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Research Article

Artificial Intelligence Solution for Relaxation Oscillation Equation

A. Kalaiyarasi[1*], V. Joseph Raj[1], S. Sindu Devi[2], V. J. Joliz Anton[3]

[1] PG & Research Department of Mathematics, Kamaraj College, Thoothukudi, Tamil Nadu 628003, India
[2] Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Chennai, Tamil Nadu 603203, India
[3] Faculty of Information and communication Engineering, Anna University, Chennai, Tamil Nadu 600025, India
E-mail: ranikalai1976@gmail.com

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Abstract: Artificial Intelligence Solution is given for Relaxation Oscillation Equation, which is one of the simplest Fractional Differential Equation. Artificial Intelligence solution has been identified as the best one for fractional differential equation. Sample of patterns of fractional different equation are extracted. The better performance of Machine Learning Models is compared with the Numerical solution of Successive Approximation Method and reported.

***Keywords*:** fractional differential equation, machine learning, patterns, random forest

MSC: 34C26

1. Introduction

Natural and Engineering soft materials are often viscoelastic, characterized by fractional frequency dependent. Memory behaviors stress relaxation and damped oscillation are their two common mechanical behaviors. The fractional power law dissipation is dependent process of the memory. For the damped oscillation, damping dependent on the frequency to the powers α can be non-integer [1]. Solution of Fractional differential equation (FDE) with initial condition can be classified into two classes namely approximative methods and analytic methods. The approximative method includes operational matrix method based on orthogonal functions, predictor-corrector methods, fractional Euler method and so on [2]. Machine learning is used for the solution of LPP with bounded variable. The patterns of LPP are generated and given to the machine learning technique for classification [3].

In this paper a numerical solution of general linear inhomogeneous fractional differential equation with constant coefficient has been obtained by using approximative method for different values of input (t and α). When the dimension of the problem is greater, it is difficult and time consuming to use these existing mathematical methods. Machine Learning Model, Random Forest is identified as the better model in the above situation.

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2. Literature survey

Daftardar-Gejji and Jafari [4] discussed the Iterative method for obtaining the numerical solution of non-linear fractional equations. Jafari et al. [5] developed iterative Laplace transforms method to obtain the numerical solution of a system of fractional partial differential equation. Mainardi [1] developed Fractional Relaxation-Oscillation and Fractional Diffusion-Wave phenomena. Chen et al. [6] discussed fractional and fractal derivative Relaxation oscillation models . Joseph Raj et al. [7] discussed Detection of Recovery of COVID-19 cases using machine learning.

3. Numerical solution of relaxation oscillation equation

Let us consider an initial value problem for the FDE which was originally formulated by Bagley et al. [8]. It concentrates on describing the method without studying the convergence of the method from the theoretical point of view. The proposed Numerical Scheme is explicit. It was experimentally verified on several examples by using MATLAB.

Consider relaxation-oscillation equation:

$$ {}_0D_t^{\alpha} y(t) + Ay(t) = H(t)\ (t > 0) \tag{1} $$

$$ y^{(k)}(0) = 0\ (k = 0,\ 1,\ 2,\ \ldots,\ n-1) \tag{2} $$

where $0 < \alpha \leq 2$, where $H(t)$ is the Heaviside function.

The first order approximation of equation (1) and equation (2)

$$ 2h^{-\alpha} \sum_{j=0}^{m} w_j^{(\alpha)} y_{m-1} + Ay_m = f_m\ \ (m = 1,\ 2,\ \ldots);\ y_0 = 0 \tag{3} $$

is

$$ t_m = mh,\ y_m = y(t_m),\ f_m = f(t_m) = H(t_m) = mh\ \ (m = 0,\ 1,\ 2,\ \ldots) \tag{4} $$

$$ w_j^{(\alpha)} = (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix}\ \ (j = 0,\ 1,\ 2,\ \ldots) \tag{5} $$

using approximation in equation (3), we derive the following algorithm for obtaining the numerical solution:

$$ y_k = 0\ \ (k = 1,\ 2,\ \ldots,\ n-1) \tag{6} $$

$$ y_m = -Ah^{\alpha} y_{m-1} - \sum_{j=1}^{m} w_j^{(\alpha)} y_{m-j} + h^{\alpha} f_m\ \ (m = n,\ n+1,\ \ldots) \tag{7} $$

our aim is to calculate y_m for different value of α, where $0 \leq \alpha \leq 2$ with $0 < t \leq 1$.

3.1 *Machine learning algorithms and performance*

For the solution of ROE (Table 1) different machine learning models are applied and compared.

3.2 *Machine learning algorithms*

3.2.1 *Support vector machine (SVM)*

Support Vector Machine is a supervised Machine learning algorithm used for both classification and regression. The advantages are effective in high dimensional spaces. Still effective in cases where the number of dimensions is greater than the number of samples.

3.2.2 *Random forest (RF)*

It is an ensemble learning method for classification, regression and other tasks that operates by constructing a multitude of decision trees at training time.

3.2.3 *Decision tree (DT)*

It is a type of supervised machine learning used to categorize or make predictions based on how a previous set of questions were answered. Decision trees imitate human thinking. So, it is easy for data scientists to understand and interpret the results.

3.2.4 *K-nearest neighbors (KNN)*

It is a non-parametric supervised learning classifier, to make classifications or predictions about the grouping of an individual data point.

3.2.5 *Lasso regression (LR)*

It is a regression analysis method that perform both variable selection and regularization to enhance the prediction accuracy.

3.2.6 *Ridge regression (RR)*

It is a model tuning method that is used to analyze any data that suffers from multi collinearity.

Here t and α are the main input and $y(t)$ is the output, where α is the order of the fractional differential equations (Figure 1). In our problem we have 325 values for training and testing 100 values. Random forest is better than SVM, DT, KNN, LR, RR and existing Mathematical Approximation Method.

3.2.7 *Pattern generation*

In the proposed work the patterns are generated for relaxation oscillation equations and given to machine learning technique for the solution (Figure 2). The vector (t, α) are the input patterns used to get the solution $y(t)$. In our problem, we find $y(t)$ with step size $h = 0.01$ and α takes 1.25, 1.5, 1.75, 2. Comparing t values with α values. The vector (t, α) are the input patterns used to get the solution $y(t)$. In the problem, the output is $y(t)$ with step size $h = 0.01$ and α takes 1.25, 1.5, 1.75 and 2.

The performance of the methods SVM, RF, DT, KNN, LR and RR using the metrices Cross validation (CV), Mean absolute error (MAE), Mean square error (MSE) and Root mean square error (RMSE) are given in the Table 2. Machine Learning Performance on CV is given Figure 3. Machine Learning Performance on MAE is given in Figure 4. Machine Learning Performance on MSE is given in Figure 4. Machine Learning Performance on RMSE is given in Figure 5.

4. Experimental results

Table 1. Solution ROE using SAM

t values	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$
0.01	0.0032	0.001	0.0003	0.0001
0.02	0.0071	0.0025	0.0009	0.003
0.03	0.0115	0.0044	0.0016	0.0006
...				
...				
0.92	0.5922	0.5429	0.4798	0.3982
0.93	0.5979	0.5561	0.4878	0.4062
...				
...				
1	0.6367	0.606	0.5444	0.4639

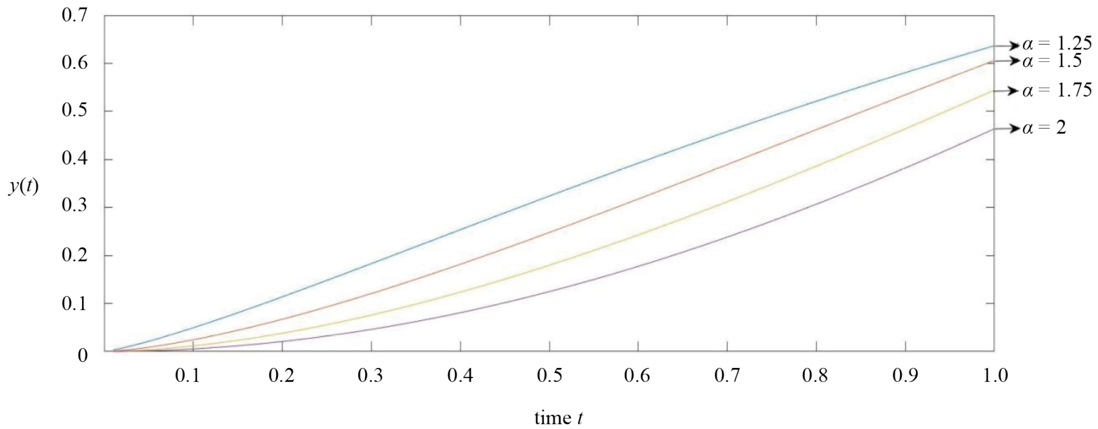

Figure 1. $y(t)$ with different orders of fractional derivative based on SAM using MATLAB

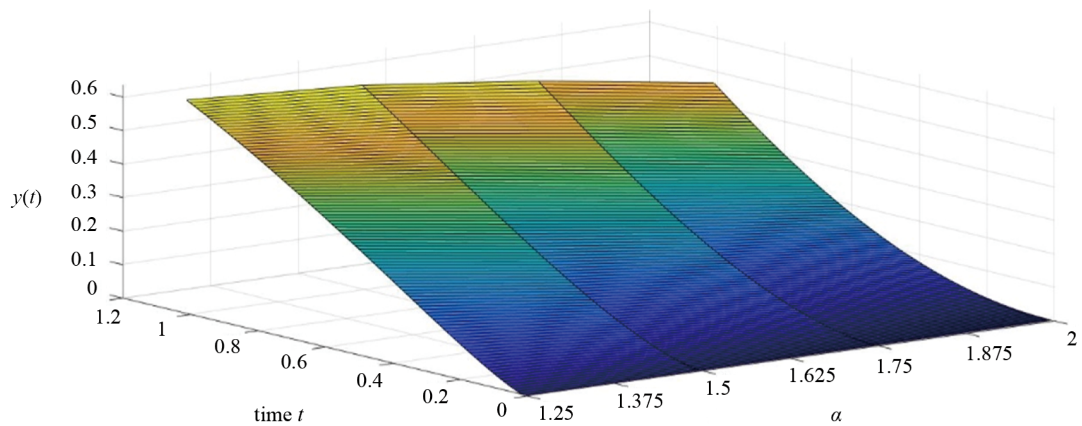

Figure 2. Solution of relaxation-oscillation equation for $1 \leq \alpha \leq 2$

Table 2. The performance of the methods SVM, RF, DT, KNN, LR, RR using the metrices CV, MAE, MSE, RMSE

	CV	MAE	MSE	RMSE
SVM	0.857979	62.879555	315703.6762	561.875143
RF	0.998883	62.822887	315695.5466	561.867909
DT	0.998163	62.824933	315694.5226	561.866997
KNN	0.998173	62.824933	315694.5226	561.866997
LR	0.998157	62.980186	315727.7903	561.896601
RR	0.998165	62.824933	315694.5226	561.866997

5. Machine learning performance analysis

The metrices of the models are also calculated. The best model is identify. Performance metrics of machine learning are cross validation (CV), Mean absolute Error (MAS), Mean Squared Error (MSE) and Root Mean Squared Error (RSME).

5.1 *Cross validation (CV)*

It is a statistical method of evaluating and comparing learning algorithms by dividing data into two segments, one used to learn or train a model and the other used to validate the model (Figure 3).

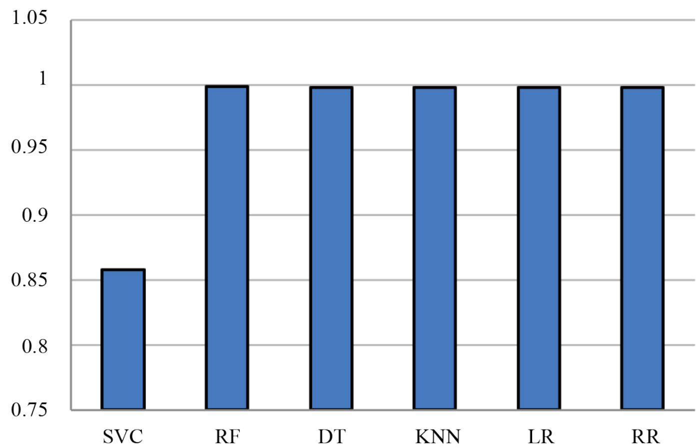

Figure 3. Machine learning performance on cross validation

5.2 *Mean absolute error (MAE)*

It is a measure of error between paired observations expressing the same phenomenon:

$$MAF = \frac{\sum_{i=1}^{n} |y_i - x_i|}{n} \quad (8)$$

where y_i is the predictor value, x_i is the true value, n_i is the total number of data points (Figure 4).

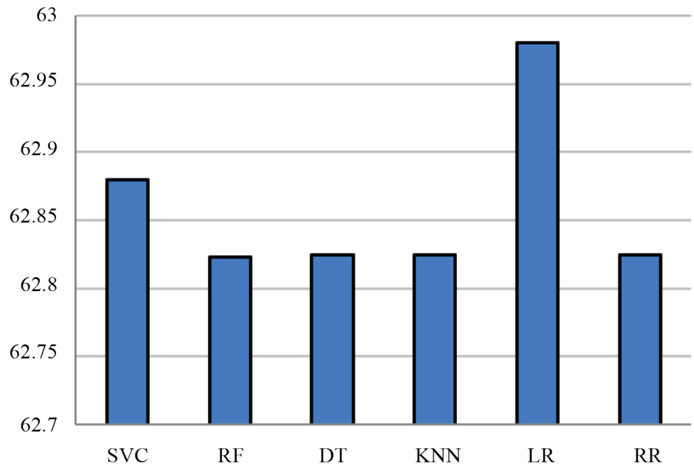

Figure 4. Machine learning performance on MAE

5.3 *Mean square error*

It is an estimator measuring the average of errors square. It is always non-negative, and values close to zero are better (Figure 5).

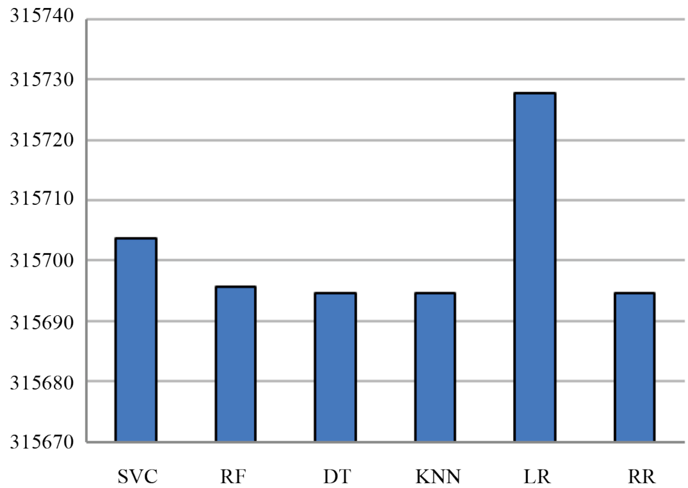

Figure 5. Machine learning performance on mean square error

5.4 *Root mean square errors (RMSE)*

It is defined as the square root of value obtained from Mean Square Error function. Usually, a RSME score of less than 180 is considered a good score for a moderately mean squared value (Figure 6).

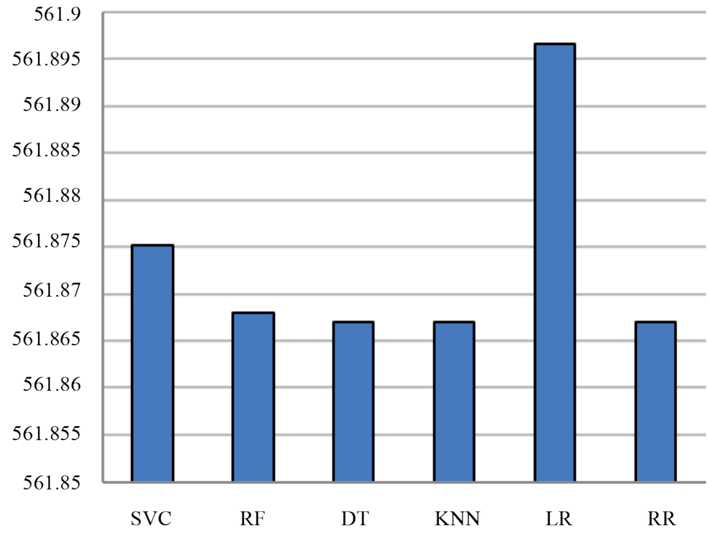

Figure 6. Machine learning performance on RMSE

5.5 *Mathematical justification*

When the database is bigger, and a quick solution is needed. The machine learning technique gives solution efficiently.

The order of complexity of ML model in SVM is $O(3n)$ and its error term is margin error + classification error.

The order of complexity of ML model in RF is $O(n\log(n)*d*m)$ where m denotes number of decision trees and its error term is difference between observed valued and calculated value.

The order of complexity of ML model in DT is $O(n\log(n*d)+O(nd))$, where d is the number of feature in data set.

The order of complexity of ML model in KNN is $O(nd)$ where d is the

The order of complexity of model in LR is $O(3k + k2n)$ and error term is L_1 Norm = sum of absolute coefficient.

The order of complexity of model in RR is $O(\max i\ \text{ter}*n^3)$ and error term is L_2 Norm = sum of the squad coefficient.

Total error of an approximation method is = round off error + truncation error, truncation error = $O(h^{n+1})$, where h is the step size, n denotes none of terms included.

6. Conclusions

In this paper, patterns are (input, output) generated from FDE.

The RF performs better than Machine Learning Techniques SVM, DT, KNN, LR, RR and SAM method. RF gives solution faster than SAM and RF solution is stable and same as SAM. Further research can be done in improving Architecture of the RF.

Conflict of interest

The authors declare no conflict of interest.

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