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Research Article

Simulation of Fractional Order 2D-Mathematical Model Using α-Fractional Differential Transform Method

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Abstract: In this paper, we will introduce a well-known transformation technique, the modified α-fractional differential transform, to the differential equation of fractional order. We derive some new results with proof using new techniques that never existed before. By using this new technique, we are attempting to solve the nonlinear fractional-order mathematical epidemic model. Furthermore, the fractional epidemic model's solution obtained by using this new technique is correlated with the solution of the same model calculated for a different fractional order by the modified α-fractional differential transform method. Moreover, using the Python software, we can numerically and graphically represent the solution of fractional differential equations.

***Keywords*:** fractional differential transform method, fractional differential equation, conformable fractional differential transform, α-fractional derivative

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1. Introduction

In recent decades, the theory of fractional derivation has gained prominence in mathematical studies. There is no standard form of fractional derivative definition. But the definitions that are most widely used are those that employ integration: The definitions of Riemann-Liouville and Caputo are found in [1].

Here, these fractional derivatives lack several properties, including an algebra of derivatives and mean value theorem (MVT). To avoid some of these and other issues, Khalil et al. [2] proposed a new concept of fractional derivative and demonstrated some outcomes using his new fractional derivative definition, which can be found in [3-10]. In [11], Almeida et al. proposed the limit definition of a fractional derivative. He also used his definition of fractional derivatives to draw several significant conclusions about fractional derivatives. This study's main objective is to offer a limited definition of a fractional derivative that adheres to traditional concepts. For undefined and unexplained concepts and terms, readers can see [12-14].

The fractional differential transform method (FDTM) is one of the numerical methods, and it is used to determine the solution to various differential equations. Zhou [15] first introduced the concept of the differential transform method (DTM), and by using this new DTM method, he solved linear as well as nonlinear initial value problem (IVP) in electrical science. Recently, to solve fractional differential equations, a novel analytical method known

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as the FDTM has been devised, which can be found in [16]. This approach, FDTM, creates fractional power series in the same way that the DTM forms the Taylor series. Many authors have done studies using FDTM about solutions for different types of fractional differential equations (FDEs).

Erturk et al. [17] utilized the principles of fractional calculus (FC) and looked into the motion of a beam on an internally bent nanowire. They start to formulate the standard Lagrangian, then the standard Euler-Lagrange equation. Furthermore, the fractional Euler-Lagrange equation offered a versatile model with more detail than the traditional explanation. Yang et al. [18], reviewed some articles in ten sub-fields: image enhancement, image denoising, image edge detection, image segmentation, image registration, image recognition, image fusion, image encryption, image compression, and image restoration. Mahatekar et al. [19] derived a new numerical method to solve FDEs containing Caputo-Fabrizio derivatives. An analysis of the fractional climate change model and the fractional delay dynamical model can be found in [20-21]. The major characteristics and applications of artificial neural networks (ANNs) have been compiled in [22], which utilized a bibliographic analysis employing FC theory. A study of a fractional boundary value problem (FBVP) for modeling the human corneal shape dynamics and a strongly nonlinear boundary value problem with multi-fractional derivatives that comes in chemical reactor theory can be found in [23-24]. Kumar et al. [25] derived a new version of the L1-predictor-corrector (L1-PC) method by using some previously given methods to solve multiple delay-type fractional differential equations. A study to simulate the dynamics of the alkali-silica reaction (ASR) by using the Caputo fractional derivative can be found in [26].

In [27], we developed the notion of α-fractional derivative and integral by making the conventional changes in the classical definition of a derivative, which is defined as follows:

Definition 1.1. α-fractional derivative [27]. If $\alpha \in (0,1]$ and $\chi : [0,\infty) \to \mathbb{R}$ then α-fractional derivative of order α is expressed as

$$T_\alpha(\chi(\zeta)) = \lim_{\mu \to 0} \frac{\chi(\zeta e^{\mu\zeta^{1-\alpha}}) - \chi(\zeta)}{\mu}. \tag{1}$$

Theorem 1.2. [27] If χ at point $\zeta > 0$, is α-differential function then

$$T_\alpha \chi(\zeta) = \zeta^{2-\alpha} \frac{d\chi(\zeta)}{d\chi}. \tag{2}$$

In this paper, we present a novel modified α-fractional differential transform method (Mα-FDTM) for modified α-fractional derivatives. Mα-FDTM represents modified α-fractional power series in the same way that FDTM represents the fractional power series and DTM represents the Taylor series.

2. Some properties of modified α-fractional derivative

By using Definition 1.1 of a modified α-fractional derivative, we obtain the following useful results:

Theorem 2.1. If χ and Ω are α-differentiable functions at point $\vartheta > 0$, then

1. If χ at point $\vartheta > 0$ is an α-differentiable function, then χ is continuous at point ϑ.
2. $T_\alpha(\beta\chi) = \beta T_\alpha(\chi)$, $\forall\ \beta \in \mathbb{R}$.
3. $T_\alpha(\chi + \Omega)(\vartheta) = T_\alpha(\chi)(\vartheta) + T_\alpha(\Omega)(\vartheta)$.
4. $T_\alpha(\chi\Omega)(\vartheta) = \chi T_\alpha(\Omega)(\vartheta) + \Omega T_\alpha(\chi)(\vartheta)$.
5. $T_\alpha\left(\frac{\chi}{\Omega}\right)(\vartheta) = \left(\frac{\Omega(\vartheta) T_\alpha \chi(\vartheta) - \chi(\vartheta) T_\alpha \Omega(\vartheta)}{\Omega(\vartheta)^2}\right)$.
6. $T_\alpha\left(\frac{1}{\Omega}\right) = -\frac{T_\alpha \Omega}{\Omega^2}$.
7. $T_\alpha(\chi \circ \Omega)(\vartheta) = T_\alpha \chi(\Omega(\vartheta)) T_\alpha \Omega(\vartheta)$.

2.1 *α-fractional derivative of some standard functions*

If $0 < \alpha \le 1$ and $c \in \mathbb{R}$, then, using Theorem 1.2, we have

1. $T_\alpha(\vartheta^n) = n\vartheta^{n+1-\alpha}$.
2. $T_\alpha\left(\frac{1}{\alpha-1}\tau^{\alpha-1}\right) = 1$.
3. $T_\alpha(\beta) = 0, \forall \beta \in \mathbb{R}$.
4. $T_\alpha(e^{c\tau}) = e^{c\tau}c\tau^{2-\alpha}$.
5. $T_\alpha(\sin c\tau) = c\tau^{2-\alpha}\cos(ct)$.
6. $T_\alpha(\cos c\tau) = -c\tau^{2-\alpha}\sin(ct)$.
7. $T_\alpha(\log \tau) = \tau^{1-\alpha}$.
8. $T_\alpha(a^\tau) = (a^\tau \log a)\tau^{2-\alpha}$.
9. $T_\alpha(\tan \tau) = \tau^{2-\alpha}\sec^2\tau$.
10. $T_\alpha(\cot \tau) = -\tau^{2-\alpha}\csc^2\tau$.

Theorem 2.2. If $\alpha \in (1,2]$, $\phi \in C^\infty[0,\infty)$ and $\xi_0 \in [0,\infty)$, then

$$\phi(\xi) = \sum_{k=0}^{\infty} \frac{(T_\alpha\phi)^{(k)}(\xi_0)(\xi-\xi_0)^{k(\alpha-1)}}{(\alpha-1)^k k!}, \quad \xi_0 < \xi < \xi_0 + R^{1/\alpha}, R > 0 \tag{3}$$

where $(T_\alpha\phi)^{(k)}(\xi_0)$ represents the kth fractional derivative.

Theorem 2.3. Let $\alpha \in (1,2]$ and ϕ be an infinitely α-differentiable function in a neighborhood of a point ξ_0, then ϕ features a depiction of the power series (3) like that $|(T_\alpha^{\xi_0}\phi)^{(n+1)}| \le M$, for some $M > 0$ and $n \in N$.
If $\xi \in (\xi_0, \xi_0 + R)$, then

$$|R_n^\alpha(\xi)| \le \frac{M}{(\alpha-1)^{n+1}(n+1)!}(\xi-\xi_0)^{\alpha-1(n+1)}, \tag{4}$$

where $R_n^\alpha(\xi) = \sum_{k=n+1}^{\infty} \frac{(T_\alpha\phi)^{(k)}(\xi_0)(t-t_0)^{k(\alpha-1)}}{(\alpha-1)^k k!} = \phi(\xi) - \sum_{k=n+1}^{\infty} \frac{(T_\alpha\phi)^{(k)}(\xi_0)(\xi-\xi_0)^{k(\alpha-1)}}{(\alpha-1)^k k!}$.

Proof. The method for proving the theorem is same as that used in conventional calculus, by applying $I_\alpha^{\xi_0}$ instead of integration.

Remark. As the functions $f(\xi) = \sin\frac{(\xi-\xi_0)^{\alpha-1}}{\alpha-1}$, $g(\xi) = \cos\frac{(\xi-\xi_0)^{\alpha-1}}{\alpha-1}$, $h(\xi) = \phi(\xi) = e^{(\xi-\xi_0)^{\alpha-1}}$ and $\phi(\xi) = \frac{1}{1-\frac{\xi^{\alpha-1}}{\alpha-1}}$ are not differential at $\xi = \xi_0$ and therefore these functions have no power series expansions about $\xi = \xi_0$ for $\alpha \in (1,2]$. By using Equation (3), the α-fractional power series expansions of these functions at point ξ_0 are as follows:

1. $\sin\frac{(\xi-\xi_0)^{\alpha-1}}{\alpha-1} = \sum_{k=0}^{\infty}(-1)^k \frac{(\xi-\xi_0)^{(2k+1)(\alpha-1)}}{(\alpha-1)^{(2k+1)}(2k+1)!}, \ \xi \in [\xi_0, \infty)$.
2. $\cos\frac{(\xi-\xi_0)^{\alpha-1}}{\alpha-1} = \sum_{k=0}^{\infty}(-1)^k \frac{(\xi-\xi_0)^{(2k)(\alpha-1)}}{(\alpha-1)^{2k}(2k)!}, \ \xi \in [\xi_0, \infty)$.
3. $e^{(\xi-\xi_0)^{\alpha-1}} = \sum_{k=0}^{\infty} \frac{(\xi-\xi_0)^{k(\alpha-1)}}{(\alpha-1)^k}$.
4. $\frac{1}{1-\frac{(\xi-\xi_0)^{\alpha-1}}{\alpha-1}} = \sum_{k=0}^{\infty}(\xi-\xi_0)^{(\alpha-1)k}, \ \xi \in [\xi_0, \xi_0+1)$.

Definition 2.4. The α-fractional exponential function is denoted by E_α and for every $\xi \ge 0$ defined as

$$E_\alpha = exp\left(c\frac{\xi^{\alpha-1}}{\alpha-1}\right). \tag{5}$$

By using the Theorem 1.2, we have

$$D_\xi^\alpha E_\alpha(c,\xi) = cE_\alpha(c,\xi), \tag{6}$$

here, $E_\alpha(-1,\xi)$ (2.4) is an eigenfunction of D_ξ^α having an eigenvalue of 1.

Lemma 2.5. Suppose that r is a continuous, non-negative function on $0 \le \xi < T$ and $a, b \in [0,\infty)$ such that

$$r(\xi) \le a + \int_a^\xi r(s)d_\alpha\sigma.$$

Then

$$r(\xi) \le aE_\alpha(c,(\xi - a)).$$

Proof. Consider $R(\xi) = a + \int_a^\xi kr(\sigma)\sigma^{\alpha-2}ds = a + I_\alpha^a(kr(\sigma))(\xi)$. Then, $R(a) = a$, $R(\xi) \ge r(\xi)$, and

$$T_\alpha^a R(\xi) - kR(\xi) = kr(\xi) - kR(\xi) \le kr(\xi) - kr(\xi) = 0. \tag{7}$$

Multiply Equation (7) by $K(\xi) = e^{-c\frac{(\xi-a)^{\alpha-1}}{\alpha-1}}$. With the help of chain rule, we see that $T_\alpha^a K(\xi) = -kK(\xi)$ and hence, by the product rule, we conclude that $T_\alpha^a K(\xi)R(\xi) \le 0$. Since $K(\xi)R(\xi)$ is differentiable on (a, b), then

$$I_\alpha^a T_\alpha^a K(\xi)R(\xi) = K(\xi)R(\xi) - K(a)R(a) = K(\xi)R(\xi) - a \le 0.$$

Hence,

$$r(\xi) \le R(\xi) \le \frac{a}{K(\xi)} = ae^{c\frac{(\xi-a)^{\alpha-1}}{\alpha-1}}.$$

3. Modified α-fractional DTM

Definition 3.1. If $\varpi(\tau) \in C^\infty[0,\infty)$ and $\alpha \in (1,2)$, then $\varpi(\tau)$ has a modified α-fractional differential transform (Mα-FDT) and is given as

$$F_\alpha(\delta) = \frac{1}{(\alpha-1)^\delta \delta!}\left[\left(T_\alpha^{\tau_0}\varpi\right)^{(\delta)}(\tau)\right]_{\tau=\tau_0}, \tag{8}$$

where $(T_\alpha^{\tau_0}\varpi)^{(\delta)}(\tau)$ represents the fractional derivative of $\varpi(\tau)$ taking δ times.

Definition 3.2. If $F_\alpha(\delta)$ is the Mα-FDT of $\varpi(\tau)$ then there is an inverse. The α-FDT has been modified to $F(\delta)$ and is defined as

$$\varpi(\tau) = \sum_{\delta=0}^\infty F_\alpha(\delta)\left(\tau-\tau_0\right)^{(\alpha-1)\delta} = \sum_{\delta=0}^\infty \frac{1}{(\alpha-1)^\delta \delta!}\left[\left(T_\alpha^{\tau_0}\varpi\right)^{(\delta)}(\tau)\right]_{\tau=\tau_0}\left(\tau-\tau_0\right)^{(\alpha-1)\delta}. \tag{9}$$

For integer-order derivatives, the initial condition Mα-FDT is defined as

$$F_\alpha(\delta) = \begin{cases} \text{if } (\alpha-1)\delta \in Z^+ & \dfrac{1}{((\alpha-1)\delta)!}\left[\dfrac{d^{(\alpha-1)\delta}\varpi(\tau)}{d\tau^{(\alpha-1)\delta}}\right]_{\tau=\tau_0}, \text{ for } \delta = 0,1,2,\ldots,\left(\dfrac{n}{(\alpha-1)}-1\right) \\ \text{if } (\alpha-1)\delta \notin Z^+ & 0 \end{cases}$$

where n is the order of the modified α-fractional ordinary differential equation (Mα-FODE).

Theorem 3.3. Let $Y_\alpha(\zeta)$, $W_\alpha(\zeta)$ and $Z_\alpha(\zeta)$ be the Mα-FDT of the relevant functions, $y(\zeta)$, $w(\zeta)$, and $z(\zeta)$. Then

1. If $\varpi(\xi) = u(\xi) \pm v(\xi)$, then $F_\alpha(\delta) = U_\alpha(\delta) \pm V_\alpha(\delta)$.
2. If $\varpi(\xi) = cu(\xi)$, then $F_\alpha(\delta) = cU_\alpha(\delta)$, where c be a constant.
3. If $\varpi(\xi) = u(\xi)v(\xi)$, then $F_\alpha(\delta) = \sum_{l=0}^{\delta} U_\alpha(l)V_\alpha(\delta - l)$.
4. If $\varpi(\xi) = T_\alpha^{\xi_0}(u(\xi))$, then $F_\alpha(\delta) = (\alpha - 1)(\delta + 1)U_\alpha(\delta + 1)$.
5. If $\varpi(\xi) = T_\beta^{\xi_0}(u(\xi))$ for $m < \beta \le m+1$, then $F_\alpha(\delta) = U_\alpha(\delta + \beta/(\alpha-1))\dfrac{\Gamma(\delta(\alpha-1)+\beta+1)}{\Gamma(\delta(\alpha-1)+\beta-m)}$.
6. If $\varpi(\zeta) = \left(T_{\beta_1}^{\xi_0}u_1\right)(\xi)\cdot\left(T_{\beta_2}^{\xi_0}u_2\right)(\xi)\ldots\left(T_{\beta_n}^{\xi_0}u_n\right)(\xi)$, then $F\ (\ ) =$ where $\dfrac{i}{(\ -1)} \in Z^+$ and $m_i <\ {}_i \le m_i + 1$ for $i = 1,2,\ldots,n$.
7. If $\varpi(\xi) = (\xi - \xi_0)^p$, then $F_\alpha(\delta) = \Delta\left(\delta - \dfrac{p}{(\alpha-1)}\right)$ where $\Delta(\delta) = \begin{cases} 1, & \text{if } \delta = 0 \\ 0, & \text{if } \delta \ne 0 \end{cases}$.
8. If λ is constant and $\varpi(\xi) = e^{\lambda\frac{(\xi-\xi_0)^{\alpha-1}}{\alpha-1}}$, then $F_\alpha(\delta) = \dfrac{\lambda^\delta}{(\alpha-1)^\delta\delta!}$.

Definition 3.4. [28] A modified α-fractional partial differential transform (Mα-FPDT) of $\phi(\zeta, \xi)$ is defined as

$$\phi_\alpha(\zeta,\xi) = \sum_{\delta=0}^{\infty}\sum_{\eta=0}^{\infty} U_{\alpha,\beta}(\delta,\eta)(\zeta-\zeta_0)^{\delta(\alpha-1)}(\xi-\xi_0)^{\eta(\beta-1)}\frac{1}{(\alpha-1)^\delta\delta!}\left[(T_\alpha^{\xi_0}\phi)^{(\delta)}(\xi)\right]_{\xi_0},$$

where $0 < \alpha, \beta \le 1$, and the Mα-FDT of a function, $\phi(\zeta, \xi)$ is

$$\Phi_{\alpha,\beta}(\delta,\eta) = \frac{1}{(\alpha-1)^\delta\delta!(\beta-1)^\eta\eta!}\left[({}_{\zeta_0}T_\zeta^\alpha\phi)^{(\delta)}(\xi)({}_{\xi_0}T_\xi^\beta\phi)^{(\eta)}(\xi)\right]_{(\zeta_0,\xi_0)}.$$

Theorem 3.5. [28] Let $Y_{\alpha,\beta}(\delta, \eta)$, $W_{\alpha,\beta}(\delta, \eta)$ and $Z_{\alpha,\beta}(\delta, \eta)$ be the Mα-FDT of the functions, $y(\zeta, \xi)$, $w(\zeta, \xi)$ and $z(\zeta, \xi)$, respectively. Then

1. If $y(\zeta,\xi) = w(\zeta,\xi) \pm z(\zeta,\xi)$, then $Y_{\alpha,\beta}(\delta, \eta) = W_{\alpha,\beta}(\delta, \eta) + Z_{\alpha,\beta}(\delta, \eta)$.
2. If $y(\zeta, \xi) = cw(\zeta, \xi)$, $c \in R$, then $Y_{\alpha,\beta}(\delta, \eta) = cW_{\alpha,\beta}(\delta, \eta)$.
3. If $y(\zeta, \xi) = w(\zeta, \xi)z(\zeta, \xi)$, then $Y_{\alpha,\beta}(\delta,\eta) = \sum_{\delta=0}^{\delta}\sum_{\eta=0}^{\eta} W_{\alpha,\beta}(r,\eta-\sigma)Z_{\alpha,\beta}(\delta-r,\sigma)$.
4. If $y(\zeta, \xi) = v(\zeta, \xi)w(\zeta, \xi)z(\zeta, \xi)$, then $Y_{\alpha,\beta}(\delta,\eta) = \sum_{r=0}^{\delta}\sum_{\xi=0}^{\delta-r}\sum_{\sigma=0}^{\eta}\sum_{p=0}^{\eta-\sigma} V_{\alpha,\beta}(r,\eta-\sigma-p)W_{\alpha,\beta}(\xi,\sigma)Z_{\alpha,\beta}(\delta-r-\xi,p)$.
5. If $y(\zeta, \xi) = (\zeta - \zeta_0)^m(\xi - \xi_0)^n$, then $Y_{\alpha,\beta}(\delta,\eta) = \Delta\left(k - \dfrac{m}{(\alpha-1)}\right)\Delta\left(\eta - \dfrac{n}{(\beta-1)}\right)$.
6. If $y(\zeta,\xi) = e^{\lambda\left(\frac{(\zeta-\zeta_0)^\alpha}{\alpha}+\frac{(\xi-\xi_0)^\beta}{\beta}\right)}$ where λ is constant, then $Y_{\alpha,\beta}(\delta,\eta) = \dfrac{\lambda^{\delta+\eta}}{\alpha^\delta\delta!\beta^\eta\eta!}$.
7. If $y(\zeta,\xi) = {}_{\zeta_0}T_\zeta^\gamma w(\zeta,\xi)$, $m < \gamma \le m+1$, then $Y_{\alpha,\beta}(\delta,\eta) = W_{\alpha,\beta}\left(\delta + \dfrac{\gamma}{\alpha}, \eta\right)\dfrac{\Gamma(\delta(\alpha-1)+\gamma+1)}{\Gamma(\delta(\alpha-1)+\gamma-m)}$.

Theorem 3.6. If $u(\zeta, \xi) = v(\zeta, \xi)w(\zeta, \xi)$, then ${}_\xi U_\alpha(\zeta,\delta) = \sum_{r=0}^{\delta} {}_\xi V_\alpha(\zeta,r)\,{}_\xi W_\alpha(\zeta,\delta-r)$.

Proof. With the assistance of Definition 3.4, $u(\zeta, \xi)$ and $v(\zeta, \xi)$ can be written as

$$v(\zeta,\xi)=\sum_{\delta=0}^{\infty}{}_{\xi}V_{\alpha}(\zeta,\delta)\left(\xi-\xi_{0}\right)^{\delta(\alpha-1)},$$

$$w(\zeta,\xi)=\sum_{\delta=0}^{\infty}{}_{\xi}W_{\alpha}(\zeta,\delta)\left(\xi-\xi_{0}\right)^{\delta(\alpha-1)}.$$

Then, $u(\zeta, \xi)$ is obtained as follows:

$$\begin{aligned}
u(\zeta,\xi)&=\left(\sum_{\delta=0}^{\infty}{}_{\xi}V_{\alpha}(\zeta,\delta)\left(\xi-\xi_{0}\right)^{\delta(\alpha-1)}\right)\times\left(\sum_{\delta=0}^{\infty}{}_{\xi}W_{\alpha}(\zeta,\delta)\left(\xi-\xi_{0}\right)^{\delta(\alpha-1)}\right)\\
&=\left({}_{\xi}V_{\alpha}(\zeta,0)+{}_{\xi}V_{\alpha}(\zeta,1)\left(\xi-\xi_{0}\right)^{(\alpha-1)}+{}_{\xi}V_{\alpha}(\zeta,2)\left(\xi-\xi_{0}\right)^{2(\alpha-1)}+\cdots\right)\\
&\times\left({}_{\xi}W_{\alpha}(\zeta,0)+{}_{\xi}W_{\alpha}(\zeta,1)\left(\xi-\xi_{0}\right)+{}_{\xi}W_{\alpha}(\zeta,2)\left(\xi-\xi_{0}\right)^{2(\alpha-1)}+\cdots\right)\\
&=V_{\alpha}(\zeta,0){}_{\xi}W_{\alpha}(\zeta,0)+\left({}_{\xi}V_{\alpha}(\zeta,0){}_{\xi}W_{\alpha}(\zeta,1)+{}_{\xi}V_{\alpha}(\zeta,1){}_{\xi}W_{\alpha}(\zeta,0)\right)\left(\xi-\xi_{0}\right)^{(\alpha-1)}\\
&+\left({}_{\xi}V_{\alpha}(\zeta,0){}_{\xi}W_{\alpha}(\zeta,2)+{}_{\xi}U_{\alpha}(\zeta,1){}_{\xi}W_{\alpha}(\zeta,1)+{}_{\xi}V_{\alpha}(\zeta,2){}_{\xi}W_{\alpha}(\zeta,0)\right)\left(\xi-\xi_{0}\right)^{2(\alpha-1)}+\cdots\\
&=\sum_{\delta=0}^{\infty}\sum_{r=0}^{\delta}{}_{\xi}V_{\alpha}(\zeta,r){}_{\xi}W_{\alpha}(\zeta,\delta-r)\left(\xi-\xi_{0}\right)^{\delta(\alpha-1)}
\end{aligned}$$

$${}_{\xi}U_{\alpha}(\zeta,\delta)=\sum_{r=0}^{\delta}V_{\alpha}(\zeta,r){}_{\xi}W_{\alpha}(\zeta,\delta-r).$$

In general, for $U(\zeta,\xi)=v_{1}(\zeta,\xi)\cdot v_{2}(x,t)\cdots v_{n}(\zeta,\xi)$, we have

$${}_{\xi}U_{\alpha}(\zeta,\delta)=\sum_{\delta_{n-1}=0}^{\delta_{n}}\cdots\sum_{\delta_{1}=0}^{\delta_{2}}{}_{\xi}V_{1\alpha}\left(\zeta,\delta_{1}\right){}_{\xi}V_{2\alpha}\left(\psi,\delta_{2}-\delta_{1}\right)\times\cdots{}_{\xi}V_{(n-1)\alpha}\left(\zeta,\delta_{n-1}\right)V_{n\alpha}\left(\zeta,\delta_{n}-\delta_{n-1}\right).$$

Theorem 3.7. If $y(\zeta,\xi)={}_{\zeta_0}T_{\zeta}^{\alpha}w(\zeta,\xi)$, $0<\alpha\le 1$, then $Y_{\alpha,\beta}(\delta,\eta)=(\alpha-1)(\delta+1)W_{\alpha,\beta}(\delta+1,\eta)$.
Proof. By applying definition to $w(\zeta,\xi)$ and ${}_{\zeta_0}T_{\zeta}^{\alpha}u(\zeta,\xi)$, $0<\alpha\le 1$, we have

$$U(\delta,\eta)=\frac{1}{(\alpha-1)^{\delta}\,\delta!(\beta-1)^{\eta}\,\eta!}\Big[({}_{\zeta_0}T_{\zeta}^{\alpha}\phi)^{(\delta)}({}_{\xi_0}T_{\xi}^{\gamma}\phi)^{(m)}\left[{}_{\zeta_0}T_{\zeta}^{\alpha}u(\zeta,\tau)\right]\Big]_{(0,0)},$$

$$W(\delta,\eta,m)=\frac{(\delta+1)}{(\alpha-1)^{\delta}(\delta+1)!(\beta-1)^{\eta}\,\eta!}\Big[({}_{\zeta_0}T_{\zeta}^{\alpha}\phi)^{(\delta+1)}({}_{\xi_0}T_{\xi}^{\gamma}\phi)^{(m)}u(\zeta,\xi)\Big]_{(0,0)}.$$

Thus,

$$W(\delta,\eta)=(\alpha-1)(\delta+1)U_{\alpha,\beta,\gamma}(\delta+1,\eta).$$

Theorem 3.8. If $w(\zeta,\xi)={}_{\xi_0}T_{\xi}^{\theta}u(\zeta,\xi)$, $n<\theta\le n+1$, then

$$W_{\alpha,\beta}(\delta,\eta)=\frac{\Gamma(\eta(\beta-1)+\theta+1)}{\Gamma(\eta(\beta-1)+\theta-n)}U_{\alpha,\beta}\left(\delta,\eta+\frac{\theta}{(\beta-1)}\right).\tag{10}$$

Proof. Consider the initial condition of the problem:

$$T_{\xi}^{\theta}u(\zeta,\xi) = {}_{\xi_0}T_{\xi}^{\theta}\left[u(\zeta,\xi) - \phi(\zeta)\times\sum_{\eta=0}^{n}\frac{\psi^{(\eta)}(0)}{\eta!}(\xi-\xi_0)^{\eta}\right]$$
$$= {}_{\xi_0}T_{\xi}^{\theta}\left[\phi(\zeta)\times\sum_{\eta=0}^{\infty}\Psi_{\beta}(\eta)(\xi-\xi_0)^{\eta\beta} - \phi(\zeta)\times\sum_{\eta=0}^{n}\frac{\psi^{(\eta)}(0)}{\eta!}(\xi-\xi_0)^{\eta}\right].$$

Substituting $\eta(\beta-1)$ in place of η in the second series and considering the initial conditions, we get

$$T_{\xi}^{\theta}u(\zeta,\xi) = {}_{\xi_0}T_{\xi}^{\theta}\left[\phi(\zeta)\times\sum_{\eta=0}^{\infty}\Psi_{\beta}(\eta)(\xi-\xi_0)^{\eta(\beta-1)} - \phi(\zeta)\times\sum_{\eta=0}^{\frac{\theta}{(\beta-1)}-1}\frac{\psi^{(\eta(\beta-1))}(0)}{(\eta(\beta-1))!}(\xi-\xi_0)^{\eta(\beta-1)}\right]$$
$$= {}_{\xi_0}T_{\xi}^{\theta}\left[\phi(\zeta)\times\sum_{\eta=0}^{\infty}\Psi_{\beta}(\eta)(\xi-\xi_0)^{\eta(\beta-1)} - \phi(\zeta)\times\sum_{\eta=0}^{\frac{\theta}{(\beta-1)}-1}\frac{\psi^{(\eta(\beta-1))}(0)}{(\eta(\beta-1))!}(\xi-\xi_0)^{\eta(\beta-1)}\right]$$
$$= {}_{\xi_0}T_{\xi}^{\theta}\left[\phi(\zeta)\times\sum_{\eta=\frac{\theta}{(\beta-1)}}^{\infty}\Psi_{\beta}(\eta)(\xi-\xi_0)^{\eta(\beta-1)}\right]$$
$$= \sum_{\delta=0}^{\infty}\Phi(\zeta)(\zeta-\zeta_0)^{\delta(\alpha-1)}\times\sum_{\eta=\frac{\theta}{(\beta-1)}}^{\infty}\Psi_{\beta}(\eta)\frac{\Gamma(\eta(\beta-1)+1)}{\Gamma(\eta(\beta-1)-n)}(\xi-\xi_0)^{\eta(\beta-1)-\theta}$$
$$= \sum_{\delta=0}^{\infty}\Phi(\zeta)(\zeta-\zeta_0)^{\delta\alpha}\times\sum_{\eta=0}^{\infty}\Psi_{\beta}(\eta)\frac{\Gamma(\eta(\beta-1)+\theta+1)}{\Gamma(\eta(\beta-1)+\theta-n)}(\zeta-\zeta_0)^{\eta(\beta-1)-\theta}$$
$$= \frac{\Gamma(\eta(\beta-1)+\theta+1)}{\Gamma(\eta(\beta-1)+\theta-n)}W_{\alpha,\beta}\left(\delta,\eta+\frac{\theta}{(\beta-1)}\right)(\zeta-\zeta_0)^{k\alpha}(\xi-\xi_0)^{\eta(\beta-1)}$$
$$\therefore W_{\alpha,\beta}(\delta,\eta) = \frac{\Gamma(\eta(\beta-1)+\theta+1)}{\Gamma(\eta(\beta-1)+\theta-n)}U_{\alpha,\beta}\left(\delta,\eta+\frac{\theta}{(\beta-1)}\right).$$

3.1 *Applications*

Example 3.9. Consider the system of nonlinear fractional differential

$$\begin{aligned} T_{\alpha}^{\xi}y_1 &= 2y_2^2, \quad 1<\alpha\le 2,\\ T_{\beta}^{\xi}y_2 &= \tau y_1, \quad 1<\beta\le 2,\\ T_{\gamma}^{\xi}y_3 &= y_2y_3, \quad 1<\gamma\le 2, \end{aligned} \tag{11}$$

subject to the conditions

$$y_1(0)=0, \quad y_2(0)=1, \quad y_3(0)=1.$$

By using Theorem 3.5, the set of equations in (11) when converted, looks like this:

$$Y_1(\delta+1)=\frac{1}{(\alpha-1)(\delta+1)}\left[2\sum_{l=0}^{\delta}Y_2(l)Y_2(\delta-l)\right]$$
$$Y_2(\delta+1)=\frac{1}{(\beta-1)(\delta+1)}\left[\sum_{l=0}^{\delta}\Delta(l-1)Y_1(\delta-l)\right]$$
$$Y_3(\delta+1)=\frac{1}{(\gamma-1)(\delta+1)}\left[\sum_{l=0}^{\delta}Y_2(l)Y_3(\delta-l)\right]. \tag{12}$$

The initial conditions of transformation are

$$Y_1(0)=0,\quad Y_2(0)=1,\quad Y_3(0)=1. \tag{13}$$

Using Equations (12) and (13), $Y_1(\delta)$ for $\delta=1,2,\ldots,n$, $Y_2(\delta)$ for $\delta=1,2,3,\ldots,n$ and $Y_3(\delta)$ for $\delta=1,2$ are computed and transformation inverse is the rule used in Equation (4), $y_1(\xi)$, $y_2(\xi)$ and $y_3(\xi)$ are calculated for various values of α, β and γ. Using Equations (12) and (13), $y_1(\xi)$, $y_2(\xi)$ and $y_3(\xi)$ are obtained up to $\delta = 40$. The following series solutions are produced with the help of inverse transformation procedure in Equation (9). $\alpha = 3/2$, $\beta = 1.4$ and $\gamma = 1.3$, the solutions $x(\xi)$, $y(\xi)$ and $z(\xi)$ are evaluated up to five terms.

$$\zeta(\xi)=4\xi^{0.5}+\frac{2\xi^2}{0.3}+\cdots,$$
$$y(\xi)=1+\frac{2\xi^{1.2}}{0.6}+\cdots,$$
$$z(\xi)=1+\frac{\xi^{0.3}}{0.3}+\frac{\xi^{0.6}}{0.18}+\frac{\xi^{0.9}}{0.162}+\frac{\xi^{1.2}}{0.0388}+\cdots$$

If we take values of $\alpha=\beta=\gamma=2$, then we find approximate series solutions

$$x(\xi)=2\xi+\frac{2\xi^4}{3}+\cdots,$$
$$y(\xi)=1+\frac{2\xi^3}{3}+\cdots,$$
$$z(\xi)=1+\xi+\frac{\xi^2}{2}+\frac{\xi^3}{6}+\frac{5\xi^4}{24}+\cdots$$

Figure 1 demonstrates the rough answers for the values of system (11): $\alpha = 3/2$, $\beta = 1.4$ and $\gamma = 1.3$.

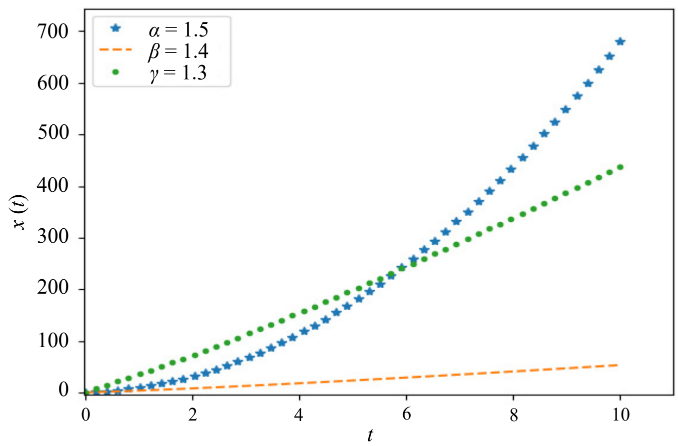


Figure 1. Graph of approximate solution of (11) for different values of α, β, γ

Example 3.10. Consider the system of linear Mα-FPDEs:

$$\begin{aligned} T_{\alpha}^{\xi} u - v_{\zeta} &= 0, \quad 1 < \alpha \le 2, \\ T_{\beta}^{\xi} v - u_{\zeta} &= 0, \quad 1 < \beta \le 2, \end{aligned} \tag{14}$$

subject to the conditions

$$u(\zeta, 0) = e^{\zeta}, \quad v(\zeta, 0) = e^{-\zeta}. \tag{15}$$

By using Mα-FDTM, we can rewrite Equation (14),

$$\begin{aligned} (\alpha - 1)(\eta + 1)U(\delta, \eta + 1) &= (\delta + 1)V(\delta + 1, \eta), \\ (\beta - 1)(\eta + 1)V(\delta, \eta + 1) &= (\delta + 1)U(\delta + 1, \eta). \end{aligned} \tag{16}$$

The initial conditions of transformation are

$$\begin{cases} U(\delta, 0) = \dfrac{1}{\delta!}, \\ V(\delta, 0) = \dfrac{(-1)^{\delta}}{\delta!}. \end{cases} \tag{17}$$

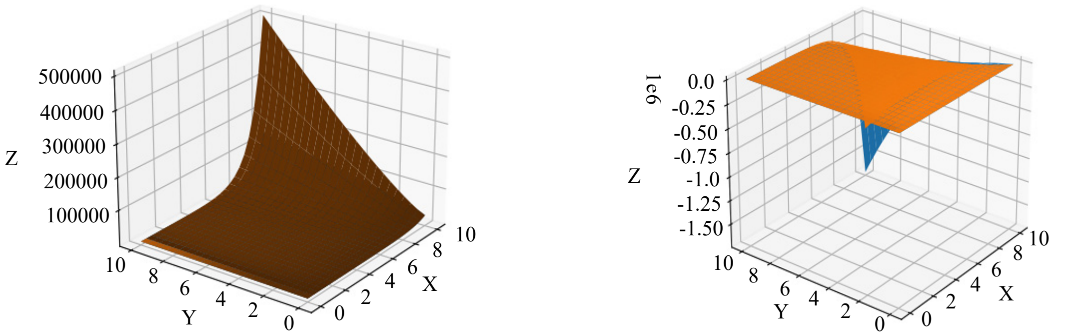
Putting (17) in (16), we obtain the closed-form solutions.

$$\begin{aligned} u(\zeta, \xi) &= e^{\zeta}\left(1 + \frac{\xi^{2(\alpha-1)}}{2!(\alpha-1)^2} + \ldots\right) + e^{-\zeta}\left(\frac{\xi^{(\alpha-1)}}{(\alpha-1)} + \frac{\xi^{3(\alpha-1)}}{3!(\alpha-1)^3} + \ldots\right), \\ v(\zeta, \xi) &= e^{-\zeta}\left(1 + \frac{\xi^{2(\beta-1)}}{2!(\beta-1)^2} + \ldots\right) - e^{\zeta}\left(\frac{\xi^{(\beta-1)}}{(\beta-1)} + \frac{\xi^{3(\beta-1)}}{3!(\beta-1)^3} + \ldots\right). \end{aligned}$$

As $\alpha = \beta = 2$ which are identical to the solutions obtained by homotopy analysis method (HAM) when it converges to closed-form solutions,

$$\begin{aligned} u(\zeta, \xi) &= e^{\zeta} \cosh \xi + e^{-\zeta} \sinh \xi, \\ v(\zeta, \xi) &= e^{-\zeta} \cosh \xi - e^{\zeta} \sinh \xi. \end{aligned}$$

In Figure 2, for α = 1.75, β = 1.85, the approximate solutions obtained by Mα-FDTM are $u(\zeta, \xi)$ and $v(\zeta, \xi)$ respectively.



(a) Approximate solution of $u(\zeta, \xi)$ for α = 1.75

(b) Approximate solution of $v(\zeta, \xi)$ for β = 1.85

Figure 2. Graph of the approximate solution of (14) for different values of α, β

Example 3.11. Consider the system of non-linear FPDEs:

$$T_\alpha^\xi u - u_{\zeta\zeta} - 2uu_\zeta + (uv)_\zeta = 0,$$
$$T_\alpha^\xi u - v_{\zeta\zeta} - 2vv_\zeta + (uv)_\zeta = 0, \quad 0 < (\alpha - 1) \le 1, \tag{18}$$

subject to the initial conditions

$$u(\zeta, 0) = \sin\zeta, \quad v(\zeta, 0) = \cos\zeta.$$

Using DTM, we can write

$$\begin{aligned}(\alpha-1)(\eta+1)U(\delta,\eta+1) &= (\delta+1)(\delta+2)U(\delta+2,\eta) + 2\sum_{r=0}^{\delta}\sum_{\sigma=0}^{\eta} U(r,\eta-\sigma)(\delta-r+1),\\ &\times U(\delta-r+1,\sigma) - \sum_{r=0}^{\delta}\sum_{\sigma=0}^{\eta}(r+1)U(r+1,\eta-\sigma)V(\delta-r,\sigma)\\ &- \sum_{r=0}^{\delta}\sum_{\sigma=0}^{\eta}(r+1)V(r+1,\eta-\sigma)U(\delta-r,\sigma),\end{aligned}$$

$$\begin{aligned}(\beta-1)(\eta+1)V(\delta,\eta+1) &= (\delta+1)(\delta+2)V(\delta+2,\eta) + 2\sum_{r=0}^{\delta}\sum_{\sigma=0}^{\eta} V(r,\eta-\sigma)(\delta-r+1),\\ &\times V(\delta-r+1,\sigma) - \sum_{r=0}^{\delta}\sum_{\sigma=0}^{\eta}(r+1)U(r+1,\eta-\sigma)V(\delta-r,\sigma)\\ &- \sum_{r=0}^{\delta}\sum_{\sigma=0}^{\eta}(r+1)V(r+1,\eta-\sigma)U(\delta-r,\sigma).\end{aligned}$$

The transformed version of the initial conditions

$$U(\delta,0) = V(\delta,0)\begin{cases} 0, & \delta = 0, 2, 4, \ldots \\ \dfrac{1}{\delta!}, & \delta = 1, 5, \ldots \\ \dfrac{1}{-\delta!}, & \delta = 3, 7, \ldots \end{cases} \tag{19}$$

We found the following closed-form answers by substituting (19) in the above.

$$u(\zeta,\xi) = \left(\frac{\zeta}{1!} - \frac{\zeta^3}{3!} + \ldots\right)\left(1 + \sum_{n=1}^{\infty}\frac{(-\xi^{(\alpha-1)})^n}{n!(\alpha-1)^n}\right),$$
$$v(\zeta,\xi) = \left(\frac{\zeta}{1!} - \frac{\zeta^3}{3!} + \ldots\right)\left(1 + \sum_{n=1}^{\infty}\frac{(-\xi^{(\alpha-1)})^n}{n!(\alpha-1)^n}\right).$$

If $\alpha = 2$, we get

$$u(\zeta,\xi) = \sin\zeta e^{-\xi},$$
$$v(\zeta,\xi) = \sin\zeta e^{-\xi}.$$

Those are the precise solutions to the equation system.

Figure 3 shows the graph of the approximate solution of Equation (18).

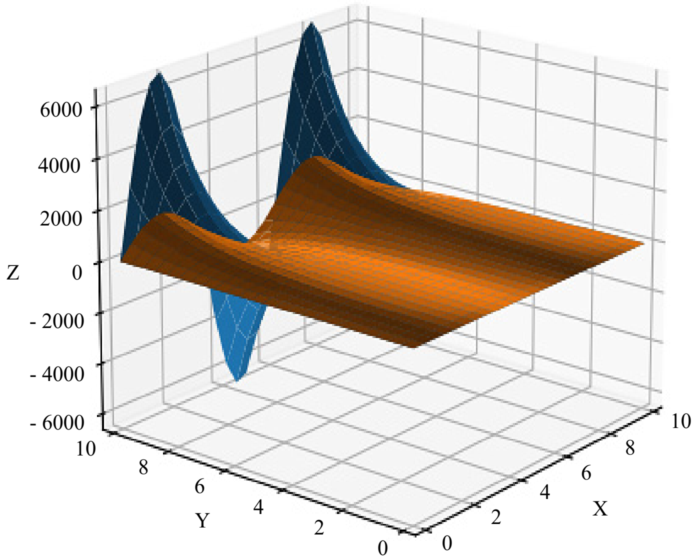


Figure 3. Graph of the approximate solution of (18)

4. Discussion and conclusion

In Figure 1, we plot the graph of the third-term approximate solution for Example 3.9 when $\alpha = 1.5$, $\beta = 1.4$ and $\gamma = 1.3$. In Figure 2(a), we plot the graph of the third-term approximate solution for Example 3.10 when $\alpha = 1.75$. In Figure 2(b), we plot the graph of the third-term approximate solution for Example 3.10 when $\beta = 1.85$. In Figure 3, we plot the graph of the third-term approximate solution for Example 3.11 when $\alpha = 1.60$ and 1.75.

In this article, we introduced Mα-FDTM to find the analytical solution for FDEs. Here, we have derived some results on Mα-FDTM, and these results are applied to various fractional differential equations. Modified fractional differential transform method (MFDTM) is found to be an efficient approach for Mα-FODE. In some cases, the series solution generated through the use of MFDTM can be compared to the exact solution. It is found that the solutions obtained by MFDTM and exact solutions are approximately equal. Otherwise, if the solutions do not match, then the number of terms is increased so that we get the most accurate solution.

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Conflict of interest

There is no conflict of interest for this study.

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