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Research Article

Certain Class of *P*-Valent Analytic Function Associated with Derivative Operator and Their Properties

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Abstract: This study aims to define a new subclass of multivalent analytic functions in the open unit disk. Jackson's derivative operator has been used to generate this subclass. Before getting coefficient characterization, we look at certain needs for the functions related to this subclass. We can see several fascinating features, including coefficient estimates, growth and distortion theorem, extreme points, and the radius of starlikeness and convexity of functions belonging to the subclass are shown using this technique.

Keywords: new subclass, p-valent function, quantum or $(i,\ j)$-calculus, $(i,\ j)$-derivative operator

MSC: 30C45

1. Introduction and definition

Let us use the notation $\hat{\mathbb{A}}$ to refer to the class of all analytic functions of the type

$$f(z)=z+\sum_{\eta=2}^{\infty}a_\eta z^\eta,$$

that are defined on the open unit disk $\underline{\mathcal{U}}=\left\{z\in\overline{\mathbb{C}}:|z|<1\right\}$ on the complex plane $\overline{\mathbb{C}}$. Let $\hat{\mathbb{A}}(p)$ $(p\in\mathbb{N}=\{1,\ 2,\ 3,\ ...\})$ be the class consisting of all functions f has the Taylor series expansion of the form

$$f(z)=z^p+\sum_{\eta=p+1}^{\infty}a_\eta z^\eta,\quad a_\eta \text{ is complex number}$$

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which are analytic and p-valent in the open disk $\underline{\mathcal{U}}$ on the complex plane $\overline{\mathbb{C}}$. We note that $\hat{\mathbb{A}}(1) = \hat{\mathbb{A}}$. Let us use the notation $S(p)$ to represent the subclass of $\hat{\mathbb{A}}(p)$ that is comprised of multivalent functions in $\underline{\mathcal{U}}$. In addition, consider the $S_p^*(\alpha)$ and $C_p(\alpha)$ to be illustrations of the classes of p-valent functions that are starlike of order α and convex of order α, respectively, given the range of values $0 \leq \alpha < p$. In specifically, the classes $S_p^*(0) = S_p^*$ and $C_p(0) = C_p$ are the well-known classes of starlike and convex p-valent functions in $\underline{\mathcal{U}}$, respectively.

Let $\mathcal{T}(p)$ $(p \in \mathbb{N} = \{1,\ 2,\ 3,\ ...\})$ be the subclass of $S(p)$, consisting of functions of the form

$$f(z) = z^p - \sum_{\eta=p+1}^{\infty} a_\eta z^\eta, \quad a_\eta > 0 \tag{1}$$

defined on the open unit disk $\underline{\mathcal{U}} = \left\{z \in \overline{\mathbb{C}} : |z| < 1\right\}$. When the function $f \in \mathcal{T}(p)$ has negative coefficients, we refer to it as a p-valent function with negative coefficients. The subclasses of $\mathcal{T}(p)$ designated by $S^*_{\mathcal{T},\ p}(\alpha)$ and $C_{\mathcal{T},\ p}(\alpha)$ for $0 \leq \alpha < p$ are p-valent functions that are starlike of order α and convex of order α, respectively.

According to Silverman [1], the class $\mathcal{T}(1) = \mathcal{T}$ was established and investigated. In [1], Silverman discovered the subclasses of $\mathcal{T}(1)$ indicated by $S^*_{\mathcal{T},\ 1}(\alpha) = S^*_{\mathcal{T}}(\alpha)$ and $C_{\mathcal{T},\ 1}(\alpha) = C_{\mathcal{T}}(\alpha)$, which respectively starlike of order α and convex of order α where $0 \leq \alpha < 1$.

Let $\mathcal{U}(k,\ \wp,\ \ell)$ be the subclass of $\mathcal{T}$ consist of functions $f \in \mathcal{T}$ satisfying the condition

$$\operatorname{Re}\left\{\frac{\wp z^3 f'''(z) + (1+2\wp)z^2 f''(z) + zf'(z)}{\wp z^2 f''(z) + zf'(z)}\right\}$$

$$> k\left|\frac{\wp z^3 f'''(z) + (1+2\wp)z^2 f''(z) + zf'(z)}{\wp z^2 f''(z) + zf'(z)} - 1\right| + \ell.$$

where $0 \leq \wp \leq 1$, $0 \leq \ell < 1$ and $k \geq 0$ for all $z \in \underline{\mathcal{U}}$. This class of functions was studied by Shanmugam et al. [2].

In this section, we recall some known concepts and basic results of $(i,\ j)$-calculus. Throughout this paper, we let $i,\ j$ be constants with $0 < j < i \leq 1$. We give some definitions and theorems for $(i,\ j)$-calculus, which will be used in these papers [3–11].

For $0 < j < i \leq 1$ the jackson's $(i,\ j)$-derivative of a function $f \in \hat{\mathbb{A}}(p)$ is, by definition, given as follow

$$\mathcal{D}_{i,\ j} f(z) := \begin{cases} \dfrac{f(iz) - f(jz)}{(i-j)z}, & z \neq 0, \\ \\ f'(0), & z = 0. \end{cases} \tag{2}$$

From (2), we have

$$\mathcal{D}_{i,\ j} f(z) = [p]_{i,\ j}\, z^{p-1} + \sum_{\eta=p+1}^{\infty} [\eta]_{i,\ j}\, a_\eta\, z^{\eta-1} \quad (0 < j < i \leq 1),$$

where $[p]_{i,\ j} = \dfrac{i^p - j^p}{i-j}$ and $[\eta]_{i,\ j} = \dfrac{i^\eta - j^\eta}{i-j}$.

Note that for $i=1$, the jackson $(i,\ j)$-derivative reduces to the jackson j-derivative operator of the function f, $\mathcal{D}_j f(z)$ (refer to [12–14]). Note also that $\mathcal{D}_{1,\ j} f(z) \to f'(z)$ when $j \to 1_-$, where f' is the classical derivative of the function f.

Clearly for a function $g(z) = z^\eta$, we obtain

$$\mathcal{D}_{i,\ j} g(z) = \mathcal{D}_{i,\ j} z^\eta = \frac{i^\eta - j^\eta}{i-j} z^{\eta-1} = [\eta]_{i,\ j} z^{\eta-1},$$

and

$$\lim_{j\to 1-} \mathcal{D}_{1,\ j} g(z) = \lim_{j\to 1-} \frac{1-j^\eta}{1-j} z^{\eta-1} = \eta z^{\eta-1} = g'(z),$$

where g' is the ordinary derivative.

The theory of j-calculus are used in describing and solving various problems in applied science such as ordinary fractional calculus, quantum physics, optimal control, hypergeometric series, operator theory, j-difference and j-integral equations, as well as geometric function theory of complex analysis. The application of j-calculus was initiated by Jackson [13]. Kanas and Raducanu [15] have used the fractional j-calculus operators in investigations of certain classes of functions which are analytic in $\underline{\mathcal{U}}$. For details on j-calculus one can refer [10, 13, 15–19] and also the reference cited therein.

Along with the development of the theory and application of j-calculus, the theory of j-calculus based on two parameters $(i,\ j)$-integers has also presented and recieved more attention during the last few dacades. In 1991, Chakrabarti and Jagannathan [20] introduced the $(i,\ j)$-calculus. Next, Sadjang [5] studied the fundamental theorem of $(i,\ j)$-calculus and some $(i,\ j)$-Taylor formulas. Recently, Tunc and Göv [10] defined the $(i,\ j)$-derivative and $(i,\ j)$-integral on finite intervals. Moreover, they studied some properties of $(i,\ j)$-calculus and $(i,\ j)$-analogue of some important integral inequalities. The $(i,\ j)$-derivative have been studied and rapidly developed during this period by many authors.

Using the above defined $(i,\ j)$-calculus, several subclasses belonging to the class $\hat{\mathbb{A}}(p)$ have already been investigated in geometric function theory. Ismail et al. [21] were the first who used the j-derivative operator $\mathcal{D}_j$ to study the j-calculus analogous of the class S^* of starlike functions in $\underline{\mathcal{U}}$.

From now on we introduce some general subclass of analytic and multivalent functions associated with $(i,\ j)$-derivative operator as follows.

Definition 1.1 For $0 \le \wp \le 1$, $0 \le \ell < 1$, $k \ge 0$, $0 < j < i \le 1$ and $p \in \mathbb{N} = \{1,\ 2,\ 3,\ ...\}$, we let $U(k,\ \wp,\ \ell,\ i,\ j,\ p)$ consist of functions $f \in \mathcal{T}(p)$ satisfying the condition

$$\operatorname{Re}\left\{\frac{\wp z^3 (\mathcal{D}_{i,\ j} f(z))''' + (1+2\wp) z^2 (\mathcal{D}_{i,\ j} f(z))'' + z(\mathcal{D}_{i,\ j} f(z))'}{\wp z^2 (\mathcal{D}_{i,\ j} f(z))'' + z(\mathcal{D}_{i,\ j} f(z))'}\right\}$$

$$> k\left|\frac{\wp z^3 (\mathcal{D}_{i,\ j} f(z))''' + (1+2\wp) z^2 (\mathcal{D}_{i,\ j} f(z))'' + z(\mathcal{D}_{i,\ j} f(z))'}{\wp z^2 (\mathcal{D}_{i,\ j} f(z))'' + z(\mathcal{D}_{i,\ j} f(z))'} - 1\right| + \ell. \tag{3}$$

The growth and distortion theorem is one of our other findings, along with the coefficient inequalities for the functions $f \in U(k,\ \wp,\ \ell,\ i,\ j,\ p)$. The extreme points are then obtained. Let's first investigate the coefficient inequalities. And the technique which studied in [22–25].

2. Coefficient inequalities

Our first finding, is a necessary and sufficient condition for the function f belongs to the class $U(k,\ \wp,\ \ell,\ i,\ j,\ p)$.

Theorem 2.1 Let $0 \le \wp \le 1$, $0 \le \ell < 1$, $k \ge 0$, $0 < j < i \le 1$, and $p \in \mathbb{N} = \{1,\ 2,\ 3,\ \ldots\}$. A function f given by (1) is in the class $U(k,\ \wp,\ \ell,\ i,\ j,\ p)$ if and only if

$$\sum_{\eta=p+1}^{\infty} \mu_\eta\, a_\eta \le \mu_p, \tag{4}$$

where

$$\mu_\eta = [(k+1)\,|2-\eta|+1-\ell]\left[(\wp(\eta-2)+1)\,(\eta-1)\,[\eta]_{i,\ j}\right]. \tag{5}$$

Proof. We have $f \in U(k,\ \wp,\ \ell,\ i,\ j,\ p)$ if and only if the condition (3) is satisfied.

Let

$$w = \frac{\wp z^3\,(\mathcal{D}_{i,\ j}f(z))''' + (1+2\wp)z^2\,(\mathcal{D}_{i,\ j}f(z))'' + z\,(\mathcal{D}_{i,\ j}f(z))'}{\wp z^2\,(\mathcal{D}_{i,\ j}f(z))'' + z\,(\mathcal{D}_{i,\ j}f(z))'},$$

so

$$w = \frac{\sum_{\eta=p+1}^{\infty} T_\eta\,(\eta-1)\,a_\eta z^{\eta-1} - T_p\,(p-1)\,z^{p-1}}{\sum_{\eta=p+1}^{\infty} T_\eta a_\eta z^{\eta-1} - T_p z^{p-1}},$$

where $T_\eta = (\wp(\eta-2)+1)\,(\eta-1)\,[\eta]_{i,\ j}$.

Considering that,

$$\mathrm{Re}(w) \ \ge\ k\,|w-1| + \ell \ \text{ if and only if } \ (k+1)\ |w-1| \ \le\ 1-\ell.$$

Now

$$
\begin{aligned}
&(k+1)\,|w-1| \\
&= (k+1)\left|\frac{\sum_{\eta=p+1}^{\infty} T_\eta\,(\eta-1)\,a_\eta z^{\eta-1} - T_p\,(p-1)\,z^{p-1}}{\sum_{\eta=p+1}^{\infty} T_\eta a_\eta z^{\eta-1} - T_p z^{p-1}} - 1\right| \\
&= (k+1)\left|\frac{\sum_{\eta=p+1}^{\infty} T_\eta\,(2-\eta)\,a_\eta z^{\eta-1} - T_p\,(2-p)\,z^{p-1}}{T_p z^{p-1} - \sum_{\eta=p+1}^{\infty} T_\eta a_\eta z^{\eta-1}}\right| \\
&\leq 1-\ell,
\end{aligned}
$$

equivalent to

$$
(k+1)\left|\frac{\sum_{\eta=p+1}^{\infty} T_\eta\,(2-\eta)\,a_\eta z^{\eta-p} - T_p\,(2-p)}{T_p - \sum_{\eta=p+1}^{\infty} T_\eta a_\eta z^{\eta-p}}\right| \leq 1-\ell. \tag{6}
$$

The above inequality reduces to

$$
\frac{(k+1)\left(\left|\sum_{\eta=p+1}^{\infty} T_\eta\,(2-\eta)\,a_\eta z^{\eta-p}\right| - |T_p\,(2-p)|\right)}{|T_p| - \left|\sum_{\eta=p+1}^{\infty} T_\eta a_\eta z^{\eta-p}\right|} \leq 1-\ell.
$$

After that

$$
\frac{(k+1)\left(\sum_{\eta=p+1}^{\infty} T_\eta\,|2-\eta|\,a_\eta - T_p\,|2-p|\right)}{T_p - \sum_{\eta=p+1}^{\infty} T_\eta a_\eta} \leq 1-\ell, \tag{7}
$$

where $|z| < 1$.

Then

$$(k+1)\left(\sum_{\eta=p+1}^{\infty} T_\eta\,|2-\eta|\,a_\eta - T_p\,|2-p|\right) \leq \left(T_p - \sum_{\eta=p+1}^{\infty} T_\eta a_\eta\right)(1-\ell). \tag{8}$$

Therefore,

$$\sum_{\eta=p+1}^{\infty}\big((k+1)\,|2-\eta|+1-\ell\big)T_\eta a_\eta \leq \big((k+1)\,|2-p|+1-\ell\big)T_p. \tag{9}$$

Which yield to (4).

Suppose that (4) holds and we have to show (3). That is equivalent to (6). From condition (4) we have (8) and then (7). Now it is suffices to show that,

$$\left|\frac{\sum_{\eta=p+1}^{\infty} T_\eta\,(2-\eta)\,a_\eta z^{\eta-p} - T_p\,(2-p)}{T_p - \sum_{\eta=p+1}^{\infty} T_\eta a_\eta z^{\eta-p}}\right|$$

$$\leq \frac{\sum_{\eta=p+1}^{\infty} T_\eta\,|2-\eta|\,a_\eta - T_p\,|2-p|}{T_p - \sum_{\eta=p+1}^{\infty} T_\eta a_\eta z^{\eta-p}}. \tag{10}$$

Since,

$$\left|T_p - \sum_{\eta=p+1}^{\infty} T_\eta a_\eta z^{\eta-p}\right| \geq |T_p| - \left|\sum_{\eta=p+1}^{\infty} T_\eta a_\eta z^{\eta-p}\right|,$$

$$\geq T_p - \sum_{\eta=p+1}^{\infty} T_\eta a_\eta,$$

where $|z| < 1$. And hence (10) obtained.

Theorem 2.2 Let $0 \leq \wp \leq 1$, $0 \leq \ell < 1$, $k \geq 0$, $0 < j < i \leq 1$, and $p \in \mathbb{N} = \{1,\ 2,\ 3,\ \ldots\}$. If the function f given by (1) be in the class $U(k,\ \wp,\ \ell,\ i,\ j,\ p)$ then

$$a_\eta \leq \frac{\mu_p}{\mu_\eta}, \quad \eta = p+1,\ p+2,\ p+3,\ \ldots, \tag{11}$$

where μ_η is given by (5).

Equality holds for the functions given by,

$$f(z) = z^P - \frac{\mu_p z^\eta}{\mu_\eta}. \tag{12}$$

Proof. Since $f \in U(k,\ \wp,\ \ell,\ i,\ j,\ p)$ Theorem 2.1 holds.

Now

$$\sum_{\eta=p+1}^{\infty} \mu_\eta a_\eta \leq \mu_p,$$

we have,

$$a_\eta \leq \frac{\mu_p}{\mu_\eta}.$$

Clearly the function given by (12) satisfies (11) and therefore f given by (12) is in $U(k,\ \wp,\ \ell,\ i,\ j,\ p)$ for this function, the result is clearly sharp.

3. Growth and distortion theorems for the subclass $U(k, \wp, \ell, i, j, p)$

The growth and distortion theorem and the covering property for functions in the class $U(k,\ \wp,\ \ell,\ i,\ j,\ p)$ will both be covered in this section.

Theorem 3.1 Let $0 \leq \wp \leq 1$, $0 \leq \ell < 1$, $k \geq 0$, $0 < j < i \leq 1$, and $p \in \mathbb{N} = \{1,\ 2,\ 3,\ \ldots\}$. If the function f given by (1) be in the class $U(k,\ \wp,\ \ell,\ i,\ j,\ p)$ then for $0 < |z| = l < 1$, we have

$$l^p - \frac{\mu_p}{\mu_{p+1}} l^{p+1} \leqslant |f(z)| \leqslant l^p + \frac{\mu_p}{\mu_{p+1}} l^{p+1}. \tag{13}$$

Equality holds for the function,

$$f(z) = z^p - \frac{\mu_p}{\mu_{p+1}} z^{p+1}, \quad (z = \pm l,\ \pm il),$$

where μ_p and μ_{p+1} are found by (5).

Proof. We only demonstrate the right side inequality in (13) since the other inequality may be supported by reasons that are comparable.

Since $f \in U(k,\ \wp,\ \ell,\ i,\ j,\ p)$ by Theorem 2.1, we have,

$$\sum_{\eta=p+1}^{\infty} \mu_\eta a_\eta \leq \mu_p.$$

Now

$$\mu_{p+1}\sum_{\eta=p+1}^{\infty} a_\eta = \sum_{\eta=p+1}^{\infty} \mu_{p+1} a_\eta,$$

$$\leq \sum_{\eta=p+1}^{\infty} \mu_\eta \, a_\eta$$

$$\leq \mu_p.$$

And therefore

$$\sum_{\eta=p+1}^{\infty} a_\eta \leqslant \frac{\mu_p}{\mu_{p+1}}, \tag{14}$$

since

$$f(z) = z^p - \sum_{\eta=p+1}^{\infty} a_\eta z^\eta, \quad a_\eta > 0,$$

we have,

$$|f(z)| = \left| z^p - \sum_{\eta=p+1}^{\infty} a_\eta z^\eta \right|,$$

$$\leq |z|^p + |z|^{p+1} \sum_{\eta=p+1}^{\infty} a_\eta \, |z|^{\eta-(p+1)},$$

$$\leq l^p + l^{p+1} \sum_{\eta=p+1}^{\infty} a_\eta.$$

With the help of inequality (14), the right-side inequality of (13) is obtained.

Theorem 3.2 If the function f given by (1) is in the class $U(k, \wp, \ell, i, j, p)$ for $0 < |z| = l < 1$ and $\eta \mu_{p+1} \leq (p+1)\mu_\eta$ where $\eta = p+1, \ p+2, \ p+3, \ \ldots$, then, we have

$$p\,l^{p-1} - \frac{(p+1)\mu_p}{\mu_{p+1}} l^p \leqslant \left| f'(\mathrm{z}) \right| \leqslant p\,l^{p-1} + \frac{(p+1)\mu_p}{\mu_{p+1}} l^p. \tag{15}$$

Equality holds for the function f given by

$$f(z) = z^p - \frac{\mu_p}{\mu_{p+1}} z^{p+1}, \quad (z = \pm l, \, \pm il),$$

where μ_p and μ_{p+1} are given by (5).

Proof. Since $f \in U(k, \wp, \ell, i, j, p)$ by Theorem 2.1 we have

$$\sum_{\eta=p+1}^{\infty} \mu_\eta \, a_\eta \leq \mu_p.$$

Now,

$$\mu_{p+1} \sum_{\eta=p+1}^{\infty} \eta \, a_\eta \leq (p+1) \sum_{\eta=p+1}^{\infty} \mu_\eta a_\eta \leq (p+1)\mu_p.$$

Hence

$$\sum_{\eta=p+1}^{\infty} \eta a_\eta \leqslant \frac{(p+1)\mu_p}{\mu_{p+1}}, \tag{16}$$

since

$$f'(z) = pz^{p-1} - \sum_{\eta=p+1}^{\infty} \eta \, a_\eta \, z^{\eta-1}.$$

Then, we have

$$p\,|z|^{p-1} - |z|^p \sum_{\eta=p+1}^{\infty} \eta a_\eta \, |z|^{\eta-1-p} \leqslant \left| f'(z) \right| \leqslant p\,|z|^{p-1} + |z|^p \sum_{\eta=p+1}^{\infty} \eta a_\eta \, |z|^{\eta-1-p},$$

where $|z| < 1$. By using the inequality (16), we now have Theorem 3.2, which concludes our demonstration.

Theorem 3.3 If the function f given by (1) is in the class $U(k, \wp, \ell, i, j, p)$ then f is starlike of order δ, where

$$\delta = 1 - \frac{\mu_p p}{-\mu_p + \mu_{p+1}}.$$

The result is sharp with

$$f(z)=z^p-\frac{\mu_p}{\mu_{p+1}}z^{p+1},$$

where μ_p and μ_{p+1} are found by (5).

Proof. It is sufficient to show that (4) implies

$$\sum_{\eta=p+1}^{\infty}a_\eta(\eta-\delta)\leq 1-\delta. \tag{17}$$

That is,

$$\frac{\eta-\delta}{1-\delta}\leq\frac{\mu_\eta}{\mu_p},\quad \eta\geq p+1. \tag{18}$$

The above inequality is equivalent to

$$\delta\leqslant 1-\frac{\mu_p(\eta-1)}{-\mu_p+\mu_\eta}=\psi(\eta),$$

where $\eta\geq p+1$.

And $\psi(\eta)\geq\psi(p+1)$, (18) holds true for any $0\leq\wp\leq 1$, $0\leq\ell<1$, $k\geq 0$, $0<j<i\leq 1$ and $p\in\mathbb{N}=\{1, 2, 3, ...\}$. This completes the proof of Theorem 3.3.

4. Extreme points of the class $U(k, \wp, \ell, i, j, p)$

The following formula shows the extreme points of the class $U(k, \wp, \ell, i, j, p)$.

Theorem 4.1 Let $f_p(z)=z^p$, and

$$f_\eta(z)=z^p-\frac{\mu_p}{\mu_\eta}z^\eta,\quad \eta=p+1,\ p+2,\ p+3,\ ...,$$

where μ_η is given by (5).

Then $f\in U(k, \wp, \ell, i, j, p)$ if and only if it can be represented in the form

$$f(z)=\sum_{\eta=p}^{\infty}y_\eta f_\eta(z) \tag{19}$$

where $y_\eta\geq 0$ and $\sum_{\eta=p}^{\infty}y_\eta=1$.

Proof. Suppose f can be represented as in (19). Our aim is to demonstrate that $f\in U(k, \wp, \ell, i, j, p)$.

By (19) we have

$$f(z)=\sum_{\eta=p}^{\infty} y_\eta \left\{z^p-\frac{\mu_p z^\eta}{\mu_\eta}\right\}.$$

Then

$$f(z)=z^p-\sum_{\eta=p+1}^{\infty} a_\eta z^\eta$$

$$=z^p-\sum_{\eta=p+1}^{\infty} \frac{\mu_p y_\eta}{\mu_\eta} z^\eta.$$

So that

$$a_\eta=\frac{\mu_p y_\eta}{\mu_\eta}, \quad \eta \geqslant p+1.$$

Now, we have

$$\sum_{\eta=p+1}^{\infty} y_\eta = 1-y_p \le 1.$$

Setting

$$\sum_{\eta=p+1}^{\infty} y_\eta \frac{\mu_p}{\mu_\eta} \times \frac{\mu_\eta}{\mu_p} = \sum_{\eta=p+1}^{\infty} y_\eta = 1-y_p \leqslant 1.$$

It follows from Theorem 2.1 that the function $f \in U(k, \wp, \ell, i, j, p)$.
Conversely, it suffices to show that

$$a_\eta=\frac{\mu_p}{\mu_\eta} y_\eta.$$

Now we have $f \in U(k, \wp, \ell, i, j, p)$ then by previous Theorem 2.2.

$$a_\eta \leqslant \frac{\mu_p}{\mu_\eta}, \quad \eta \geqslant p+1.$$

That is,

$$\frac{\mu_\eta a_\eta}{\mu_p} \leqslant 1,$$

but $y_\eta \leq 1$.

Setting,

$$y_\eta = \frac{\mu_\eta a_\eta}{\mu_p}, \quad \eta \geqslant p+1.$$

Which yields to the desired result. This completes the proof of the theorem.

Corollary 4.2 The extreme point of the class $U(k,\ \wp,\ \ell,\ i,\ j,\ p)$ are the function

$$f_p(z) = z^p,$$

and

$$f_\eta(z) = z^p - \frac{\mu_p}{\mu_\eta} z^\eta, \quad \eta = p+1,\ p+2,\ p+3,\ ...,$$

where μ_η is given by (5).

Theorem 4.3 The class $U(k,\ \wp,\ \ell,\ i,\ j,\ p)$ is closed under convex linear combinations.

Proof. Suppose that the functions $f_1(z)$ and $f_2(z)$ defined by

$$f_i(z) = z^p - \sum_{\eta=p+1}^{\infty} a_{\eta,\ i} z^\eta, \quad (i = 1,\ 2;\ z \in \underline{\mathcal{U}}) \tag{20}$$

are in the class $U(k,\ \wp,\ \ell,\ i,\ j,\ p)$.

Setting $f(z) = cf_1(z) + (1-c)f_2(z)$ $(0 \leq c \leq 1)$, we find from (20) that

$$f(z) = z^p - \sum_{\eta=p+1}^{\infty} (ca_{\eta,\ 1} + (1-c)a_{\eta,\ 2})z^\eta, \quad (0 \leq c \leq 1;\ z \in \underline{\mathcal{U}}).$$

In view of Theorem 2.1, we have

$$\sum_{\eta=p+1}^{\infty} \mu_\eta\,(c a_{\eta,\ 1} + (1-c)\,a_{\eta,\ 2}) = c \sum_{\eta=p+1}^{\infty} \mu_\eta\, a_{\eta,\ 1} + (1-c) \sum_{\eta=p+1}^{\infty} \mu_\eta\, a_{\eta,\ 2}$$

$$\leq c\mu_p + (1-c)\mu_p = \mu_p,$$

which shows shat $f(z) \in U(k,\ \wp,\ \ell,\ i,\ j,\ p)$. Hence the theorem.

Finally, in this paper we consider the radius of starlikeness and convexity.

5. Radius of starlikeness and convexity

The following theorems specify the radius of starlikeness and convexity for the class $U(k,\ \wp,\ \ell,\ i,\ j,\ p)$.

Theorem 5.1 If the function f given by (1) is in the class $U(k,\ \wp,\ \ell,\ i,\ j,\ p)$, then f is starlike of order $\delta\ (0 \leq \delta < p)$, in the disk $|z| < R$, where

$$R = \inf\left[\frac{\mu_\eta}{\mu_p} \times \left(\frac{p-\delta}{\eta-\delta}\right)\right]^{\frac{1}{\eta-p}}, \quad \eta = p+1,\ p+2,\ p+3,\ \ldots, \tag{21}$$

where μ_η is given by (5).

Proof. Here (21) implies

$$\mu_p\,(\eta-\delta)\,|z|^{\eta-P} \leq \mu_\eta\,(p-\delta)\,.$$

It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p-\delta,$$

for $|z| < R$, we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum\limits_{\eta=p+1}^{\infty} (\eta-p)a_\eta\,|z|^{\eta-p}}{1-\sum\limits_{\eta=p+1}^{\infty} a_\eta\,|z|^{\eta-p}}. \tag{22}$$

By aid of (11), we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leqslant \frac{\sum\limits_{\eta=p+1}^{\infty} \dfrac{\mu_p\,(\eta-p)\,|z|^{\eta-p}}{\mu_\eta}}{1-\sum\limits_{\eta=p+1}^{\infty} \dfrac{\mu_p\,|z|^{\eta-p}}{\mu_\eta}}.$$

The final expression is bounded above by the $p-\delta$ if

$$\sum_{\eta=p+1}^{\infty} \frac{\mu_p (\eta - p) |z|^{\eta-p}}{\mu_\eta} \le \left[1 - \sum_{\eta=p+1}^{\infty} \frac{\mu_p |z|^{\eta-p}}{\mu_\eta}\right](p-\delta),$$

and it follows that

$$|z|^{\eta-p} \leqslant \left[\frac{\mu_\eta}{\mu_p}\left(\frac{p-\delta}{\eta-\delta}\right)\right], \quad \eta \geqslant p+1$$

which is equivalent to our condition (21) of the theorem.

Theorem 5.2 If the function f given by (1) is in the class $U(k, \wp, \ell, i, j, p)$, then f is convex of order ε $(0 \le \varepsilon < p)$, in the disk $|z| < w$, where

$$w = \inf\left[\frac{\mu_\eta}{\mu_p} \times \left(\frac{p(p-\varepsilon)}{\eta(\eta-\varepsilon)}\right)\right]^{\frac{1}{\eta-p}}, \quad \eta = p+1,\ p+2,\ p+3,\ \ldots,$$

where μ_η is given by (5).

Proof. By using the same technique in the proof of Theorem 5.1, we can show that

$$\left|\frac{zf''(z)}{f'(z)} - (p-1)\right| \le p - \varepsilon, \quad \text{for} \quad |z| \le w,$$

with the aid of (11). Thus we have the assertion of Theorem 5.2.

6. Conclusion

This article proposes to identify a significant subclass of multivalent analytic functions in the open unit disk, that have been characterized using Jackson's derivative operator. Futher to study certain sufficient requirements for the functions belonging to this class, one of the main requirements needed to satisfy coefficient characterization. This approach, for example, can provide several many fascinating features.

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Conflict of interest

The author declares no competing financial interest.

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