ \frac{1}{s^\alpha} L \{ a_4 I_s(t) + a_5 I_u(t) - a_6 R(t) - \mu R(t) \} \right].
\end{aligned} \tag{65}$$

Now, we applying HPM [11] to equation (65),

$$\begin{aligned}
\sum_{n=0}^{\infty} p^n S_n(t) &= S_0 + pL^{-1} \left[ \frac{1}{s^\alpha} L \left\{ \beta - a_1 \sum_{n=0}^{\infty} p^n S_n(t) E_n(t) + a_6 \sum_{n=0}^{\infty} p^n R_n(t) - \mu \sum_{n=0}^{\infty} p^n S_n(t) \right\} \right], \\
\sum_{n=0}^{\infty} p^n E_n(t) &= E_0 + pL^{-1} \left[ \frac{1}{s^\alpha} L \left\{ a_1 \sum_{n=0}^{\infty} p^n S_n(t) E_n(t) - a_2 \sum_{n=0}^{\infty} p^n E_n(t) - a_3 \sum_{n=0}^{\infty} p^n E_n(t) - \mu \sum_{n=0}^{\infty} p^n E_n(t) \right\} \right], \\
\sum_{n=0}^{\infty} p^n I_{s_n}(t) &= I_{s_0} + pL^{-1} \left[ \frac{1}{s^\alpha} L \left\{ a_2 \sum_{n=0}^{\infty} p^n E_n(t) - a_4 \sum_{n=0}^{\infty} p^n I_{s_n}(t) - \mu \sum_{n=0}^{\infty} p^n I_{s_n}(t) \right\} \right], \\
\sum_{n=0}^{\infty} p^n I_{u_n}(t) &= I_{u_0} + pL^{-1} \left[ \frac{1}{s^\alpha} L \left\{ a_3 \sum_{n=0}^{\infty} p^n E_n(t) - a_5 \sum_{n=0}^{\infty} p^n I_{u_n}(t) - \mu \sum_{n=0}^{\infty} p^n I_{u_n}(t) \right\} \right], \\
\sum_{n=0}^{\infty} p^n R_n(t) &= R_0 + pL^{-1} \left[ \frac{1}{s^\alpha} L \left\{ a_4 \sum_{n=0}^{\infty} p^n I_{s_n}(t) + a_5 \sum_{n=0}^{\infty} p^n I_{u_n}(t) - a_6 \sum_{n=0}^{\infty} p^n R_n(t) - \mu \sum_{n=0}^{\infty} p^n R_n(t) \right\} \right],
\end{aligned} \tag{66}$$

In above equation (66), nonlinear terms are decomposed using He's [26] polynomial  $H_n$ ,

$$\begin{aligned}
H_0 &= S_0 E_0, \\
H_1 &= S_0 E_1 + S_1 E_0, \\
H_2 &= S_0 E_2 + S_1 E_1 + S_2 E_0, \\
H_3 &= S_0 E_3 + S_1 E_2 + S_2 E_1 + S_3 E_0, \\
&\vdots
\end{aligned} \tag{67}$$



Comparing “ $p$ ” terms of equation (66), yields

$$p^0 : S_0(t) = S_0$$

$$E_0(t) = E_0$$

$$Is_0(t) = Is_0$$

$$Iu_0(t) = Iu_0$$

$$R_0(t) = R_0$$

$$p^1 : S_1(t) = L^{-1} \left[ \frac{1}{s^\alpha} L \{ \beta + a_6 R_0 - \mu S_0 - a_1 H_0 \} \right],$$

$$= (\beta + a_6 R_0 - \mu S_0 - a_1 S_0 E_0) \frac{t^\alpha}{\Gamma(1 + \alpha)},$$

$$E_1(t) = L^{-1} \left[ \frac{1}{s^\alpha} L \{ -a_2 E_0 - a_3 E_0 - \mu E_0 + a_1 H_0 \} \right],$$

$$= (-a_2 E_0 - a_3 E_0 - \mu E_0 + a_1 S_0 E_0) \frac{t^\alpha}{\Gamma(1 + \alpha)},$$

$$Is_1(t) = L^{-1} \left[ \frac{1}{s^\alpha} L \{ a_2 E_0 - a_4 Is_0 - \mu Is_0 \} \right],$$

$$= (a_2 E_0 - a_4 Is_0 - \mu Is_0) \frac{t^\alpha}{\Gamma(1 + \alpha)},$$

$$Iu_1(t) = L^{-1} \left[ \frac{1}{s^\alpha} L \{ a_3 E_0 - a_5 Iu_0 - \mu Iu_0 \} \right]$$

$$= (a_3 E_0 - a_5 Iu_0 - \mu Iu_0) \frac{t^\alpha}{\Gamma(1 + \alpha)},$$

$$R_1(t) = L^{-1} \left[ \frac{1}{s^\alpha} L \{ a_4 Is_0 + a_5 Iu_0 - a_6 R_0 - \mu R_0 \} \right]$$

$$= (a_4 Is_0 + a_5 Iu_0 - a_6 R_0 - \mu R_0) \frac{t^\alpha}{\Gamma(1 + \alpha)},$$

$$\begin{aligned}
p^2 : S_2(t) &= L^{-1} \left[ \frac{1}{s^\alpha} L \{ \beta + a_6 R_1 - \mu S_1 - a_1 H_1 \} \right], \\
&= \beta \frac{t^\alpha}{\Gamma(1+\alpha)} + \left[ a_6 (a_4 I s_0 + a_5 I u_0 - a_6 R_0 - \mu R_0) - \mu (\beta + a_6 R_0 - \mu S_0 - a_1 S_0 E_0) \right. \\
&\quad - a_1 S_0 (-a_2 E_0 - a_3 E_0 - \mu E_0 + a_1 S_0 E_0) \\
&\quad \left. - a_1 E_0 (\beta + a_6 R_0 - \mu S_0 - a_1 S_0 E_0) \right] \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\
E_2(t) &= L^{-1} \left[ \frac{1}{s^\alpha} L \{ \beta + a_6 R_1 - \mu S_1 - a_1 H_1 \} \right], \\
&= L^{-1} \left[ \frac{1}{s^\alpha} L \{ -a_2 E_1 - a_3 E_1 - \mu E_1 + a_1 H_1 \} \right], \\
&\quad - (a_2 + a_3 + \mu + a_1 S_0) (-a_2 E_0 - a_3 E_0 - \mu E_0 + a_1 S_0 E_0) \left] \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\
I s_2(t) &= L^{-1} \left[ \frac{1}{s^\alpha} L \{ a_2 E_1 - a_4 I s_1 - \mu I s_1 \} \right], \\
&= \left[ a_2 (-a_2 E_0 - a_3 E_0 - \mu E_0 + a_1 S_0 E_0) - (a_4 + \mu) (a_2 E_0 - a_4 I s_0 - \mu I s_0) \right] \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\
I u_2(t) &= L^{-1} \left[ \frac{1}{s^\alpha} L \{ a_3 E_1 - a_5 I u_1 - \mu I u_1 \} \right], \\
&= \left[ a_3 (-a_2 E_0 - a_3 E_0 - \mu E_0 + a_1 S_0 E_0) - (a_5 + \mu) (a_3 E_0 - a_5 I u_0 - \mu I u_0) \right] \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\
R_2(t) &= L^{-1} \left[ \frac{1}{s^\alpha} L \{ a_4 I s_1 + a_5 I u_1 - a_6 R_1 - \mu R_1 \} \right], \\
&= \left[ a_4 (a_2 E_0 - a_4 I s_0 - \mu I s_0) + a_5 (a_3 E_0 - a_5 I u_0 - \mu I u_0) \right. \\
&\quad \left. - (a_6 + \mu) (a_4 I s_0 + a_5 I u_0 - a_6 R_0 - \mu R_0) \right] \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \\
&\quad \vdots
\end{aligned} \tag{68}$$

Thus, the solution of equation (62) can be obtained using (61) as

$$S(t) = S_0(t) + S_1(t) + S_2(t) + \dots$$

$$= S_0 + (2\beta + a_6 R_0 - \mu S_0 - a_1 S_0 E_0) \frac{t^\alpha}{\Gamma(1+\alpha)} + [a_6 (a_4 I_0 + a_5 I u_0 - a_6 R_0 - \mu R_0) - \mu (\beta + a_6 R_0 - \mu S_0 - a_1 S_0 E_0) - a_1 S_0 (-a_2 E_0 - a_3 E_0 - \mu E_0 + a_1 S_0 E_0) - a_1 E_0 (\beta + a_6 R_0 - \mu S_0 - a_1 S_0 E_0)] \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \dots$$

$$E(t) = E_0(t) + E_1(t) + E_2(t) + \dots$$

$$= E_0 + (-a_2 E_0 - a_3 E_0 - \mu E_0 + a_1 S_0 E_0) \frac{t^\alpha}{\Gamma(1+\alpha)} + [a_1 E_0 (\beta + a_6 R_0 - \mu S_0 - a_1 S_0 E_0) - (a_2 + a_3 + \mu + a_1 S_0) (-a_2 E_0 - a_3 E_0 - \mu E_0 + a_1 S_0 E_0)] \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \dots$$

$$I_s(t) = I_{s0}(t) + I_{s1}(t) + I_{s2}(t) + \dots$$

$$= I_{s0} + (a_2 E_0 - a_4 I_{s0} - \mu I_{s0}) \frac{t^\alpha}{\Gamma(1+\alpha)} + [a_2 (-a_2 E_0 - a_3 E_0 - \mu E_0 + a_1 S_0 E_0) - (a_4 + \mu) (a_2 E_0 - a_4 I_{s0} - \mu I_{s0})] \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \dots$$

$$I_u(t) = I_{u0}(t) + I_{u1}(t) + I_{u2}(t) + \dots$$

$$= I_{u0} + (a_3 E_0 - a_5 I_{u0} - \mu I_{u0}) \frac{t^\alpha}{\Gamma(1+\alpha)} + [a_3 (-a_2 E_0 - a_3 E_0 - \mu E_0 + a_1 S_0 E_0) - (a_5 + \mu) (a_3 E_0 - a_5 I_{u0} - \mu I_{u0})] \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \dots$$

$$R(t) = R_0(t) + R_1(t) + R_2(t) + \dots$$

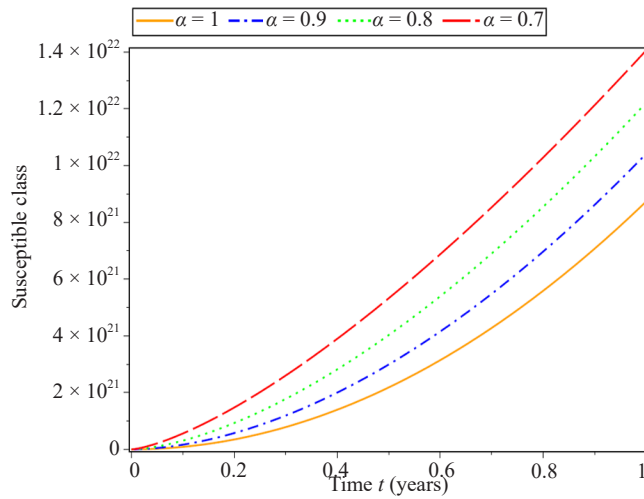
$$= R_0 + (a_4 I_{s0} + a_5 I_{u0} - a_6 R_0 - \mu R_0) \frac{t^\alpha}{\Gamma(1+\alpha)} + [a_4 (a_2 E_0 - a_4 I_{s0} - \mu I_{s0}) + a_5 (a_3 E_0 - a_5 I_{u0} - \mu I_{u0}) - (a_6 + \mu) (a_4 I_{s0} + a_5 I_{u0} - a_6 R_0 - \mu R_0)] \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \dots \quad (69)$$

## 8. Conclusions

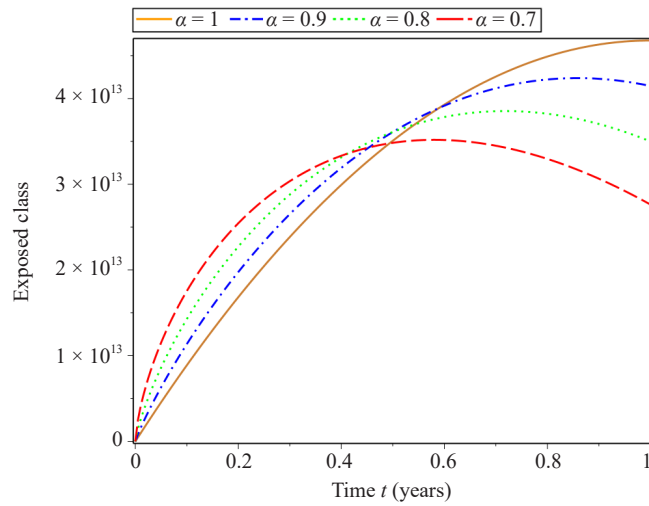
In this work, we have studied a nonlinear mathematical model of chlamydia transmission using FDEs in the Caputo sense. As a means of speculating about potential chlamydia epidemics, we have run several simulations under a wide range of parameters and studied the memory properties of the system using the different noninteger orders of derived FDEs. By using this method, we can attain a more comprehensive understanding of the chlamydia model. The effectiveness of the technique is shown by the findings in (69). The research has considerable significance in predicting the future of the disease and its treatment. The data shows that homotopy perturbation Laplace transform method may outperform conventional approaches. Figures 2-6 show the results for various fractional orders  $\alpha$  for  $S(t)$ ,  $E(t)$ ,  $I_s(t)$ ,  $I_u(t)$ , and  $R(t)$ . Table 2 depicts the approximate solutions to the chlamydia model for five separate  $\alpha$  values: 1, 0.9, 0.8, and 0.7. This study is beneficial for medical research institutions to track and understand the spread of disease.

**Table 2.** The approximate solution of chlamydia model for distinct order  $\alpha$

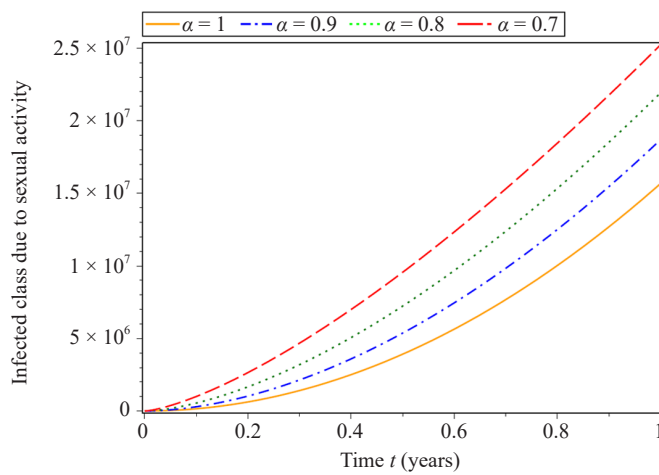
Order	The approximate solution of chlamydia model
$\alpha = 1$	$S(t) = 233824096 - 9.352964073 \times 10^{13}t - 8.729087285 \times 10^{21}t^2 + \dots$
	$E(t) = 500000 + 9.352963790 \times 10^{13}t - 4.676481895 \times 10^{13}t^2 + \dots$
	$I_s(t) = 200904 + 1.4815928 \times 105t + 3.133242862 \times 10^{13}t^2 + \dots$
	$I_u(t) = 250000 - 80000t + 1.496474210 \times 10^{13}t^2 + \dots$
$\alpha = 0.9$	$R(t) = 225000 + 1.671433426 \times 10^{11}t + 6.399838705 \times 10^8t^2 + \dots$
	$S(t) = 233824096 - 9.724783063 \times 10^{13}t^{0.9} - 1.041352252 \times 10^{22}t^{1.8} + \dots$
	$E(t) = 500000 + 9.724782769 \times 10^{13}t^{0.9} - 5.578893637 \times 10^{13}t^{1.8} + \dots$
	$I_s(t) = 200904 + 1.540492239 \times 10^5t^{0.9} + 3.737858728 \times 10^{13}t^{1.8} + \dots$
$\alpha = 0.8$	$I_u(t) = 250000 - 83180.33075 \times t^{0.9} + 1.785245969 \times 10^{13}t^{1.8} + \dots$
	$R(t) = 225000 + 4.250844295 \times 10^5t^{0.9} + 21340.19289t^{1.8} + \dots$
	$S(t) = 233824096 - 1.004200885 \times 10^{14}t^{0.8} - 1.221171983 \times 10^{22}t^{1.6} + \dots$
	$E(t) = 500000 + 1.004200855 \times 10^{14}t^{0.8} - 6.542251762 \times 10^{13}t^{1.6} + \dots$
$\alpha = 0.7$	$I_s(t) = 200904 + 1.590743629 \times 10^5t^{0.8} + 4.383308670 \times 10^{13}t^{1.6} + \dots$
	$I_u(t) = 250000 - 85893.70192 \times t^{0.8} + 2.093520570 \times 10^{13}t^{1.6} + \dots$
	$R(t) = 225000 + 4.389508307 \times 105t^{0.8} + 25025.19740 \times t^{1.6} + \dots$
	$S(t) = 233824096 - 1.029338034 \times 10^{14}t^{0.7} - 1.405458494 \times 10^{22}t^{1.4} + \dots$
$\alpha = 0.7$	$E(t) = 500000 + 1.029338003 \times 10^{14}t^{0.7} - 7.529540016 \times 10^{13}t^{1.4} + \dots$
	$I_s(t) = 200904 + 1.630563112 \times 10^5t^{0.7} + 5.044791799 \times 10^{13}t^{1.4} + \dots$
	$I_u(t) = 250000 - 88043.79244 \times t^{0.7} + 2.409452812 \times 10^{13}t^{1.4} + \dots$
	$R(t) = 225000 + 4.499386447 \times 10^5t^{0.7} + 28801.73862t^{1.4} + \dots$



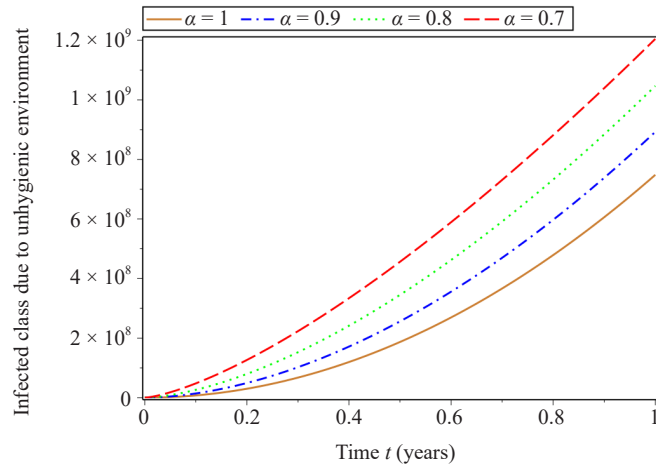
**Figure 2.** The behavior of the susceptible class  $S(t)$  at various order  $\alpha$



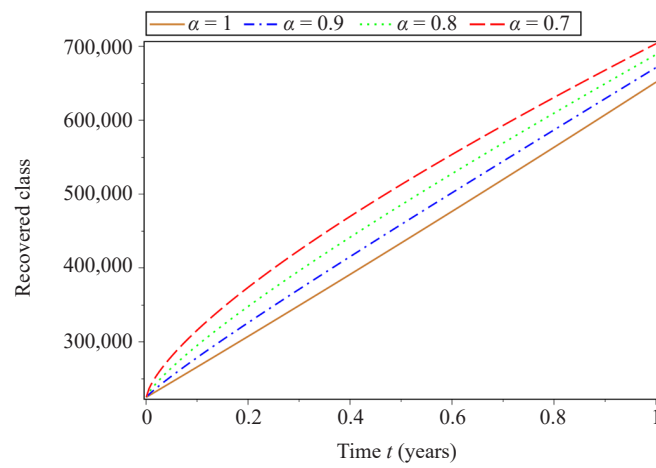
**Figure 3.** The behavior of the exposed class  $E(t)$  at various order  $\alpha$



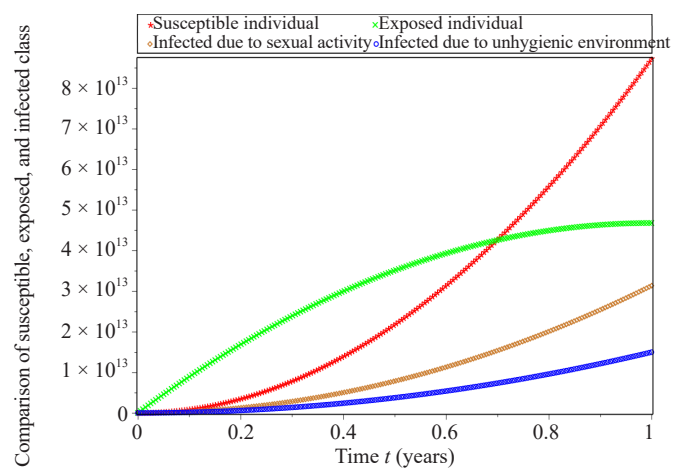
**Figure 4.** The behavior of the infected class due to sexual activity  $I_s(t)$  at various order  $\alpha$



**Figure 5.** The behavior of infected class due to unhygienic environment  $I_u(t)$  at various order  $\alpha$



**Figure 6.** The behavior of the recovered class  $R(t)$  at various order  $\alpha$



**Figure 7.** The comparison of  $S(t)$ ,  $E(t)$ ,  $I_s(t)$ , and  $I_u(t)$

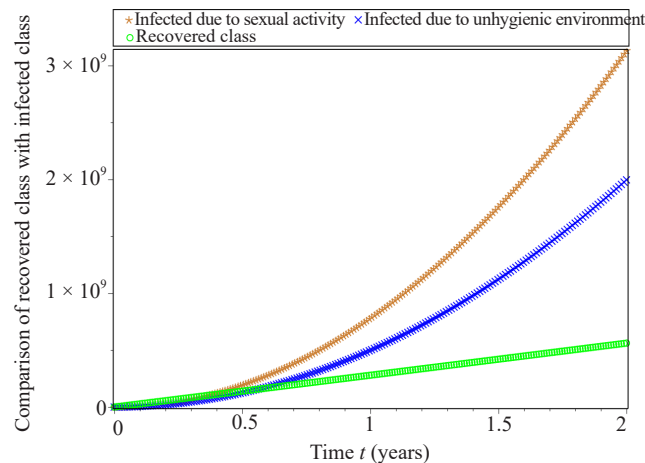


Figure 8. The comparison of  $I_s(t)$ ,  $I_u(t)$ , and  $R(t)$

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## Conflict of interest

The authors have no conflict of interest to declare.

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