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Research Article

Solving Nonlinear Time-Fractional Partial Differential Equations Using Conformable Fractional Reduced Differential Transform with Adomian Decomposition Method

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Abstract: In this article, we use a new technique called conformable fractional reduced differential transform (CFRDT) with Adomian decomposition to estimate the solution of one and two-dimensional time-fractional partial linear and nonlinear differential equations with initial values. We explain the convergence analysis of this technique. The obtained results illustrate that the novel method is efficient and easy to use to find approximate solutions for the time-fractional partial differential equations (PDEs). Thus, the suggested method has a significant impact on how engineering, physics, and other disciplines solve fractional PDEs. Furthermore, we analyze the solution of problems with a 2D or 3D graphical representation by using Mathematica software.

***Keywords*:** time-fractional partial differential equation, conformable fractional reduced differential transform method, conformable fractional differential transform method, Adomian decomposition method

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1. Introduction

The area of fractional calculus is one of the numerous applied mathematics disciplines. Instead of integer order, it deals with arbitrarily ordered derivatives and integrals, as well as their applications [1-3]. A system of fractional partial differential equations (FPDEs) can be used to simulate many processes in a variety of disciplines, including fluid mechanics, biology, finance, and material science. Even with linear fractional differential equations (FDEs), finding the precise solution is challenging. Therefore, approximate solutions are required. Many authors, including Ertürk et al. [4], who used the differential transform (DT) approach, pointed out the solution of the system of FPDEs. The method of fractional-complex transformation was studied by Ghazanfari et al. [5]. An iterative Laplace transform was given by Jafari et al. [6]. The Laplace transform and the Adomian decomposition technique are combined to generate the Laplace-Adomian decomposition method (LADM) [7-8]. Numerous different kinds of nonlinear equations, including differential equations of integer and fractional order, can be solved using this method. This approach was used by Jafari et al. [6] to solve the linear and nonlinear fractional diffusion and wave equations. The precise solution of nonlinear fractional differential equations (NLFDEs) has been determined using a variety of mathematical techniques that have been developed and examined. For example, the Adomian decomposition method (ADM) makes it possible

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to solve nonlinear ordinary or partial fractional differential equations analytically and effectively without the need for linearization or perturbation techniques found in [9-12], while Momani et al. [13] and Kharrat [14] have used ADM in order to resolve fractional Riccati differential equations. In the Caputo notion, the fractional derivatives are defined, and besides the homotopy analysis method (HAM) [15], variational iteration method (VIM) [16], finite difference method (FDM) [17], differential transform method (DTM) [18-20], LADM [21], Teppawar et al. [22-23], developed the conformable fractional differential transform method (CFDTM) with Adomian polynomials has been used to solve nonlinear and singular Lane-Emden FDEs. Tamboli et al. [24], used the fractional reduced differential transform method (FRDTM) to evaluate the time-fractional generalized Burger-Fisher equation (TF-GBFE) and the modified fractional differential transform method (FDTM) for solving nonlinear FDEs which can be found in [25]. Recently, Keskin et al. [26] have introduced the reduced differential transform method (RDTM) for partial differential equations and the FRDTM found in [26-27]. Muneshwar et al. [28] have found the solution of linear and nonlinear FPDEs considering partial differential equations of integer order that involve derivatives with regard to time or space variables. In [29], Kumar et al. developed a new version of the L1-predictor-corrector (L1-PC) approach for solving multiple delay-type FDEs. Mahatekar et al. [30] derived a new numerical method to solve FDEs containing Caputo-Fabrizio derivatives. Along with the derivation of the algorithm of the method, error and stability were analyzed, and the validity and effectiveness of the method were briefly explored. Marasi et al. [31] provided two arrays that contained the coefficients of the fractional Adams-Bashforth and Adams-Moulton techniques, as well as recursive relations to generate the members of these arrays. Kumar et al. [32] were to propose generalized forms of three well-known fractional numerical methods, namely Euler, Runge-Kutta 2-step, and Runge-Kutta 4-step, respectively. The new versions they present of these methods are derived from concern with a non-uniform grid, which is slightly different from previous versions of these algorithms. Odibat et al. [33] used the generalized differential transform scheme to simulate impulsive differential equations with non-integer order. Thabet et al. [34] have provided a novel iterative approach for finding an analytical solution to nonlinear fractional partial differential equations (NFPDEs). Thorat et al. [35] addressed the geometrical meaning of the modified alpha-derivative by using the notion of fractional cords.

2. Preliminaries

In [1-3] and the references referenced within, you may find a variety of definitions and theorems of fractional integrals and derivatives.

Definition 2.1. [3] The Riemann-Liouville time-fractional partial derivative of function $u(\xi,\zeta)$ of order α for $\alpha \in \mathbb{R}$, $m-1 \le \alpha < m \in \mathbb{N}$ is defined as follows:

$$ {}^{RL}\mathcal{D}_{\zeta}^{\alpha} u(\xi,\zeta) = \frac{\partial^m}{\partial \zeta^m} \int_0^{\zeta} \frac{(\zeta-\tau)^{m-\alpha-1}}{\Gamma(m-\alpha)} u(x,\tau)\, d\tau, \quad \zeta > 0. $$

Definition 2.2. [3] The Caputo time-fractional partial derivative of the function $u(\xi,\zeta)$ of order α, for $\alpha, \zeta \in \mathbb{R}$ and $m-1 < \alpha < m \in \mathbb{N}, t > 0$ is defined as follows:

$$ \begin{cases} \mathcal{D}_{\zeta}^{\alpha} u(\xi,\zeta) = \int_0^{\zeta} \dfrac{(\zeta-\tau)^{m-\alpha-1}}{\Gamma(m-\alpha)} \dfrac{\partial^m u(\xi,\tau)}{\partial \tau^m} d\tau, \\ \mathcal{D}_{\zeta}^{\alpha} u(\xi,\zeta) = \dfrac{\partial^m u(\xi,\zeta)}{\partial \zeta^m}, \alpha = m \in \mathbb{N}. \end{cases} $$

Theorem 2.3. [3, 34] Let $\alpha_1, \alpha_2 \in \mathbb{R}$, such that $n-1 < \alpha_1 < n$, $m-1 < \alpha_2 < m$, $n \neq m$ for $n, m \in \mathbb{N}$. Then, in general

$$ \begin{cases} \mathcal{D}_{\zeta}^{\alpha_1} \mathcal{D}_{\zeta}^{\alpha_2} u(\xi,\zeta) = \mathcal{D}_{\zeta}^{\alpha_2} \mathcal{D}_{\zeta}^{\alpha_1} u(\xi,\zeta) = \mathcal{D}_{\zeta}^{\alpha_1+\alpha_2} u(\xi,\zeta), \\ \mathcal{D}_{\zeta}^{\alpha_1} \mathcal{D}_{\zeta}^{m} u(\xi,\zeta) \neq \mathcal{D}_{\zeta}^{m} \mathcal{D}_{\zeta}^{\alpha_1} u(\xi,\zeta). \end{cases} $$

Theorem 2.4. If $(\xi,\zeta_0) \in I$, $\zeta_0 > 0$ and $u(\xi,\zeta) \in C^{\infty}[I]$ with $\alpha \in (0,1]$, then

$$u(\xi,\zeta)=\sum_{k=0}^{\infty}\frac{1}{\alpha^k k!}\left[\mathcal{D}_{\zeta}^{k\alpha}u(\xi,\zeta)\right]_{(x,\zeta_0)}\left(\zeta-\zeta_0\right)^{k\alpha},$$

where, $I=\mathbb{R}\times(0,\infty)$ and $\mathcal{D}_{\zeta}^{k\alpha}$ denotes time-fractional partial derivative for k-times.

Definition 2.5. If $(\xi,\zeta_0)\in I$, $\zeta_0>0$ and $u(\xi,\zeta)\in C^{\infty}[I]$ with $\alpha\in(0,1]$, then the time-fractional partial differential transform (TFPDT) of $u(\xi,\zeta)$ is defined as:

$${}_{\zeta}\bar{U}_{\alpha}(\xi,k)=\frac{1}{\alpha^k k!}\left[\mathcal{D}_{\zeta}^{k\alpha}u(\xi,\zeta)\right]_{(\xi,\zeta_0)},\ \zeta_0>0,$$

where $I=\mathbb{R}\times(0,\infty)$.

2.1 *Conformable fractional partial derivative (FPD)*

Definition 2.6. Given a function $u(\xi,\zeta):\mathbb{R}\times(0,\infty)\to\mathbb{R}$. Then the conformable time-fractional partial derivative (CTFPD) of order α for a function $u(\xi,\zeta)$ is defined as:

$$\mathcal{T}_{\alpha}u(\xi,\zeta)=\lim_{\epsilon\to 0}\frac{u(\xi,\zeta+\epsilon\zeta^{1-\alpha})-u(\xi,\zeta)}{\epsilon},\tag{1}$$

for all $\zeta>0$, $\alpha\in(0,1]$.

Table 1 shows the function ϕ and $T_{\alpha}(\phi(\vartheta))$ when $\alpha\in(0,1]$ and $u(\xi,\zeta)$, $v(\xi,\zeta)$ be α-differentiable functions at $(\xi,\zeta)\in\mathbb{R}\times(0,\infty)$.

Table 1. Conformable derivative of function ϕ at $(\xi,\zeta)\in\mathbb{R}\times(0,\infty)$

No	Function ϕ	$T_{\alpha}(\phi(\vartheta))$
1	${}_{\zeta}\mathcal{T}_{\alpha}(au+bv)$	$a\,{}_{\zeta}\mathcal{T}_{\alpha}u+b\,{}_{\zeta}\mathcal{T}_{\alpha}v$
2	${}_{\zeta}\mathcal{T}_{\alpha}\left(\zeta^{p}\right)$	$p\zeta^{p-\alpha},\forall p\in\mathbb{R}$
3	$u(\xi,\zeta)=\lambda$	${}_{\zeta}\mathcal{T}_{\alpha}(\lambda)=0$
4	${}_{\zeta}\mathcal{T}_{\alpha}(uv)$	$u\,{}_{\zeta}\mathcal{T}_{\alpha}v+v\,{}_{\zeta}\mathcal{T}_{\alpha}u$
5	${}_{\zeta}\mathcal{T}_{\alpha}(u/v)$	$\frac{v({}_{\zeta}\mathcal{T}_{\alpha}u)-u({}_{\zeta}\mathcal{T}_{\alpha}v)}{v^2}$
6	${}_{\zeta}\mathcal{T}_{\alpha}u(\xi,\zeta)$	$\zeta^{1-\alpha}\frac{\partial u(\xi,\zeta)}{\partial\zeta}$
7	${}_{\zeta}\mathcal{T}_{\alpha}^{a}\left((\zeta-a)^{p}\right)$	$p(\zeta-a)^{p-\alpha},\forall\, p\in\mathbb{R}$
8	${}_{\zeta}\mathcal{T}_{\alpha}^{a}\left(e^{\lambda\left(\frac{(\zeta-a)^{\alpha}}{\alpha}+\xi\right)}\right)$	$\lambda e^{\lambda\left(\frac{(\zeta-a)^{\alpha}}{\alpha}+\xi\right)}$
9	${}_{\zeta}\mathcal{T}_{\alpha}^{a}\left(\frac{(\zeta-a)^{\alpha}}{\alpha}\right)$	1

Lemma 2.7. Let $u(\xi,\zeta)$ to be k-times differentiable at $(\xi,\zeta)\in\mathbb{R}\times(0,\infty)$, $\alpha\in(0,1]$ and define $v_0(\xi,\zeta)=u(\xi,\zeta)$. Then, the k-times CTFPD for a function $u(\xi,\zeta)$ at the point (ξ,ζ) can be represented as

$$v_k(\xi,\zeta)={}_{\zeta}\mathcal{T}_{k\alpha}v_{k-1}(\xi,\zeta)=\zeta^{1-\alpha}\frac{\partial^k v_{k-1}(\xi,\zeta)}{\partial\zeta^k}.$$

Definition 2.8. Let function $u:\mathbb{R}\times(0,\infty)\to\mathbb{R}$, $\alpha\in(n,n+1]$ and $\beta=\alpha-n$. Then, the CTFPD for u of order α, such

that $\frac{\partial^n u}{\partial \zeta^n}$ exists, is defined by

$$ {}_{\zeta}\mathcal{T}_{\alpha}^{a} u(\xi,\zeta) = {}_{\zeta}\mathcal{T}_{\beta}^{a} \frac{\partial^n u}{\partial \zeta^n}(\xi,\zeta). $$

Theorem 2.9. If $u(\xi,\zeta)$ is infinitely α-differentiable function, for $\alpha \in (0,1]$ at a point (ξ,ζ_0). Then,

$$ u(\xi,\zeta) = \sum_{k=0}^{\infty} \frac{1}{\alpha^k k!} \left[{}_{\zeta}\mathcal{T}_{k\alpha}^{\zeta_0} u(\xi,\zeta) \right]_{(\xi,\zeta_0)} \left(\zeta - \zeta_0\right)^{k\alpha} $$

for $0 < \zeta_0 \le \zeta \le \zeta_0 + R^{\frac{1}{\alpha}}$, $R > 0$, where ${}_{\zeta}\mathcal{T}_{k\alpha}^{\zeta_0}$ k-times CTFPD.

Definition 2.10. Let $0 < \alpha \le 1, u(\xi,\zeta) \in C^{\infty}[I]$ at a point (ξ,ζ_0). Then, the conformable time-fractional reduced differential transform (CTFRDT) of $u(\xi,\zeta)$ is defined as

$$ {}_{\zeta}U_{\alpha k}(\xi) = \frac{1}{\alpha^k k!} \left[{}_{\zeta}\mathcal{T}_{k\alpha}^{\zeta_0} u(\xi,\zeta) \right]_{(\xi,\zeta_0)}. $$

Definition 2.11. The CTFRDT of initial conditions for integer order derivative is defined as:

$$ {}_{\zeta}U_{k}^{\alpha}(\xi) = \begin{cases} \frac{1}{(\alpha k)!} \left[\frac{\partial^{k\alpha} u}{\partial \zeta^{k\alpha}} \right]_{(\xi,\zeta_0)} & \text{if } k\alpha \in \mathbb{Z}^{+}, \\ 0 & \text{if } k\alpha \notin \mathbb{Z}^{+}, \end{cases} $$

for $k = 0, 1, 2, \ldots, \left(\frac{n}{\alpha} - 1\right)$ where n is the order of conformable time-fractional partial differential equation.

Definition 2.12. Let ${}_{\zeta}U_{\alpha k}(\xi)$ be the CTFRDT of the function $u(\xi,\zeta)$. Then, inverse of ${}_{\zeta}U_{\alpha k}(\xi)$ is defined as

$$ \begin{aligned} u(\xi,\zeta) &= \sum_{k=0}^{\infty} {}_{\zeta}U_{\alpha k}(\xi) \left(\zeta - \zeta_0\right)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\alpha^k k!} \left[{}_{\zeta}\mathcal{T}_{k\alpha}^{\zeta_0} u(\xi,\zeta) \right]_{(\xi,\zeta_0)} \left(\zeta - \zeta_0\right)^{k\alpha}, \end{aligned} $$

where $k = 0, 1, 2, \ldots, n$ and $0 < \alpha \le 1$.

Definition 2.13. Let $(\xi,\zeta_0) \in I, \zeta_0 > 0$ and $u(\xi,\zeta) \in C^{\infty}[I]$ with at a point $\alpha \in (0,1]$. Then, the CTFRDT of $u(\xi,\zeta)$ is defined as

$$ {}_{\zeta}U_{\alpha k}(\xi) = \frac{1}{\alpha^k k!} \left[{}_{\zeta}\mathcal{T}_{k\alpha}^{\zeta_0} u(\xi,\zeta) \right]_{(\xi,\zeta_0)}. $$

Theorem 2.14. If $v(\xi,\zeta) = au(\xi,\zeta)$, then we have

$$ {}_{\zeta}V_{k}^{\alpha}(\xi) = a\,{}_{\zeta}U_{k}^{\alpha}(\xi), $$

where a is a constant.

Proof.

$$
\begin{aligned}
{}_{\zeta}V_k^{\alpha}(\xi) &= \frac{1}{\alpha^k k!}\Big[{}_{\zeta}\mathcal{T}_{k\alpha}^{\zeta_0} v(\xi,\zeta)\Big]_{(\xi,\zeta_0)} \\
&= \frac{1}{\alpha^k k!}\Big[{}_{\zeta}\mathcal{T}_{k\alpha}^{\zeta_0} au(\xi,\zeta)\Big]_{(\xi,\zeta_0)} \\
&= \frac{a}{\alpha^k k!}\Big[{}_{\zeta}\mathcal{T}_{k\alpha}^{\zeta_0} u(\xi,\zeta)\Big]_{(\xi,\zeta_0)} \\
&= a\,{}_{\zeta}U_k^{\alpha}(x).
\end{aligned}
$$

Theorem 2.15. If $u(\xi,\zeta) = v(\xi,\zeta)w(\xi,\zeta)$, then ${}_{\zeta}U_k^{\alpha}(\xi) = \sum_{r=0}^{k} {}_{\zeta}V_r^{\alpha}(\xi)\,{}_{\zeta}W_{k-r}^{\alpha}(\xi)$.

Proof. By using the Definition 2.12, $u(\xi,\zeta)$ and $v(\xi,\zeta)$ can be written as:

$$
\begin{aligned}
v(\xi,\zeta) &= \sum_{k=0}^{\infty} {}_{\zeta}V_k^{\alpha}(\xi)\big(\zeta-\zeta_0\big)^{k\alpha}, \\
w(\xi,\zeta) &= \sum_{k=0}^{\infty} {}_{\zeta}W_k^{\alpha}(\xi)\big(\zeta-\zeta_0\big)^{k\alpha}.
\end{aligned}
$$

Then, $u(\xi,\zeta)$ is as follows:

$$
\begin{aligned}
u(\xi,\zeta) &= \left(\sum_{k=0}^{\infty} {}_{\zeta}V_k^{\alpha}(\xi)\big(\zeta-\zeta_0\big)^{k\alpha}\right)\times\left(\sum_{k=0}^{\infty} {}_{\zeta}W_k^{\alpha}(\xi)\big(\zeta-\zeta_0\big)^{k\alpha}\right) \\
&= \Big({}_{\zeta}V_0^{\alpha}(\xi) + {}_{\zeta}V_1^{\alpha}(\xi)\big(\zeta-\zeta_0\big)^{\alpha} + {}_{\zeta}V_2^{\alpha}(\xi)\big(\zeta-\zeta_0\big)^{2\alpha} + \cdots\Big) \\
&\times\Big({}_{\zeta}W_0^{\alpha}(\xi) + {}_{\zeta}W_1^{\alpha}(\xi)\big(\zeta-\zeta_0\big) + {}_{\zeta}W_2^{\alpha}(\xi)\big(\zeta-\zeta_0\big)^{2\alpha} + \cdots\Big) \\
&= V_0^{\alpha}(\xi)\,{}_{\zeta}W_0^{\alpha}(\xi) + \Big({}_{\zeta}V_0^{\alpha}(\xi)\,{}_{\zeta}W_1^{\alpha}(x) + {}_{\zeta}V_1^{\alpha}(\xi)\,{}_{\zeta}W_0^{\alpha}(\xi)\Big)\big(\zeta-\zeta_0\big)^{\alpha} \\
&+\Big({}_{\zeta}V_0^{\alpha}(\xi)\,{}_{\zeta}W_2^{\alpha}(\xi) + {}_{\zeta}U_1^{\alpha}(\xi)\,{}_{\zeta}W_1^{\alpha}(\xi,1) + {}_{\zeta}V_2^{\alpha}(\xi)\,{}_{\zeta}W_0^{\alpha}(\xi)\Big)\big(\zeta-\zeta_0\big)^{2\alpha} + \cdots \\
&= \sum_{k=0}^{\infty}\sum_{r=0}^{k} {}_{\zeta}V_r^{\alpha}(\xi)\,{}_{\zeta}W_{k-r}^{\alpha}(\xi)\big(\zeta-\zeta_0\big)^{k\alpha},
\end{aligned}
$$

$$
{}_{\zeta}U_k^{\alpha}(\xi) = \sum_{r=0}^{k} {}_{\zeta}V_r^{\alpha}(\xi)\,{}_{\zeta}W_{k-r}^{\alpha}(\xi).
$$

In general, for ${}_{\zeta}U(\xi,\zeta) = v_1(\xi,\zeta)\cdot v_2(\xi,\zeta)\cdots v_n(\xi,\zeta)$, we have

$$
{}_{\zeta}U_k^{\alpha}(\xi) = \sum_{k_{n-1}=0}^{k_n}\cdots\sum_{k_1=0}^{k_2} {}_{\zeta}V_{k_1}^{1\alpha}(\xi)\,{}_{\zeta}V_{k_2-k_1}^{2\alpha}(\xi)\times\cdots{}_{\zeta}V_{k_{n-1}}^{(n-1)\alpha}(\xi)\,{}_{\zeta}V_{k_n-k_{n-1}}^{n\alpha}(\xi).
$$

Theorem 2.16. If $u(\xi,\zeta) = {}_{\zeta}\mathcal{T}_{\alpha}^{\zeta_0} v(\xi,\zeta)$, for $0<\beta\le 1$. Then, ${}_{\zeta}U_k^{\alpha}(\xi) = \alpha(k+1)\,{}_{\zeta}V_{k+1}^{\alpha}(x)$.

Proof. Assuming the following is $v(\xi,\zeta)$ is conformable time-fractional partial differential transform (CTFPDT):

$$
{}_{\zeta}V_k^{\alpha}(\xi) = \frac{1}{\alpha^k k!}\Big[{}_{\zeta}\mathcal{T}_{k\alpha}^{\zeta_0} v(\xi,\zeta)\Big]_{(\xi,\zeta_0)}.
$$

For $u(\xi,\zeta) = {}_{\zeta}\mathcal{T}_{\alpha}^{\zeta_0} v(\xi,\zeta)$, we have

$$
\begin{aligned}
{}_{\zeta}U_k^{\alpha}(\xi) &= \frac{1}{\alpha^k k!}\Big[{}_{\zeta}\mathcal{T}_{k\alpha}^{\zeta_0}\left({}_{\zeta}\mathcal{T}_{\alpha}^{\zeta_0}v(\xi,\zeta)\right)\Big]_{(x,\zeta_0)} \\
&= \frac{1}{\alpha^k k!}\Big[{}_{\zeta}\mathcal{T}_{(k+1)\alpha}^{\zeta_0}v(\xi,\zeta)\Big]_{(\xi,\zeta_0)} \\
&= \frac{\alpha(k+1)}{\alpha^{k+1}(k+1)!}\Big[{}_{\zeta}\mathcal{T}_{(k+1)\alpha}^{\zeta_0}v(\xi,\zeta)\Big]_{(\xi,\zeta_0)} \\
&= \alpha(k+1)V_{k+1}^{\alpha}(\xi).
\end{aligned}
$$

Theorem 2.17. If $u(\xi,\zeta) = {}_{\zeta}\mathcal{T}_{\beta}^{\zeta_0}v(\xi,\zeta)$ for $\beta \in (m-1,m]$. Then ${}_{\zeta}U_k^{\alpha}(\xi) = \dfrac{\Gamma(k\alpha+\beta+1)}{\Gamma(k\alpha+\beta-m)_{\zeta}}\ V_{k+\beta/\alpha}^{\alpha}(\xi)$.

Proof. We look for CTFRDT of function $u(\xi,\zeta)$. We have

$$
{}_{\zeta}\mathcal{T}_{\beta}^{\zeta_0}v(\xi,\zeta) = {}_{\zeta}\mathcal{T}_{\beta}^{\zeta_0}\left(v(\xi,\zeta) - \sum_{k=0}^{m-1}\frac{(\zeta-\zeta_0)^k}{k!}{}_{\zeta}\mathcal{T}_k^0 v(\xi,\zeta)\right).
$$

$$
u(\xi,\zeta) = {}_{\zeta}\mathcal{T}_{\beta}^{\zeta_0}\left(\sum_{k=0}^{\infty}{}_{\zeta}V_k^{\alpha}(\xi)(\zeta-\zeta_0)^{k\alpha} - \sum_{k=0}^{m-1}\frac{(\zeta-\zeta_0)^k}{k!}{}_{\zeta}\mathcal{T}_k^0 v(\xi,\zeta)\right).
$$

$$
u(\xi,\zeta) = {}_{\zeta}\mathcal{T}_{\beta}^{\zeta_0}\left(\sum_{k=0}^{\infty}{}_{\zeta}V_k^{\alpha}(\xi)(\zeta-\zeta_0)^{k\alpha} - \sum_{k=0}^{\frac{\beta}{\alpha}-1}\frac{(\zeta-\zeta_0)^{k\alpha}}{k\alpha!}{}_{\zeta}\mathcal{T}_{k\alpha}^0 v(\xi,\zeta)\right)
$$

$$
= {}_{\zeta}\mathcal{T}_{\beta}^{t_0}\left(\sum_{k=0}^{\infty}{}_{\zeta}V_k^{\alpha}(\xi)(\zeta-\zeta_0)^{k\alpha} - \sum_{k=0}^{\frac{\beta}{\alpha}-1}{}_{\zeta}V_k^{\alpha}(\xi)(\zeta-\zeta_0)^{k\alpha}\right)
$$

$$
= {}_{\zeta}\mathcal{T}_{\beta}^{\zeta_0}\left(\sum_{k=\frac{\beta}{\alpha}}{}_{\zeta}V_k^{\alpha}(\xi)(\zeta-\zeta_0)^{k\alpha}\right)
$$

$$
= \sum_{k=\frac{\beta}{\alpha}}^{\infty}\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha-m)}{}_{\zeta}V_k^{\alpha}(\xi)(\zeta-\zeta_0)^{k\alpha-\beta}
$$

$$
= \sum_{k=0}^{\infty}\frac{\Gamma(k\alpha+\beta+1)}{\Gamma(k\alpha+\beta-m)}{}_{\zeta}V_{k+\frac{\beta}{\alpha}}^{\alpha}(\xi)(\zeta-\zeta_0)^{k\alpha},
$$

$$
{}_{\zeta}U_k^{\alpha}(\xi) = \frac{\Gamma(k\alpha+\beta+1)}{\Gamma(k\alpha+\beta-m)}{}_{\zeta}V_{k+\beta/\alpha}^{\alpha}(\xi).
$$

Theorem 2.18. If $u(\xi,\zeta) = (\zeta-\zeta_0)^p\, g(\xi)$, then ${}_{\zeta}U_k^{\alpha}(\xi) = \delta\left(k-\dfrac{p}{\alpha}\right)g(\xi)$, where

$$
\delta(k) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}
$$

3. Modified conformable fractional reduced differential transform

Consider the following system of FPDE:

$$
\mathcal{D}^{\beta}u_j(\xi,\tau) + \mathcal{L}u_j(\xi,\tau) + \mathcal{N}u_j(\xi,\tau) = r_j(\xi,\tau),\ \xi,\tau \geq 0,\ m-1 < \beta \leq m,\ j = 1,2. \tag{2}
$$

The conformable sense is used in the expression of Equation (2). $\mathcal{L}$ and $\mathcal{N}$ stand for the linear and nonlinear terms, respectively and $r_j(\xi,\tau)$ is the remaining terms with the initial condition.

$$u_{j,\tau}^{(k)}(\xi,0) = f_{j,k}(\xi), \quad k = 0,1,2,\ldots,m-1. \tag{3}$$

Applying the conformable reduced differential transform on both sides of Equation (2), we get

$$\frac{\Gamma(k\alpha+\beta+1)}{\Gamma(k\alpha+\beta-m)}\,{}_\tau U_{j,k+\beta/\alpha}^{\alpha}(\xi) = G_{j,k}^{\alpha}(\xi) - [\mathcal{L}_\tau U_{j,k}^{\alpha}(\xi) + \mathcal{N}_\tau U_{j,k}^{\alpha}(\xi)]. \tag{4}$$

Where ${}_\tau U_{j,k}^{\alpha}(\xi), \mathcal{N}_\tau U_{j,k}^{\alpha}(\xi)$ and $G_{j,k}^{\alpha}(\xi)$ are transformation of the functions $\mathcal{L}u_j(\xi,\tau)$, $\mathcal{N}u_j(\xi,\tau)$ and $r_j(\xi,\tau)$ respectively.

By representing the solution as an infinite series given by in the second phase of the conformable reduced differential decomposition approach, we can:

$$u_j(\xi,\tau) = \sum_{k=0}^{\infty} {}_\tau U_{j,k}(\xi). \tag{5}$$

In the problem, the nonlinear term is given as

$$\mathcal{N}u_j(\xi,\tau) = \sum_{k=0}^{\infty} A_{j,k}, \tag{6}$$

where

$$A_{j,k} = \frac{1}{k!}\left[\frac{d^k}{d\lambda}\left[\mathcal{N}\sum_{k=0}^{\infty}\left(\lambda^k {}_\tau U_{j,k}\right)\right]\right]_{\lambda=0}, \; j = 1,2 \tag{7}$$

known as Adomian polynomials. By replacing Equation (4) with Equations (5) and (6), we obtain

$$\sum_{k=0}^{\infty} {}_\tau U_{j,k}(\xi) = G_{j,k}^{\alpha}(x) - \left(\left[\mathcal{L}\sum_{k=0}^{\infty} {}_\tau U_{j,k}(\xi) + \sum_{k=0}^{\infty} A_{j,k}\right]\right).$$

Reduced differential decomposition method, we get

$${}_\tau U_{j,0}(\xi) = G_{j,k}^{a}(\xi),$$

$${}_\tau U_{j,k+1}(\xi) = \left(\left[\mathcal{L}\sum_{k=0}^{\infty} {}_\tau U_{j,k}(\xi) + \sum_{k=0}^{\infty} A_{j,k}\right]\right), \quad k \geq 0. \tag{8}$$

Then, the inverse transformation of the set of values $\{{}_\tau U_{j,k}^{\alpha}(\xi)\}_{k=0}^{n}$ gives the n-term approximation solution as follows

$$u_n(\xi,\tau) = \sum_{k=0}^{n} {}_\tau U_{j,k}^{\alpha}(\xi)\tau^{k\alpha}.$$

Consequently, the precise solution to the problem is provided by

$$u(\xi,\tau) = \lim_{n\to\infty} u_n(\xi,\tau).$$

3.1 *Analysis of convergence and error estimate*

In this section, we analyze convergence and error estimates for this new technique in approximately solved problems. Particularly in close proximity to convergence, when nonlinear systems exhibit linear behavior and error estimation is most required.

Theorem 3.1. If $\mathcal{B}$ be a Banach space, then the series solution of the system (2) converges to $S_j \in \mathcal{B}$, for $j = 1\ldots, n$ if $\exists \sigma \in [0,1)$ such that, $\|U_{jm}\| \le \sigma \|U_{j(m-1)}\| \forall\, m \in \mathbb{N}$.

Proof. Let the sequences S_{jm}, be a partial sum of the series given by the system (8) as

$$\begin{cases} S_{j0} = U_{j0}(\xi) \\ S_{j1} = U_{j0}(\xi) + U_{j1}(\xi) \\ S_{j2} = U_{j0}(\xi) + U_{j1}(\xi) + U_{j2}(\xi) \\ \vdots \\ S_{jm} = U_{j0}(\xi) + U_{j1}(\xi) + U_{j2}(\xi) + \cdots + U_{jm}(\xi), \end{cases} \tag{9}$$

then we must prove that in Banach space $\mathcal{B}$, $\{S_{jm}\}_{m=0}^{\infty}$ are Cauchy sequences. We examine the following factors in this regard:

$$\|S_{j(m+1)} - S_{jm}\| = \|U_{j(m+1)}(\xi)\| \le \sigma \|U_{jm}(\xi)\| \le \sigma^2 \|U_{j(m-1)}(\xi)\| \le \cdots \le \sigma^{m+1} \|U_{j0}(\xi)\|. \tag{10}$$

For every $m, r \in \mathbb{N}$, $m \ge r$, by using the system (9) and triangle inequality successively, we have

$$\begin{aligned} \|S_{jm} - S_{jr}\| &= \|S_{jm} - S_{j(m-1)} + S_{j(m-1)} - S_{j(m-2)} + \cdots + S_{j(r+1)} - S_{jr}\| \\ &\le \|S_{jm} - S_{j(m-1)}\| + \|S_{j(m-1)} - S_{j(m-2)}\| + \cdots + \|S_{j(r+1)} - S_{jr}\| \\ &\le \sigma^m \|U_{j0}(\xi)\| + \sigma^{m-1} \|U_{j0}(\xi)\| + \cdots + \sigma^{r+1} \|U_{j0}(\xi)\| \\ &= \sigma^{r+1} \left(1 + \sigma + \cdots + \sigma^{m-r-1}\right) \|U_{j0}(\xi)\| \\ &\le \sigma^{r+1} \left(\frac{1-\sigma^{m-r}}{1-\sigma}\right) \|U_{j0}(\xi)\|. \end{aligned}$$

As $0 < \sigma < 1$, so $1 - \sigma^{m-r} \le 1$ then

$$\|S_{jm} - S_{jr}\| \le \frac{\sigma^{r+1}}{1-\sigma} \|U_{j0}(\xi)\|.$$

for $j = 1, 2, \ldots, n$. Since $U_{j0}(\xi)$ is bounded then

$$\lim_{m,r \to \infty} \|S_{jm} - S_{jr}\| = 0, \quad j = 1, 2, \ldots, n.$$

As a result, the sequences $\{S_{jm}\}_{m=0}^{\infty}$ in the Banach space, $\mathcal{B}$ are Cauchy sequences, and the series solution specified in system (5) converges.

Theorem 3.2. The series solution (5) of the nonlinear fractional differential system (2) is determined to have a maximum absolute truncation error of (5).

$$\sup_{(\xi,\tau) \in \Theta} \left| u_j(\xi, \tau) - \sum_{k=0}^{r} U_{jk}(\xi) \right| \le \frac{\sigma^{r+1}}{1-\sigma} \sup_{\xi \in \Theta} |U_{j0}(\xi)|, \quad j = 1, 2, \ldots, n.$$

where the region $\Theta \subset \mathbb{R}^{n+1}$.

Proof. We deduce the following from Theorem 3.1:

$$\left\| S_{jm} - \ S_{jr} \right\| \leq \frac{\sigma^{r+1}}{1-\sigma} \sup_{(\xi)\in\Theta} \left| U_{j0}(\xi) \right|. \tag{11}$$

However, we suppose that $S_{jm} = \sum_{k=0}^{n} U_{jk}(\xi)$ and since $m \to \infty$, we obtain $S_{jm} \to u_j(\xi,\tau)$, so the system (11) can be rephrased as

$$\left\| u_j(\xi,\tau) - S_{jr} \right\| = \left\| u_j(\xi,\tau) - \sum_{k=0}^{r} U_{jk}(\xi) \right\| \leq \frac{\sigma^{r+1}}{1-\sigma} \sup_{\xi\in\Theta} \left| U_{j0}(\xi) \right|, \ j = 1,2,\ldots,n.$$

As a result, in the Θ region, the maximum absolute truncation error is

$$\sup_{(\xi,\tau)\in\Theta} \left| u_j(\xi,\tau) - \sum_{k=0}^{r} U_{jk}(\xi) \right| \leq \frac{\sigma^{r+1}}{1-\sigma} \sup_{(\xi)\in\Theta} \left| U_{j0}(\xi) \right|, j = 1,2,\ldots,n$$

and this completes the proof.

4. Solution of the one and two-dimensional partial fractional differential equations (PFDEs)

Example 4.1. Considering the following fractional damped Burgers' equation

$${}_{\varsigma}\mathcal{T}_{\alpha}^{\zeta_0} u + u u_{\xi} + u_{\xi\xi} + \frac{1}{5} u = 0, \quad 0 < \alpha \leq 1, \tag{12}$$

subject to the initial conditions

$$u(\xi,0) = \frac{1}{5}\xi. \tag{13}$$

Now, apply the conformable fractional reduced differential transform method (CFRDTM) can be expressed as follows:

$${}_{\varsigma}U_{k+1}^{\alpha}(\xi) = \frac{1}{\alpha(k+1)} \left[-A_k - \frac{\partial^2 {}_{\varsigma}U_k^{\alpha}}{\partial \xi^2} - \frac{1}{5} {}_{\varsigma}U_k^{\alpha} \right], \tag{14}$$

${}_{\varsigma}U_k(\xi)$ are the transformed functions. $A_k(\xi)$ is transformed form of the nonlinear terms are as follows:

$$A_0 = {}_{\varsigma}U_0 \frac{\partial}{\partial \xi} {}_{\varsigma}U_0,$$

$$A_1 = {}_{\varsigma}U_1 \frac{\partial}{\partial \xi} {}_{\varsigma}U_0 + {}_{\varsigma}U_0 \frac{\partial}{\partial \xi} {}_{\varsigma}U_1,$$

$$A_2 = {}_{\varsigma}U_2 \frac{\partial}{\partial \xi} {}_{\varsigma}U_0 + {}_{\varsigma}U_1 \frac{\partial}{\partial \xi} {}_{\varsigma}U_1 + {}_{\varsigma}U_0 \frac{\partial}{\partial \xi} {}_{\varsigma}U_2,$$

$$A_3 = {}_\zeta U_3 \frac{\partial}{\partial \xi} {}_\zeta U_0 + {}_\zeta U_2 \frac{\partial}{\partial \xi} {}_\zeta U_1 + {}_\zeta U_1 \frac{\partial}{\partial \xi} {}_\zeta U_2 + {}_\zeta U_0 \frac{\partial}{\partial \xi} {}_\zeta U_3,$$

$$\vdots$$

from initial condition

$${}_\zeta U_0^\alpha = \frac{1}{5}\xi. \tag{15}$$

Now, substituting (15) with (14) respectively, we obtain

$${}_\zeta U_1^\alpha(\xi) = \frac{-2}{25\alpha}\xi,\ {}_\zeta U_2^\alpha(\xi) = \frac{3}{125\alpha^2}\xi,\ {}_\zeta U_3^\alpha(\xi) = \frac{13}{18735\alpha^3}\xi, \cdots$$

Taking the inverse transformation of the set of values $\left\{U_k^\alpha(\xi)\right\}_{k=0}^4$ gives the fourth-term approximation solutions as follows

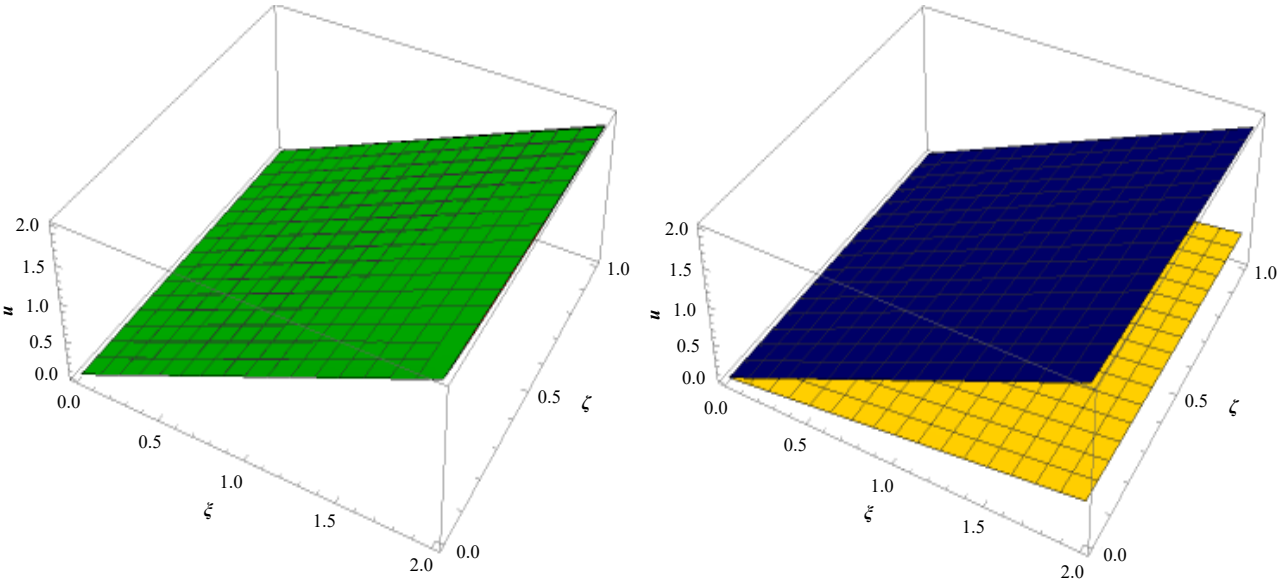
$$\begin{aligned}\tilde{u}_4(\xi,\zeta) &= \sum_{k=0}^{4} {}_\zeta U_k^\alpha(\xi)\zeta^{k\alpha} = {}_\zeta U_0^\alpha(\xi) + {}_\zeta U_1^\alpha(\xi)\zeta^\alpha + {}_\zeta U_2^\alpha(\xi)\zeta^{2\alpha} + vU_3^\alpha(\xi)\zeta^{3\alpha} \\ &= \frac{1}{5}\xi + \frac{-2}{25\alpha}\xi\zeta^\alpha + \frac{3}{125\alpha^2}\xi\zeta^{2\alpha} - \frac{13}{18735\alpha^3}\xi\zeta^{3\alpha}.\end{aligned}$$

The error between the exact and fourth-order approximation solution u of system by Laplace reduced differential transform method (LRDTM) is shown in Table 2, when $\alpha = 1, \zeta = 0.05$ and $0.01, 0 \le \xi \le 2$.

Table 2. The error between the exact and fourth-order approximation solution u of system by LRDTM for $\alpha = 1, \zeta = 0.05$ and $0.01, 0 \le \xi \le 2$

ξ	ζ	Exact	Approximate $\alpha = 1$	Approximate $\alpha = 0.9$	Absolute error u for $\alpha = 1$ $\lvert u_{exact} - u_{appr} \rvert$
0	0.05	0	0	0	0
0.5		0.09802957	0.10202996	0.10306569	0.00400038
1		0.20405991	0.20405991	0.20613138	0.00800077
1.5		0.29408872	0.30608987	0.30919707	0.01200115
2		0.39211829	0.40811983	0.41226275	0.01600154
0	0.01	0	0	0	0
0.5		0.0996012	0.1004012	0.10070812	0.0008
1		0.19920239	0.2008024	0.20141623	0.00160001
1.5		0.29880359	0.3012036	0.30212435	0.00240001
2		0.39840479	0.4016048	0.40283247	0.00320001

Figure 1 shows the approximate and exact solutions of $u(\xi,\zeta)$ for different α values for Example 4.1.



(a) For $\alpha = 0.7$ and 0.9 approximate solutions (red, green colored of surfaces, respectively) of $u(\xi,\zeta)$

(b) For $\alpha = 1$ approximate and exact solutions (blue, yellow-colored surfaces, respectively) of $u(\xi,\zeta)$

Figure 1. 4th-order of approximation solutions for different values of α and exact solution of $u(\xi,\zeta)$ for Example 4.1

Example 4.2. Considering the following nonlinear fractional partial differential equation

$$ {}_{\zeta}\mathcal{T}_{\alpha}^{\zeta_0} u - u u_{\xi\xi} - u_{\xi}^2 - u = 0, \quad 0 < \alpha \le 1, \tag{16} $$

with initial condition

$$ u(\xi,0) = \sqrt{\xi}. \tag{17} $$

Now, apply the CFRDTM can be expressed as follows:

$$ {}_{\zeta}U_{k+1}^{\alpha}(\xi) = \frac{1}{\alpha(k+1)}\left[A_k + B_k + {}_{\zeta}U_k^{\alpha}, \right] \tag{18} $$

${}_{\zeta}U_k(\xi)$ are the transformed functions. $A_k(\xi)$ and $B_k(\xi)$ are transformed form of the nonlinear terms, then the first few nonlinear terms are as follows

$$ \begin{vmatrix} A_0(u,u) = {}_{\zeta}U_0 \dfrac{\partial^2}{\partial\xi^2}{}_{\zeta}U_0, \\ A_1(u,u) = {}_{\zeta}U_1 \dfrac{\partial^2}{\partial\xi^2}{}_{\zeta}U_0 + {}_{\zeta}U_0 \dfrac{\partial^2}{\partial\xi^2}{}_{\zeta}U_1, \\ A_2(u,u) = {}_{\zeta}U_2 \dfrac{\partial^2}{\partial\xi^2}{}_{\zeta}U_0 + {}_{\zeta}U_1 \dfrac{\partial^2}{\partial\xi^2}{}_{\zeta}U_1 + {}_{\zeta}U_0 \dfrac{\partial^2}{\partial\xi^2}{}_{\zeta}U_2, \\ \vdots \end{vmatrix} \quad \begin{vmatrix} B_0(u,v) = \left(\dfrac{\partial}{\partial\xi}{}_{\zeta}U_0 \right)^2, \\ B_1(u,v) = 2\dfrac{\partial}{\partial\xi}{}_{\zeta}U_0 \dfrac{\partial}{\partial\xi}{}_{\zeta}U_1, \\ B_2(u,v) = 2\dfrac{\partial}{\partial\xi}{}_{\zeta}U_0 \dfrac{\partial}{\partial\xi}{}_{\zeta}U_1 + \left(\dfrac{\partial}{\partial\xi}{}_{\zeta}U_0 \right)^2, \\ \vdots \end{vmatrix} $$

from initial condition, ${}_{\zeta}U_0(\xi)=\sqrt{\xi}$.

Now, substituting (17) with (18) respectively, we obtain

$${}_{\zeta}U_1^{\alpha}(\xi)=\frac{\sqrt{\xi}}{\alpha},\ {}_{\zeta}U_2^{\alpha}(\xi)=\frac{\sqrt{\xi}}{2\alpha^2},\ {}_{\zeta}U_3^{\alpha}(\xi)=\frac{\sqrt{\xi}}{3!\alpha^3},\cdots$$

Taking the inverse transformation of the set of values $\left\{{}_{\zeta}U_k^{\alpha}(\xi)\right\}_{k=0}^{4}$ gives the fourth-term approximation solutions as follows

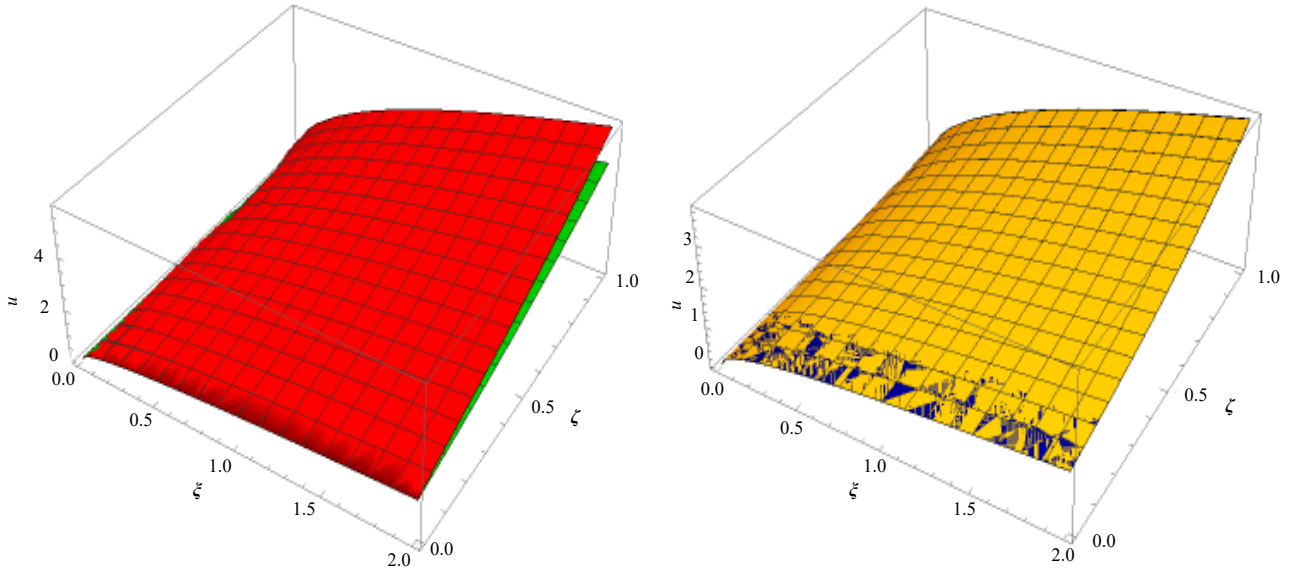
$$\begin{aligned}\tilde{u}_4(\xi,\zeta)&=\sum_{k=0}^{4}{}_{\zeta}U_k^{\alpha}(\xi)\zeta^{k\alpha}={}_{\zeta}U_0^{\alpha}(\xi)+{}_{\zeta}U_1^{\alpha}(\xi)\zeta^{\alpha}+{}_{\zeta}U_2^{\alpha}(\xi)\zeta^{2\alpha}+vU_3^{\alpha}(\xi)\zeta^{3\alpha}\\&=\sqrt{\xi}+\frac{\sqrt{\xi}}{\alpha}\zeta^{\alpha}+\frac{\sqrt{\xi}}{2\alpha^2}\zeta^{2\alpha}+\frac{\sqrt{\xi}}{3!\alpha^3}\zeta^{3\alpha}.\end{aligned}$$

The error between the exact and fourth-order approximation solution u of system by LRDTM is shown in Table 3, when $\alpha=1$, $\zeta=0.05$ and 0.01, $0\le\xi\le 2$.

Table 3. The error between the exact and fourth-order approximation solution u of system by LRDTM for $\alpha=1$, $\zeta=0.05$ and 0.01, $0\le\xi\le 2$

ξ	ζ	Exact	Approximate $\alpha=1$	Approximate $\alpha=0.9$	Absolute error u for $\alpha=1$ $\lvert u_{exact}-u_{appr}\rvert$
0	0.05	0	0	0	0
0.5		0.74336092	0.74336092	0.76214881	1.85687932e-09
1		1.0512711	1.05127109	1.07784119	2.62602406e-09
1.5		1.28753888	1.28753888	1.32008047	3.21620930e-09
2		1.48672184	1.48672184	1.52429762	3.71375863e-09
0	0.01	0	0	0	0
0.5		0.71421332	0.71421332	0.71966916	5.89972515e-13
1		1.01005017	1.01005017	1.01776589	8.34443625e-13
1.5		1.23705376	1.23705376	1.24650356	1.02207132e-12
2		1.42842664	1.42842664	1.43933833	1.17994503e-12

Figure 2 shows the approximate and exact solutions of $u(\xi,\zeta)$ for different α values for Example 4.2.



(a) For $\alpha = 0.7$ and 0.9 approximate solutions (red, green colored surfaces, respectively) of $u(\xi, \zeta)$

(b) For $\alpha = 1$, approximate and exact solutions (blue, yellow colored surfaces, respectively) of $u(\xi,\zeta)$

Figure 2. 4th-order of approximation solutions for different value of α and exact solution $u(\xi,\zeta)$ of Example 4.2

Example 4.3. Consider the system of fractional-order partial differential equations (PDEs)

$$\begin{aligned} &\frac{\partial^{\beta} u}{\partial \zeta^{\beta}} + v\frac{\partial u}{\partial \xi} + u = 1, \\ &\frac{\partial^{\beta} v}{\partial \zeta^{\beta}} - u\frac{\partial v}{\partial \xi} - v = 1, \quad 0 < \beta \le 1, \end{aligned} \tag{19}$$

with the initial conditions

$$u(\xi,0) = e^{\xi}, \quad v(\xi,0) = e^{-\xi}. \tag{20}$$

Now apply the CFRDTM can be expressed as follows:

$$\begin{aligned} {}_{\zeta}U^{\alpha}_{k+1}(\xi) &= \frac{1}{\alpha(k+1)}\left[\delta(k) + A_k + {}_{\zeta}U^{\alpha}_k\right]. \\ {}_{\zeta}V^{\alpha}_{k+1}(\xi) &= \frac{1}{\alpha(k+1)}\left[\delta(k) - B_k - {}_{\zeta}V^{\alpha}_k\right]. \end{aligned} \tag{21}$$

${}_{\zeta}U_k(\xi)$ are the transformed functions. $A_k(\xi)$ and $B_k(\xi)$ are transformed form of the nonlinear terms. For the convenience of the reader, the first few nonlinear terms are as follows

$$\begin{vmatrix} A_0(v,u) = {}_{\zeta}V_0\dfrac{\partial_{\zeta}U_0}{\partial\xi}, & B_0(u,v) = {}_{\zeta}U_0\dfrac{\partial_{\zeta}V_0}{\partial\xi}, \\ A_1(v,u) = {}_{\zeta}V_0\dfrac{\partial_{\zeta}U_1}{\partial\xi} + {}_{\zeta}V_1\dfrac{\partial_{\zeta}U_0}{\partial\xi}, & B_1(u,v) = {}_{\zeta}U_0\dfrac{\partial_{\zeta}V_1}{\partial\xi} + {}_{\zeta}U_1\dfrac{\partial_{\zeta}V_0}{\partial\xi}, \end{vmatrix}$$

$$\left| A_2(v,u) = {}_\varsigma V_0 \frac{\partial_\varsigma U_2}{\partial \xi} + {}_\varsigma V_1 \frac{\partial_\varsigma U_1}{\partial \xi} + {}_\varsigma V_2 \frac{\partial_\varsigma U_0}{\partial \xi}, \right| \left| B_2(u,v) = {}_\varsigma U_0 \frac{\partial_\varsigma V_2}{\partial \xi} + {}_\varsigma U_1 \frac{\partial_\varsigma V_1}{\partial \xi} + {}_\varsigma U_2 \frac{\partial_\varsigma V_0}{\partial \xi}, \right|$$
$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

from initial conditions, ${}_\varsigma U_0(\xi) = e^{\xi}, {}_\varsigma V_0(\xi) = e^{-\xi}$.
Now, substituting (20) with (21) respectively, we obtain

$${}_\varsigma U_1^\alpha(\xi) = \frac{-e^{\xi}}{\alpha},\ {}_\varsigma U_2^\alpha(\xi) = \frac{e^{\xi}}{2\alpha^2},\ {}_\varsigma U_3^\alpha(\xi) = \frac{-e^{\xi}}{3!\alpha^3}, \cdots$$
$${}_\varsigma V_1^\alpha(x) = \frac{e^{-\xi}}{\alpha},\ {}_\varsigma V_2^\alpha(\xi) = \frac{e^{-\xi}}{2\alpha^2},\ {}_\varsigma V_3^\alpha(\xi) = \frac{e^{-\xi}}{3!\alpha^3}, \cdots$$

Now, taking inverse transformation of the set of values $\left\{U_k^\alpha(\xi)\right\}_{k=0}^4$ and $\left\{U_k^\alpha(x)\right\}_{k=0}^4$ gives 4-terms approximation solutions as follows

$$\tilde{u}_4(\xi,\zeta) = \sum_{k=0}^{4} {}_\varsigma U_k^\alpha(\xi)\zeta^{k\alpha} = {}_\varsigma U_0^\alpha(\xi) + {}_\varsigma U_1^\alpha(\xi)\zeta^\alpha + {}_\varsigma U_2^\alpha(\xi)\zeta^{2\alpha} + vU_3^\alpha(x)\zeta^{3\alpha}$$
$$= e^{\xi} - \frac{e^{\xi}}{\alpha}\zeta^\alpha + \frac{e^{\xi}}{2!\alpha^2}\zeta^{2\alpha} - \frac{e^{\xi}}{3!\alpha^3}\zeta^{3\alpha} + \frac{e^{\xi}}{4!\alpha^4}\zeta^{4\alpha}.$$

$$\tilde{v}_4(\xi,\zeta) = \sum_{k=0}^{4} {}_\varsigma V_k^\alpha(\xi)\zeta^{k\alpha} = {}_\varsigma V_0^\alpha(\xi) + {}_\varsigma V_1^\alpha(\xi)\zeta^\alpha + {}_\varsigma V_2^\alpha(\xi)\zeta^{2\alpha} + {}_\varsigma V_3^\alpha(\xi)\zeta^{3\alpha}$$
$$= e^{-\xi} + \frac{e^{-\xi}}{\alpha}\zeta^\alpha + \frac{e^{-\xi}}{2!\alpha^2}\zeta^{2\alpha} + \frac{e^{-\xi}}{3!\alpha^3}\zeta^{3\alpha} + \frac{e^{-\xi}}{4!\alpha^4}\zeta^{4\alpha}.$$

The error between the exact and fourth-order approximation solution u of system by LRDTM is shown in Table 4, when $\alpha = 1, \zeta = 0.01$ and $0.05, -2 \le \xi \le 2$.

Table 4. The error between the exact and fourth order approximation solution u of system by LRDTM for $\alpha = 1$, $\zeta = 0.01$ and $0.05, -2 \le \xi \le 2$

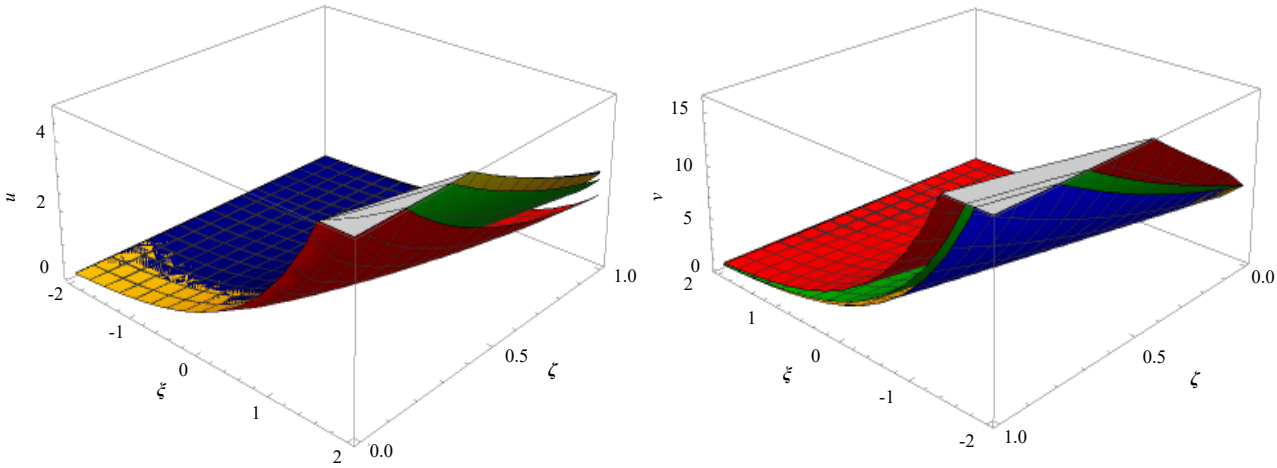
ξ	ζ	Exact	Approximate $\alpha = 1$	Approximate $\alpha = 0.9$	Absolute error u for $\alpha = 1$ $\lvert u_{exact} - u_{appr} \rvert$
-2	0.01	0.13398867	0.1329729	0.1261997	1.12548859e-13
-1		0.36421898	0.36421898	0.36145782	3.06032977e-13
0		0.99004983	0.99004983	0.98254422	8.31890112e-13
1		2.69123447	2.69123447	2.67083211	2.26174635e-12
2		7.31553376	7.31553376	7.26007439	6.14708284e-12
-2	0.05	0.13398867	0.1287349	0.12556143	3.49519469e-10
-1		0.36421898	0.34993775	0.34131136	9.50092505e-10
0		0.99004983	0.95122942	0.92778047	2.58261923e-09
1		2.69123447	2.58570966	2.52196879	7.02028702e-09
2		7.31553376	7.02868758	6.85542194	1.90831191e-08

The error between the exact and fourt-order approximation solution v of system by CFRDTM is shown in Table 5, when $\alpha = 1, \zeta = 0.01$ and $0.05, -2 \le \xi \le 2$.

Table 5. The error between the exact and fourth-order approximation solution v of system by CFRDTM for $\alpha = 1$, $\zeta = 0.05$ and $0.01, -2 \le \xi \le 2$

ξ	ζ	Exact	Approximate $\alpha = 1$	Approximate $\alpha = 0.9$	Absolute error u for $\alpha = 1$ $\lvert u_{exact} - u_{appr} \rvert$
-2	0.01	7.46331735	7.46331735	7.52032928	6.16484641e-12
-1		2.74560102	2.74560102	2.76657453	2.26840768e-12
0		1.01005017	1.01005017	1.01776589	8.34443625e-13
1		0.37157669	0.37157669	0.37441515	3.07032177e-13
2		0.13669543	0.13669543	0.13773964	1.12965193e-13
-2	0.05	7.76790111	7.964229	8.80623461	1.94038376e-08
-1		2.85765111	2.92987611	3.23963267	7.13827353e-09
0		1.05127109	1.07784119	1.19179425	2.62602406e-09
1		0.38674102	0.39651561	0.4384366	9.66060232e-10
2		0.14227407	0.14586994	0.16129181	3.55393714e-10

Figure 3 shows the approximate and exact solutions of $u(\xi,\zeta)$ and $v(\xi,\zeta)$ for different α values for Example 4.3.



(a) For $\alpha = 0.7$, 0.9 and 1 approximate and exact solution (red, green, blue and yellow-colored surfaces, respectively) of $u(\xi,\zeta)$

(b) For $\alpha = 0.7$, 0.9 and 1 approximate and exact solution (red, green, blue and yellow-colored surfaces, respectively) of $v(\xi,\zeta)$

Figure 3. 4th-order of approximation solutions for different value of α and exact solution $u(\xi,\zeta)$ and $v(\xi,\zeta)$ of Example 4.3

Example 4.4. Consider the following system of nonlinear FPDEs.

$$T_\alpha^\zeta u - u_{\xi\xi\xi} - 2uu_\xi + (uv)_\xi = 0,$$
$$T_\alpha^\zeta u - v_{\xi\xi\xi} - 2vv_\xi + (uv)_\xi = 0, \quad 0 < \alpha \le 1, \tag{22}$$

subject to the initial conditions

$$u(\xi,0) = \sin\xi, \quad v(\xi,0) = \cos\xi. \tag{23}$$

Now, the CFRDTM can be expressed as follows:

$$\begin{aligned} {}_\zeta U_{k+1}^\alpha(\xi) &= \frac{1}{\alpha(k+1)}\left[\frac{\partial^2 {}_\zeta U_k^\alpha}{\partial\xi^2} + A_k - B_k - C_k\right] \\ {}_\zeta V_{k+1}^\alpha(x) &= \frac{1}{\alpha(k+1)}\left[\frac{\partial^2 {}_\zeta U_k^\alpha}{\partial\xi^2} + D_k - B_k - C_k\right] \end{aligned} \tag{24}$$

${}_\zeta U_k(\xi)$ are the transformed functions. $A_k(\xi)$, $B_k(\xi)$, $C_k(\xi)$ and $D_k(\xi)$ are transformed form of the nonlinear terms. For the convenience of the reader, the first few nonlinear terms are as follows

$$\begin{aligned}
&A_0(u,u) = 2\,{}_\varsigma U_0 \frac{\partial_\varsigma U_0}{\partial\xi}, \\
&A_1(u,u) = 2\left({}_\varsigma U_0 \frac{\partial_\varsigma U_1}{\partial\xi} + {}_\varsigma U_1 \frac{\partial u_0}{\partial\xi}\right), \\
&A_2(u,u) = 2\left({}_\varsigma U_0 \frac{\partial_\varsigma U_2}{\partial\xi} + {}_\varsigma U_1 \frac{\partial_\varsigma U_1}{\partial\xi} + {}_\varsigma U_2 \frac{\partial_\varsigma U_0}{\partial\xi}\right), \\
&A_3(u,u) = 2\left({}_\varsigma U_0 \frac{\partial_\varsigma U_3}{\partial\xi} + {}_\varsigma U_1 \frac{\partial_\varsigma U_2}{\partial\xi} + {}_\varsigma U_2 \frac{\partial_\varsigma U_1}{\partial\xi} + {}_\varsigma U_3 \frac{\partial_\varsigma U_0}{\partial\xi}\right), \\
&\vdots \\
&C_0(u,v) = \frac{\partial_\varsigma U_0}{\partial\xi}\,{}_\varsigma V_0, \\
&C_1(u,v) = \frac{\partial_\varsigma U_0}{\partial\xi}\,{}_\varsigma V_1 + \frac{\partial_\varsigma U_1}{\partial\xi}\,{}_\varsigma V_0, \\
&C_2(u,v) = \frac{\partial_\varsigma U_3}{\partial\xi}\,{}_\varsigma V_2 + \frac{\partial_\varsigma U_1}{\partial\xi}\,{}_\varsigma V_1 + \frac{\partial_\varsigma U_2}{\partial\xi}\,{}_\varsigma V_0, \\
&C_3(u,v) = \frac{\partial_\varsigma U_0}{\partial\xi}\,{}_\varsigma V_3 + \frac{\partial_\varsigma U_1}{\partial\xi}\,{}_\varsigma V_2 + \frac{\partial_\varsigma U_2}{\partial\xi}\,{}_\varsigma V_1 + \frac{\partial_\varsigma U_3}{\partial\xi}\,{}_\varsigma V_0, \\
&\vdots
\end{aligned}
\qquad
\begin{aligned}
&B_0(u,v) = {}_\varsigma U_0 \frac{\partial_\varsigma V_0}{\partial\xi}, \\
&B_1(u,v) = {}_\varsigma U_0 \frac{\partial_\varsigma V_1}{\partial\xi} + {}_\varsigma U_1 \frac{\partial_\varsigma V_0}{\partial\xi}, \\
&B_2(u,v) = {}_\varsigma U_0 \frac{\partial_\varsigma V_2}{\partial\xi} + {}_\varsigma U_1 \frac{\partial_\varsigma V_1}{\partial\xi} + {}_\varsigma U_2 \frac{\partial_\varsigma V_0}{\partial\xi}, \\
&B_2(u,v) = {}_\varsigma U_0 \frac{\partial_\varsigma U_3}{\partial\xi} + {}_\varsigma U_1 \frac{\partial_\varsigma U_2}{\partial\xi} + {}_\varsigma U_2 \frac{\partial_\varsigma U_1}{\partial\xi} + {}_\varsigma U_3 \frac{\partial_\varsigma U_0}{\partial\xi}, \\
&\vdots \\
&D_0(v,v) = 2\,{}_\varsigma V_0 \frac{\partial_\varsigma U_0}{\partial\xi}, \\
&D_1(v,v) = 2\left({}_\varsigma U_0 \frac{\partial_\varsigma U_1}{\partial\xi} + {}_\varsigma U_1 \frac{\partial_\varsigma U_0}{\partial\xi}\right), \\
&D_2(v,v) = 2\left({}_\varsigma V_0 \frac{\partial_\varsigma V_2}{\partial\xi} + {}_\varsigma V_1 \frac{\partial_\varsigma V_1}{\partial\xi} + {}_\varsigma V_2 \frac{\partial_\varsigma V_0}{\partial\xi}\right), \\
&D_3(v,v) = 2\left({}_\varsigma V_0 \frac{\partial_\varsigma V_3}{\partial\xi} + {}_\varsigma V_1 \frac{\partial_\varsigma V_2}{\partial\xi} + {}_\varsigma V_2 \frac{\partial_\varsigma V_1}{\partial\xi} + {}_\varsigma V_3 \frac{\partial_\varsigma V_0}{\partial\xi}\right),
\end{aligned}$$

from initial conditions, ${}_\zeta U_0(\xi) = \sin\xi$, ${}_\zeta V_0(\xi) = \sin\xi$.
Now, substituting (23) with (24) respectively, we obtain

$${}_\zeta U_1^\alpha(x) = \frac{-\sin\xi}{\alpha},\ {}_\zeta U_2^\alpha(x) = \frac{\sin\xi}{2\alpha^2},\ {}_\zeta U_3^\alpha(x) = \frac{-\sin\xi}{3!\alpha^3},\cdots$$
$${}_\zeta V_1^\alpha(\xi) = \frac{-\cos\xi}{\alpha},\ {}_\zeta V_2^\alpha(\xi) = \frac{\cos\xi}{2\alpha^2},\ {}_\zeta V_3^\alpha(\xi) = \frac{-\cos\xi}{3!\alpha^3},\cdots$$

Taking the inverse transformation of the set of values $\left\{U_k^\alpha(\xi)\right\}_{k=0}^4$ and $\left\{U_k^\alpha(x)\right\}_{k=0}^4$ gives the fourth-term approximation

solutions as follows

$$\tilde{u}_4(\xi,\zeta)=\sum_{k=0}^{4}{}_{\zeta}U_k^{\alpha}(\xi)\zeta^{k\alpha}={}_{\zeta}U_0^{\alpha}(x)+{}_{\zeta}U_1^{\alpha}(\xi)\zeta^{\alpha}+{}_{\zeta}U_2^{\alpha}(\xi)\zeta^{2\alpha}+vU_3^{\alpha}(x)\zeta^{3\alpha}$$
$$=\sin\xi-\frac{\sin\xi}{\alpha}\zeta^{\alpha}+\frac{\sin\xi}{2!\alpha^2}\zeta^{2\alpha}-\frac{\sin\xi}{3!\alpha^3}\zeta^{3\alpha}+\frac{\sin\xi}{4!\alpha^4}\zeta^{4\alpha},$$

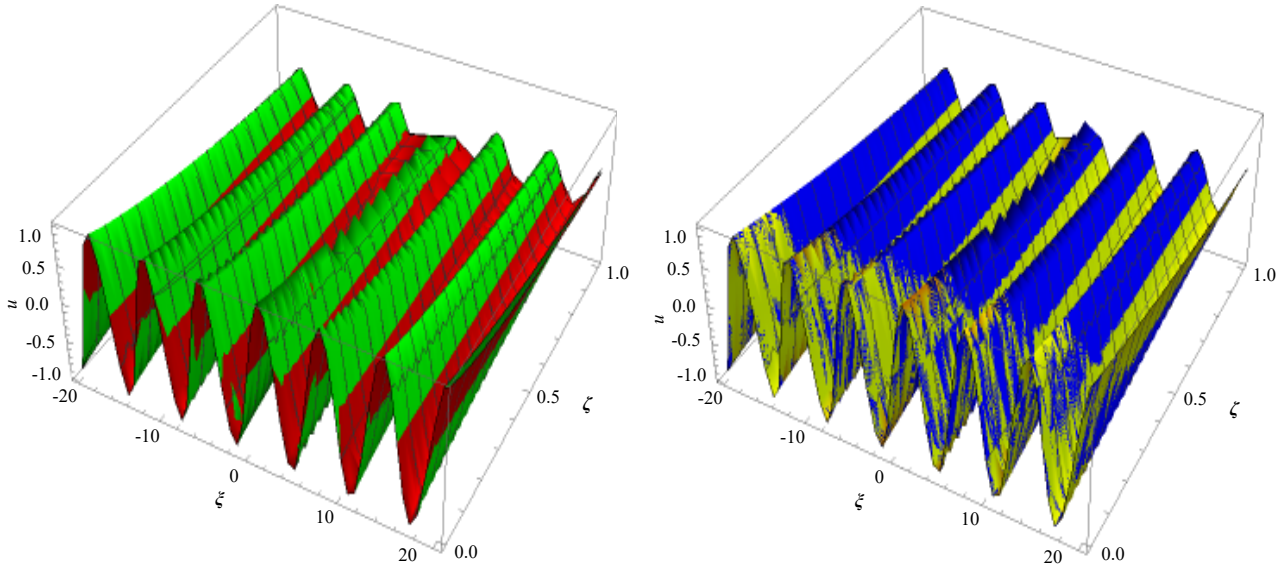
$$\tilde{v}_4(\xi,\zeta)=\sum_{k=0}^{4}{}_{\zeta}V_k^{\alpha}(x)\zeta^{k\alpha}={}_{\zeta}V_0^{\alpha}(\xi)+{}_{\zeta}V_1^{\alpha}(\xi)\zeta^{\alpha}+{}_{\zeta}V_2^{\alpha}(\xi)\zeta^{2\alpha}+V_3^{\alpha}(\xi)\zeta^{3\alpha}$$
$$=\sin\xi-\frac{\sin\xi}{\alpha}\zeta^{\alpha}+\frac{\sin\xi}{2!\alpha^2}\zeta^{2\alpha}-\frac{\sin\xi}{3!\alpha^3}\zeta^{3\alpha}+\frac{\sin\xi}{4!\alpha^4}\zeta^{4\alpha}.$$

The error between the exact and fourth-order approximation solution u and v of system by LRDTM is shown in Table 6, when $\alpha=1$, $\zeta=0.01, 0.05$ and $0.1-2\le\xi\le 2$.

Table 6. The error between the exact and fourth-order approximation solution u and v of system by LRDTM for $\alpha=1$, $\zeta=0.01$, 0.05 and 0.1, $-20\le\xi\le 20$

ξ	ζ	Exact	Approximate $\alpha=1$	Approximate $\alpha=0.9$	Absolute error u and v for $\alpha=1$ $\lvert u_{exact}-u_{appr}\rvert$
-20	0.01	-0.90386129	-0.90386129	-0.89700908	7.59503571e-13
-10		0.53860801	0.53860801	0.5345248	4.52526905e-13
0		0	0	0	0.00000000e+00
10		-0.5386080	-0.53860801	-0.5345248	4.52526905e-13
20		0.90386129	0.90386129	0.89700908	7.59503571e-13
-20	0.05	-0.86842039	-0.86842039	-0.84701277	2.35778996e-09
-10		0.51748889	0.51748889	0.50473216	1.40499934e-09
10		-0.51748889	-0.51748889	-0.50473216	1.40499934e-09
20		0.86842039	0.86842039	0.84701277	2.35778996e-09
-20	0.1	-0.82606702	-0.8260671	-0.79377164	7.48286815e-08
-10		0.49225066	0.4922507	0.47300594	4.45901684e-08
0		0	0	0	0.00000000e+00
10		-0.49225066	-0.4922507	-0.47300594	4.45901684e-08
20		0.82606702	0.8260671	0.79377164	7.48286815e-08

Figure 4 shows the approximate and exact solutions $u(\xi,\zeta)$ and $v(\xi,\zeta)$ for different α values for Example 4.4.



(a) For $\alpha = 0.7$ and 0.9 approximate solutions (red, green colored surfaces, respectively) of $u(\xi,\zeta)$ and $v(\xi,\zeta)$

(b) For $\alpha = 1$ approximate and exact solutions (blue, yellow colored surfaces, respectively) of $u(\xi,\zeta)$ and $v(\xi,\zeta)$

Figure 4. 4th-order of approximate and exact solution $u(\xi,\zeta)$ and $v(\xi,\zeta)$ of Example 4.4 for different values of α

5. Conclusion and discussion

In this article, we explored the use of the conformable RDTM in conjunction with the Adomian decomposition to solve one and two-dimensional NFPDEs. We used this novel approach to estimate the solutions of nonlinear systems of equations and discussed convergence analysis and absolute error for the proposed technique to solve these problems. To illustrate the theoretical aspects and the effectiveness of the numerical approximation, certain numerical examples are provided as well as a 2D or 3D graphical presentation using the Mathematica software.

In Tables 2,3, and 6, the numerical values of the approximate solutions for $\alpha = 0.9$ and 1 and exact solutions for Example 4.1, Example 4.2 and Example 4.4 respectively, show the accuracy and efficiency of our technique at different values of ξ, τ. In Tables 4 and 5, the numerical values of the approximate solutions for $\alpha = 0.9$ and 1 and the exact solutions for Example 4.3, show the accuracy and efficiency of our technique at different values of ξ, τ. The absolute error between an exact and a fourth-order approximate solution. In Figure 1(a), we plot the graph in different colors of fourth-term approximate solutions for Example 4.1, when $\alpha = 0.7$ and 0.9 and Figure 1(b), we plot the fourth-term approximate solution when $\alpha = 1$ and the exact solution. In Figure 2(a), we show the graph of the fourth-term approximate solution for Example 4.2, when $\alpha = 0.7$ and 0.9 and Figure 1(b), we plot the fourth-term approximate solutions for Example 4.2, when $\alpha = 1$ and exact solutions are shown in different colors. In Figure 3(a), we plot the graph of approximate solutions for Example 4.3, when $\alpha = 0.7$ and 0.9 and in Figure 3(b), we plot fourth-order approximate solutions for Example 4.3, when $\alpha = 1$ and exact solutions are shown in different colors. In Figure 4(a), we plot a graph of the fourth-term approximate solutions for Example 4.4, when $\alpha = 0.7$ and 0.9, and in Figure 4(b), we plot the fourth-order approximate solutions for Example 4.4, when $\alpha = 1$ and the exact solution is shown in different colors.

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Conflict of interest

There is no conflict of interest for this study.

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