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Research Article

On Statistical Riemann-Stieltjes Integrablility and Deferred Cesàro Summability

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Abstract: In this paper, we first introduce and study the notion of statistical Riemann-Stieltjes sum for the sequence of functions and establish some elementary results based on this notion. Subsequently, we extend this notion to the probability space and demonstrate some new results for sequence of distribution functions. Furthermore, we suggest the deferred Cesàro summability method for Riemann-Stieltjes sum. We then establish various inclusion theorems based on our proposed methods in association with the Riemann-Stieltjes sum for the sequence of usual functions as well as distribution functions in $\mathbb{R}^n$.

***Keywords*:** Riemann-Stieltjes integral, statistical Riemann-Stieltjes integral, deferred Cesàro mean, distribution functions, random variables

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1. Introduction, preliminaries and motivation

The gradual improvement in the study of sequence spaces leads to the advancement of the concept statistical convergence, which is quite more prevailing than the usual convergence. The credit of such development goes to two eminent mathematicians Fast [1] and Steinhaus [2], and this concept makes the convergence analysis much wider. Now-a-days, this potential idea has been applied in numerous disciplines of pure and applied Mathematics and analytical statistics as well. In particular, it is very much useful in the study of Machine Learning, Soft Computing, Number Theory, Measure theory and Probability theory, etc. For some latest works, the interested learners may refer [3] and [4].

Suppose $\mathcal{Y} \subseteq \mathbb{N}$, and setting $\mathcal{Y}_k = \{\zeta : \zeta \leqq k \text{ and } \zeta \in \mathcal{Y}\}$, we define the asymptotic (natural) density $d(\mathcal{Y})$ of $\mathcal{Y}$ by

$$d(\mathcal{Y}) = \lim_{k \to \infty} \frac{|\mathcal{Y}_k|}{k} = \rho,$$

where the number $\rho \in \mathbb{R}$ is finite, and $|\mathcal{Y}_k|$ denotes the cardinal number of the set $\mathcal{Y}_k$.

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It is familiar that, a sequence (y_k) converges statistically to a if, for every $\epsilon > 0$,

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \text{ and } |y_\zeta - a| \geqq \varepsilon\}$$

ensures the natural (asymptotic) density zero (see [1] and [2]). Hence, for every $\varepsilon > 0$,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0,$$

and let we write it as

$$\text{stat} \lim_{k\to\infty} y_k = a.$$

The principal edition monograph of Zygmund [5], printed in the year 1935, served as the foundation for the statistical convergence concept and subsequently, Fast [1] was investigated and studied such concepts in a new direction over sequence space and presented a note on that basis. Later on Schoenberg [6], independently developed the same concepts on sequence space with some specific fundamental limit concepts. In recent trends of sequence space, the rudimentary idea of statistical convergence has been expanded to a wider class and has becoming a very active research area in the study of various spheres of Mathematical Analysis such as, theory of approximation, Banach spaces, measure theory, locally convex spaces, summability theory and Fourier analysis etc.

In the second half of nineteenth centaury, so many works stated on statistical convergence by a few researchers, such as in the year 1980, Šalát [7] investigated the theory of statistically convergent real numbers sequences and studied the boundedness properties of such statistical convergence. After that, Fridy [8] discussed the concrete definition of Cauchy criterions of statistical convergence and accordingly established some rudimentary results based on summability means. Subsequently, in the year 1988, Maddox [9] considered the locally convex space for the extensive study of statistical convergence and accordingly established certain relevant results. Gradually, in view of more advance study in the such direction, Fridy and Orhan [10] presented the lacunary statistical summability means for sequence of real numbers and obtained some prominent results.

The notion of the fundamental limit conception on statistical Cesàro summability and its applications was first introduced by the eminent mathematician Móricz [11]. Again, Mohiuddine et al. [12] obtained a nice outcome on statistical Cesàro summability mean with an illustrative example and further proved some associated Korovkin-type theorems. Afterwards, Karakaya and Chishti [13] popularised the elementary idea of statistical convergence via weighted summability mean, and later in the year 2018, Mursaleen et al. [14] clearly modified this concept and established some fundamental limit theorems. Recently, Baliarsingh et al. [15] introduced and deliberated the notion of advance version of uncertain sequences via statistical deferred A-convergence and proved some inclusion theorems. Again, in that year Saini et al. [16] also studied the results on equi-statistical convergence via the product deferred Cesàro and deferred Euler summability means with associated Korovkin-type theorems. Also, Saini et al. [17] again studied deferred Riesz statistical convergence of a complex uncertain sequences with its applications and also in that year, Sharma et al. [18] demonstrated the implementations of statistical deferred Cesàro convergence of fuzzy number valued sequences of order (ξ, ω). For another generalized result in this direction, the responsive learners may refer the recent work of Parida et al. [19].

Let $\{(I, \sigma, \mu) : I \subseteq \mathbb{R}^n\}$ be a measurable space, and let $\mathcal{G} : \mathbb{R}^n \to \mathbb{R}$ be a measurable monotone increasing function, and also let (h_k) be the measurable step functions sequence having measure (I, μ) over

$$I_i = [a_1^i,\ b_1^i] \times [a_2^i,\ b_2^i] \times \cdots \times [a_n^i,\ b_n^i] \in \mathbb{R}^n, \quad (i = 1,\ 2,\ \dots,\ n)$$

where I_i is a partition such that $I = \cup_{i=1}^n I_i$.

Now, we propose the Riemann-Stieltjes integral over a measurable space $(I,\ \sigma,\ \mu)$ as

$$\int_I h_k(u) d\mathcal{G}(u) = \sum_{i=1}^{k} c_i |\mathcal{G}(I_i)|,$$

where $\mu(\mathcal{G}_i) = |\mathcal{G}(I_i)|$ is the measure $(I,\ \mu)$ of the transformed rectangle (region) $\mathcal{G}(I_i)$.

Next, the measure $(I,\ \mu)$ of the closed region (rectangle) I_i is assumed as the product terms in the following form:

$$\mu(I_i) = (b_1^i - a_1^i) \times (b_2^i - a_2^i) \times \dots \times (b_n^i - a_n^i),$$

and accordingly we calculate the transformed measure $\mathcal{G}(I_i)$ as mentioned below.

In one dimensional case, consider the interval (closed and bounded) $I = [a,\ b]$ which has the measure $\mu(I) = |I| = b - a$, and the transformed measure as

$$\mu(\mathcal{G}(I)) = |\mathcal{G}(I)| = \mathcal{G}(b) - \mathcal{G}(a).$$

Similarly, for two dimensional case the measure of the closed region (rectangle) in $\mathbb{R}^2$ is given by

$$\mu(I_i) = |I_i| = (b_1^i - a_1^i)(b_2^i - a_2^i)$$

$$= b_1^i b_2^i - a_1^i b_2^i - b_1^i a_2^i + a_1^i a_2^i,$$

and in the same lines, we designate the measure of the transformed rectangle (region) $\mathcal{G}(I_i)$ as

$$\mu(\mathcal{G}(I_i)) = |\mathcal{G}(I_i)| = \mathcal{G}(b_1^i,\ b_2^i) - \mathcal{G}(a_1^i,\ b_2^i) - \mathcal{G}(b_1^i,\ a_2^i) + \mathcal{G}(a_1^i,\ a_2^i).$$

Successively, for nth dimensional case the measure of the closed region in $\mathbb{R}^n$ is given by

$$\mu(I_i) = |I_i| = (b_1^i - a_1^i)(b_2^i - a_2^i) \cdot \dots \cdot (b_n^i - a_n^i),$$

and the corresponding measure of the nth transformed region $\mathcal{G}(I_i)$ is

$$\mu(\mathcal{G}(I_i)) = |\mathcal{G}(I_i)| = \Delta_j \mathcal{G}(I_i) \quad (j = 1,\ 2,\ \dots,\ n),$$

where

$$\Delta_j\mathcal{G}(I_i)=\mathcal{G}(x_1,\ ...,\ x_{j-2},\ x_{j-1},\ b_j^i,\ x_{j+1},...,\ x_n)-\mathcal{G}(x_1,\ ...,\ x_{j-2},\ x_{j-1},\ a_j^i,\ x_{j+1},\ ...,\ x_n). \tag{1}$$

Let $f_k:\mathbb{R}^n\to\mathbb{R}$ be the measurable sequence functions over $(I,\ \sigma,\ \mu)$. We now define the Riemann-Stieltjes sum of the measurable functions (f_k) over $(I,\ \sigma,\ \mu)$ allied with a tagged partition $\dot{\mathcal{P}}\in\mathbb{R}^n$ of the form

$$\delta(f_k;\ \dot{\mathcal{P}}):=\sum_{i=1}^{k} f_i(\gamma_j{}^i)\Delta_j\mathcal{G}(I_i),\ \ (j=1,\ 2,\ ...,\ n),$$

where $\Delta_j\mathcal{G}(I_i)$ is mentioned as above in (1).

Let $\mathcal{G}$ be a measurable increasing function, and let the sequence of measurable functions (f_k) be defined over the interval $I\in\mathbb{R}^n$. The given measurable functions (f_k) is integrable in Riemann-Stieltjes sense corresponding to $\mathcal{G}$, if for each $\epsilon>0$, $\exists$ a sequence of measurable step functions (h_k') and (h_k'') for which

$$\int_I h_k''(u)d\mathcal{G}(u)-\int_I h_k'(u)d\mathcal{G}(u)<\epsilon \quad (h_k'<g_k<h_k'')$$

and

$$\int_I f_k(u)d\mathcal{G}(u)=\sup\int_I h_k(u)d\mathcal{G}(u),$$

where $h_k<g_k$ and (h_k) is a sequence of measurable step functions.

Note For $\mathcal{G}(I)=I$ (identity transformation), the integral in Riemann sense on $(I,\ \sigma,\ \mu)$ is a particular case of the integral Riemann-Stieltjes sense on $(I,\ \sigma,\ \mu)$ with

$$\int f_k(t)d\mathcal{G}(t)=\int f_k(t)dt. \tag{2}$$

Consequently, in the same line of equation (2), we obtain

$$\int f_k(t)d\mathcal{G}(t)=\sum_{i=1}^{n}c_i|I_i|$$

$$=\int f_k(t)dt.$$

Similarly, it can be extended for nth dimensional case.

We now propose the definition of statistical Riemann-Stieltjes (stat_{RS}) integrability for measurable functions sequence and prove the below-mentioned elementary.

Definition 1 Let $\mathcal{G}$ be a measurable increasing function, and let the sequence (f_k) of measurable functions be defined over the interval $I \subseteq \mathbb{R}^n$. The given sequence (f_k) of measurable functions is statistically Riemann-Stieltjes (stat_{RS}) integrable to a measurable function f with respect to $\mathcal{G}$, if for each $\epsilon > 0$, $\exists$ measurable step functions sequences (h'_k) and (h''_k) for which

$$\int_I h''_k(u)d\mathcal{G}(u) - \int_I h'_k(u)d\mathcal{G}(u) < \epsilon \quad (h'_k < f_k < h''_k)$$

and the set

$$\mathcal{Y}_\varepsilon = \left\{ \zeta : \zeta \in \mathbb{N} \text{ and } \left| \int_I f_k(u)d\mathcal{G}(u) - f \right| \geqq \varepsilon \right\}$$

ensures the natural (asymptotic) density zero (see [1] and [2]). Hence, for all $\varepsilon > 0$,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0,$$

and let we write it as

$$\text{stat}_{\text{RS}} \lim_{k\to\infty} \int_I f_k(u)d\mathcal{G}(u) = f.$$

We now present the following theorem, based on our Definition 1.

Theorem 1 Let $\mathcal{G}$ be a bounded and increasing measurable function on $I \subseteq \mathbb{R}^n$, and let the sequence of measurable functions (f_k) be integrable on $I \subseteq \mathbb{R}^n$. Then

$$\text{stat}_{\text{RS}} \int_I f_k d\mathcal{G} = \lim_{\max_i |I_i| \to 0} \sum_{i=1}^{n} f_k(\xi_i)|\mathcal{G}(I_i)|,$$

where I_i is a partition with $I = \cup_{i=1}^n I_i$ and $\xi \in I_i$.

Proof. Given $\mathcal{G}$ is a bounded and increasing measurable function on $I \subseteq \mathbb{R}^n$, that is,

$$-\infty < \ell = \mathcal{G}(x_1, \, ..., \, x_{j-1}, \, a^i_j, \, x_{j+1}, \, ..., \, x_n) = \inf_I \mathcal{G}$$

$$\leqq \sup_I \mathcal{G} = \mathcal{G}(x_1, \, ..., \, x_{j-1}, \, b^i_j, \, x_{j+1}, \, ..., \, x_n) = \ell < +\infty.$$

Suppose that I is finite, and since sequence of measurable (f_k) functions is integrable on $I \subseteq \mathbb{R}^n$ to a function (measurable) f, so for all $\epsilon > 0$ there possibly exists a natural number $N(\epsilon)$ with $k \geqq N(\epsilon)$ for which

$$\|f_k(t) - f(t)\| < \epsilon \qquad (\forall \, t \in I \subseteq \mathbb{R}^n).$$

Next, let

$$\cup_{i=1}^{k} I_i = I \subseteq \mathbb{R}^n$$

be a finite partition (arbitrary) of I with $|\mathcal{G}(I_i)| \leqq \delta$ ($\delta > 0$), and let

$$m_i = \inf_{t \in I_i} f_k(t) \text{ and } M_i = \sup_{t \in I_i} f_k(t),$$

where $M_i - m_i < \epsilon$.

Furthermore, the sequence of measurable step functions are

$$h'_k(t) = \sum_{i=1}^{k} m_i 1\{t \in I_i\} \text{ and } h''_k(t) = \sum_{i=1}^{k} M_i 1\{t \in I_i\}$$

with $h'_k < g_k < h''_k$.

Thus,

$$\int_I h'_k(u) d\mathcal{G}(u) = \sum_{i=1}^{n} m_i |\mathcal{G}(I_i)| \leqq \sum_{i=1}^{n} M_i |\mathcal{G}(I_i)| = \int_I h''_k(u) d\mathcal{G}(u),$$

which clearly implies

$$\int_I h''_k(u) d\mathcal{G}(u) - \int_I h'_k(u) d\mathcal{G}(u) \leqq \sum_{i=1}^{n} (M_i - m_i) |\mathcal{G}(I_i)|$$

$$\leqq \epsilon \sum_{i=1}^{n} |\mathcal{G}(I_i)| = \epsilon |G(I)|,$$

where $\mathcal{G}(I_i)$'s are disjoint.

Thus, for each $\epsilon > 0$, (h'_k) and (h''_k) are sequences of measurable step functions with (f_k) is integrable on $I \in \mathbb{R}^n$, we get

$$\text{stat}_{\text{RS}} \int_I f_k d\mathcal{G} = \lim_{\max_i |I_i| \to 0} \sum_{i=1}^{n} f_k(\xi_i) |\mathcal{G}(I_i)|.$$

Again, let $I \subseteq \mathbb{R}^n$ is infinite. Since $\mathcal{G}$ is a measurable increasing and continuous (piecewise) function. Thus, for every $\epsilon > 0$ there possibly exists a finite interval $\tilde{I}$ with $\tilde{I} \subseteq I \subseteq \mathbb{R}^n$ for which

$$\max\left(\sup_{I}\mathcal{G}-\sup_{\tilde{I}}\mathcal{G};\ \inf_{\tilde{I}}\mathcal{G}-\inf_{I}\mathcal{G}\right)<\ \epsilon.$$

Also, since (f_k) is bounded, we have

$$\sup_{I\backslash\tilde{I}}|f_k|\leqq\mathcal{H},$$

and this implies

$$\int_{I\backslash\tilde{I}}\mathcal{H}d\mathcal{G}(u)-\int_{I\backslash\tilde{I}}-\mathcal{H}d\mathcal{G}(u)\leqq 2\mathcal{H}\epsilon,$$

assuming finite number of points of I and $\tilde{I}$.

Successively, we choose the sequences of measurable step functions $\tilde{h}'_k=(-\mathcal{H},\ h'_k,\ \mathcal{H})$ and $\tilde{h}''_k=(-\mathcal{H},\ h'{}_k,\ \mathcal{H})$ such that $\tilde{h}'_k<(f_k)<\tilde{h}''_k$, that is, (f_k) is bounded over I, and as such

$$\int_I\tilde{h}''_k(u)d\mathcal{G}(u)-\int_I\tilde{h}'_k(u)d\mathcal{G}(u)=\int_{\tilde{I}}h''_k(u)d\mathcal{G}(u)-\int_{\tilde{I}}h'_k(u)d\mathcal{G}(u)$$

$$+\int_{I\backslash\tilde{I}}\mathcal{H}d\mathcal{G}(u)-\int_{I\backslash\tilde{I}}-\mathcal{H}d\mathcal{G}(u)$$

$$\leqq\epsilon|\mathcal{G}(\tilde{I})|+2\mathcal{H}\epsilon.$$

Hence,

$$\text{stat}_{\text{RS}}\int_I f_k d\mathcal{G}=\lim_{\max_i|I_i|\to 0}\sum_{i=1}^{n}f_k(\xi_i)|\mathcal{G}(I_i)|.$$

□

We next use Theorem 1 to adopt the following two special cases in the form of corollaries.

Corollary 1 Suppose that $\mathcal{G}:\mathbb{R}^n\to\mathbb{R}$ be a measurable increasing function on $I\subseteq\mathbb{R}^n$, with

$$\frac{\partial^n}{\partial_1,\ \ldots,\ \partial_n}\mathcal{G}=g,$$

exists and is continuous, and let (f_k) be the sequence of statistical Riemann-Stieltjes measureble functions. Then

$$\text{stat}_{\text{RS}} \lim_{k\to\infty} \int_I f_k(t) d\mathcal{G}(t) = \text{stat}_{\text{RS}} \int_I f_k g dt.$$

Proof. In view of statistical Riemann-Stieltjes sums

$$\text{stat}_{\text{RS}} \sum_{i=1}^{n} f_k(\xi_i)[\mathcal{G}(I_i)] = \text{stat}_{\text{RS}} \sum_{i=1}^{n} f_k(\xi_i) \frac{\partial^n}{\partial_1,\ \ldots,\ \partial_n} \mathcal{G}(\eta_i)|I_i|$$

$$= \text{stat}_{\text{RS}} \sum_{i=1}^{n} f_k(\xi_i) g(\eta_i)|I_i|$$

with $\eta_i \in I_i$.

Thus, the statistical Riemann-Stieltjes sums for the integral

$$\text{stat}_{\text{RS}} \int_I f_k g dt$$

with g being continuous, yields the sum function to converge to the integral

$$\text{stat}_{\text{RS}} \int_I f_k g dt.$$

□

Corollary 2 Let $\mathcal{G}$ be a piece-wise continuous measurable function on I with $\Delta_j \mathcal{G}(\gamma) = g_\gamma$, and if (f_k) is measurable, then

$$\text{stat}_{\text{RS}} \lim_{k\to\infty} \int_I f_k(t) d\mathcal{G}(t) = \text{stat}_{\text{RS}} \sum_{\gamma\in\mathcal{F}} f_k(\gamma) g_\gamma.$$

Proof. Let I_k be a finer partition. For each interval $I_k \in \mathbb{R}^n$, say $\gamma \in \mathcal{G}$. Suppose that $\mathcal{J}$ be the set of indices for such I_k's which involve exactly one point.

Consequently,

$$\Delta_j \mathcal{G}(I_j) = g_\gamma$$

for some $f_k \in \mathcal{G}$, $j \in \mathcal{J}$. On the other hand if, $j \notin \mathcal{J}$, then

$$\Delta_j \mathcal{G}(I_j) = 0.$$

So, if the partition is finer, the statistical Riemann-Stieltjes sum is

$$\text{stat}_{\text{RS}} \sum_{i=1}^{m} f_k(\xi_i)\Delta_j \mathcal{G}(I_j) = \text{stat}_{\text{RS}} \sum_{\gamma\in\mathcal{F}} f_k(\xi_\gamma) g_\gamma.$$

Hence,

$$\text{stat}_{\text{RS}} \lim_{k\to\infty} \int_I f_k(t) d\mathcal{G}(t) = \text{stat}_{\text{RS}} \sum_{\gamma\in\mathcal{F}} f_k(\gamma) g_\gamma.$$

□

The statistical versions of convergence in probability space is evidently more general than the statistical convergence. In the year 2011, Şençimen [20] considered random variable sequences and studied statistical probability convergence of such sequences and also proved some elementary theorems with numerical illustrative examples. Later on in the year 2015, Das et al. [21] considered the statistical probability convergence of order α and established certain valuable results. Recently, Srivastava et al. [22] proved some approximation theorems via certain aspects of statistical probability convergence. Subsequently, Jena et al. [23] considered the deferred Cesàro summability means and proved certain rtesults based on statistical convergence of random variables sequence. Also, Jena et al. [3] further used product means of probability convergence (statistical) and proved some approximation theorems.

In view of some advance studies in this direction, we here discuss the Riemann-Stieltjes sum over the probability sequence space in $\mathbb{R}^n$ and prove some elementary results for sequence of distribution functions. Moreover, we introduce the deferred Cesàro summability method for the Riemann-Stieltjes sum. Finally, we establish various inclusion theorems based on our proposed methods in association with the Riemann-Stieltjes sum for the sequence of usual functions as well as distribution functions in $\mathbb{R}^n$.

2. Riemann-Stieltjes sum over a probability space

The Riemann integral is the most easily defined integral, and it enables the integration of all continuous functions as well as a few reasonably discontinuous functions. Here, we introduce the Riemann-Stieltjes integral of a sequence of continuous function over a measurable probability space. We then present some useful definitions by considering certain statistical aspects of this integral and accordingly establish some elementary results.

Let $(X_n)_{n\in\mathbb{N}}$ be the random variables sequence over the probability space $(\Omega,\ \mathbb{F},\ \mathbb{P})$ with distribution functions $F_n : \mathbb{R}^n \to \mathbb{R}$ specified by

$$F_n(x) = \mathbb{P}\{\omega \in \Omega : X_n(\omega) \leqq x\} \quad (\forall\, x \in \mathbb{R}^n).$$

Definition 2 Let $(X_n)_{n\in\mathbb{N}}$ be the random variables sequence having distribution functions $(F_n(x))$. Then the expectation $E(X_n)$ is

$$E(X_n) = \int (x_1,\ x_2,\ \ldots,\ x_n) dF_n(x_1,\ x_2,\ \ldots,\ x_n).$$

Next, we present the statistical versions of Definition 2.

Definition 3 Let $E(X_n)$ be the expectation of (X_n). Then, for each $\varepsilon > 0$, we define the statistical expectation $E(X_n)$ as

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \text{ and } |E(X_n) - h| \geqq \varepsilon\}$$

having the natural (asymptotic) density zero (see [1] and [2]). Hence,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0,$$

and let we write it as

$$\text{stat}_{\text{SE}} \lim_{k\to\infty} E(X_n) = h.$$

Now we are capable to adopt the expectation $E(X_n)$ in the form of Riemann-Stieltjes integral, that is,

$$E(X_n;\, \rho) = \sum_{i=1}^{k} \rho_j^i \Delta_j \mathcal{G}(I_i), \quad (j = 1,\, 2,\, \dots,\, n) \tag{3}$$

where

$$x_1,\, \dots,\, x_{j-1},\, a_j^i,\, x_{j+1},\, \dots,\, x_n < \rho_j^i \leqq x_1,\, \dots,\, x_{j-1},\, b_j^i,\, x_{j+1},\, \dots,\, x_n.$$

Next, we present below the statistical versions of (3).

Definition 4 Let $F_n(x)$ be a sequence of distribution function, and let $E(X_n)$ be the expectation of (X_n). Then, for each $\varepsilon > 0$, we define the statistical Riemann-Stieltjes integral of $E(X_n)$ as

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \text{ and } |E(X_n;\, \rho) - h| \geqq \varepsilon\}$$

ensuring the natural (asymptotic) density zero (see [1] and [2]). Hence,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0,$$

and let we write it as

$$\text{stat}_{\text{ERS}} \lim_{k\to\infty} E(X_n;\, \rho) = h.$$

Now, we easily capable to derive the the following propositions from the earlier established Corollaries 1 and 2, and Definition 4.

Proposition 1 If $(\mathcal{G}_i)$ is differentiable with

$$\frac{\partial^n}{\partial_1,\ \dots,\ \partial_n}\mathcal{G} = g,$$

then

$$\text{stat}_{\text{ERS}} E(X_n) = \text{stat}_{\text{ERS}} \int x f_n(x) dx \quad (x \in \mathbb{R}^n).$$

Proposition 2 If $(\mathcal{G}_n)$ is a sequence of step functions with jump at (ρ_i), then

$$\begin{aligned} &\text{stat}_{\text{ERS}} E(X_n;\ \rho) \\ &= \text{stat}_{\text{ERS}} \sum_{i=1}^{k} \rho_j^i [\mathcal{G}(\rho_1,\ \dots,\ \rho_{j-1},\ b_j^i,\ \rho_{j+1},\ \dots,\ \rho_n) - \mathcal{G}(\rho_1,\ \dots,\ \rho_{j-1},\ a_j^i,\ \rho_{j+1},\ \dots,\ \rho_n)]. \end{aligned}$$

Let $f \in \mathbb{R}^n$ and

$$\int f(x_1,\ \dots,\ x_n) dF_n(x_1,\ \dots,\ x_n) < \infty$$

be such that

$$\{\omega : f(X_n(\omega)) \leqq u\},\ \ u \in \mathbb{R}^n.$$

We let designate the distribution function F_n of Y as

$$F_n(y) = \mathbb{P}\{\omega : Y_n(\omega) \leqq y\} \quad (y \in \mathbb{R}^n).$$

Then the expectation of (Y_n) is

$$E(Y_n) = \int (y_1,\ \dots,\ y_n) dF_n(y_1,\ \dots,\ y_n)$$

(exists and finite).

Next, we define below the statistical versions of $E(Y_n)$.

Definition 5 Let $E(Y_n)$ be the expectation of (Y_n). Then, for each $\varepsilon > 0$, we define the statistical expectation of $E(Y_n)$ as

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \text{ and } |E(Y_n) - h| \geqq \varepsilon\}$$

having natural (asymptotic) density zero (see [1] and [2]). Hence,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0,$$

and let we write it as

$$\text{stat}_{\text{SE}'} \lim_{k\to\infty} E(Y_n) = h.$$

Now we reform the expectation $E(Y_n)$ in the form of Riemann-Stieltjes integral,

$$E(Y_n;\, \rho) = \sum_{i=1}^{k} \rho_j^i \Delta_j \mathcal{G}(I_i) \tag{4}$$

where

$$y_1,\, \dots,\, y_{j-1},\, a_j^i,\, y_{j+1},\, \dots,\, y_n < \rho_j^i \leqq y_1,\, \dots,\, y_{j-1},\, b_j^i,\, y_{j+1},\, \dots,\, y_n.$$

Next, we present the statistical versions of (4).

Definition 6 Let $(F_n(y))$ be a sequence of distribution function, and let $E(Y_n)$ be the expectation of (Y_n). Then, for each $\varepsilon > 0$, we define the statistical Riemann-Stieltjes integral of $E(Y_n)$ as

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \text{ and } |E(Y_n;\, \rho) - h| \geqq \varepsilon\}$$

ensuring the natural (asymptotic) density zero (see [1] and [2]). Hence,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0,$$

and let we write it as

$$\text{stat}_{\text{E}'\text{RS}} \lim_{k\to\infty} E(Y_n;\, \rho) = h.$$

In view of Definition 6, we establish a theorem as mentioned below.

Theorem 2 Let X_n be a sequence of random variables associated with distribution functions $F_n(x)$ $(x \in \mathbb{R}^n)$, and let $f \in \mathbb{R}^n$. Then the of random variables sequence $(Y_n) = f(X_n)$ has the statistical expectation

$$\mathrm{stat}_{\mathrm{SE'}} E(Y_n) = \mathrm{stat}_{\mathrm{SE'}} \int f(x_1, \ldots, x_n) dF_n(x_1, \ldots, x_n).$$

Proof. Following the Riemann-Stieltjes integral,

$$\mathrm{stat}_{\mathrm{E'RS}} \sum_{i=1}^{k} \rho_j^i \Delta_j \mathcal{G}(I_i) = \mathrm{stat}_{\mathrm{E'RS}} \sum_{i=1}^{n} \rho_i \mathbb{P}(Y_n \in (y_{i-1}, y_i])$$

$$= \mathrm{stat}_{\mathrm{E'RS}} \sum_{i=1}^{n} \rho_i \mathbb{P}(f(X_n) \in (y_{i-1}, y_i])$$

$$= \mathrm{stat}_{\mathrm{E'RS}} \sum_{i=1}^{n} \rho_i \mathbb{P}(X_n \in f^{-1}(y_{i-1}, y_i]),$$

where $\rho_i \in (y_{i-1}, y_i)]$. Also, recall that

$$\rho_i \in (y_{i-1}, y_i)] \Longleftrightarrow \eta_i = f^{-1}(\rho_i) \in f^{-1}\{(y_{i-1}, y_i)]\}$$

$$\Longleftrightarrow f(\eta_i) \in (y_{i-1}, y_i)].$$

Consequently, we have

$$\mathrm{stat}_{\mathrm{SE'}} \sum_{i=1}^{n} f(\eta_i) \mathbb{P}(X_n \in f^{-1}(y_{i-1}, y_i]) \tag{5}$$

with $\eta_i \in f^{-1}\{(y_{i-1}, y_i)\}$.

Next, if $(y_{i-1}, y_i]$ patterns a partition, then so also in $\mathbb{R}^n$ the intervals $I_i = f^{-1}\{(y_{i-1}, y_i]\}$ pattern a partition.

Thus, (5) can be written as

$$\mathrm{stat}_{\mathrm{SE'}} \sum_{i=1}^{n} f(\eta_i) \mathbb{P}(X_n \in I_i), \tag{6}$$

where $\eta_i \in (x_{i-1}, x_i)$, and that the Riemann-Stieltjes sum. □

3. Deferred Cesàro Riemann-Stieltjes sum

It is well known that nearly all of the transformation techniques used in the summability theory have many undesirable characteristics. In particular, the Cesàro summability technique of any given positive order having usual bounds and oscillations usually does not always preserve continuous convergence or convergence in uniform sense. However, the modified Cesàro transformation technique (or deferred Cesàro summability technique) has very useful properties as regards to uniform convergence of sequence of functions. In this section, we consider the deferred Cesàro mean to discuss the statistical aspects of Riemann-Stieltjes ($\mathrm{DCRS_{stat}}$) integrability as well as Riemann-Stieltjes ($\mathrm{stat_{DCRS}}$) summability over a measurable probaility space.

Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$ with $\phi_k < \varphi_k$ and $\lim_{k\to\infty} \varphi_k = +\infty$. We define the deferred Cesàro summability (DCS) mean [24] of the Riemann-Stieltjes sum $\delta(f_k;\, \dot{\mathcal{P}})$ of (f_k) allied with tagged partition $\dot{\mathcal{P}}$ of the form

$$\mathcal{E}_k = \frac{1}{(\varphi_k - \phi_k)} \sum_{\lambda=\phi_k+1}^{\varphi_k} \delta(f_\lambda;\, \dot{\mathcal{P}}). \tag{7}$$

We now present two definitions by using the DCS mean.

Definition 7 Let $\mathcal{G}$ be a measurable increasing function, and let (f_k) be the measurable functions defined over the interval $I \subseteq \mathbb{R}^n$. The given sequence (f_k) of functions (measurable) is deferred Cesàro statistically Riemann-Stieltjes ($\mathrm{DCRS_{stat}}$) integrable to a function (measurable) f with respect to $\mathcal{G}$, if for all $\epsilon > 0$, $\exists$ measurable step functions h'_k and h''_k with $h'_k < f_k < h''_k$ such that

$$\int_I h''_k(u)d\mathcal{F}(u) - \int_I h'_k(u)d\mathcal{F}(u) < \epsilon,$$

and the set

$$\{\zeta : \phi_k < \zeta \leqq \varphi_k \text{ and } |\delta(f_\zeta;\, \dot{\mathcal{P}}) - f| \geqq \epsilon\}$$

ensures natural (asymptotic) density zero. That is,

$$\lim_{k\to\infty} \frac{|\{\zeta : \phi_k < \zeta \leqq \varphi_k \text{ and } |\delta(f_\zeta;\, \dot{\mathcal{P}}) - f| \geqq \epsilon\}|}{(\varphi_k - \phi_k)} = 0.$$

We write

$$\mathrm{DCRS_{stat}} \lim_{k\to\infty} \delta(f_k;\, \dot{\mathcal{P}}) = f.$$

Definition 8 Let $\mathcal{G}$ be a measurable increasing function, and let the sequence of measurable functions (f_k) be a defined over the interval $I \subseteq \mathbb{R}^n$. The given sequence (f_k) of measurable functions is statistically deferred Cesàro Riemann-Stieltjes ($\mathrm{stat_{DCRS}}$) summable to a measurable f with respect to $\mathcal{G}$, if for all $\epsilon > 0$, $\exists$ measurable step functions h'_k and h''_k with $h'_k < f_k < h''_k$ such that

$$\int_I h''_k(u)d\mathcal{F}(u) - \int_I h'_k(u)d\mathcal{F}(u) < \epsilon,$$

and the set

$$\{\zeta : \phi_k < \zeta \leqq \varphi_k \text{ and } |\mathcal{E}_\zeta - f| \geqq \epsilon\}$$

ensures natural (asymptotic) density zero. That is,

$$\lim_{k\to\infty} \frac{|\{\zeta : \phi_k < \zeta \leqq \varphi_k \text{ and } |\mathcal{E}_\zeta - f| \geqq \epsilon\}|}{(\varphi_k - \phi_k)} = 0.$$

We write

$$\text{stat}_{\text{DCRS}} \lim_{k\to\infty} \mathcal{E}_k = f.$$

Now we wish to connect the above definitions by proving a theorem as follows.

Theorem 3 The deferred Cesàro statistically Riemann-Stieltjes $(\text{DCRS}_{\text{stat}})$ integrability implies the statistically deferred Cesàro Riemann-Stieltjes $(\text{stat}_{\text{DCRS}})$ summability of a sequence $(f_k)_{k\in\mathbb{N}}$ of measurable functions in $\mathbb{R}^n$, and also the limiting function is unique. However, the converse statement is not necessarily true.

Proof. Suppose $(f_k)_{k\in\mathbb{N}}$ is $(\text{DCRS}_{\text{stat}})$ integrable to f in $\mathbb{R}^n$, then under our Definition 7, we have

$$\lim_{k\to\infty} \frac{|\{\zeta : \phi_k < \zeta \leqq \varphi_k \text{ and } |\delta(f_\zeta;\, \dot{\mathcal{P}}) - f| \geqq \epsilon\}|}{(\varphi_k - \phi_k)} = 0.$$

Now under the assumption of the below-mentioned two sets,

$$\mathcal{R}_\epsilon = \{\zeta : \phi_k < \zeta \leqq \varphi_k \text{ and } |\delta(f_\zeta;\, \dot{\mathcal{P}}) - f| \geqq \epsilon\}$$

and

$$\mathcal{R}_\epsilon^c = \{\zeta : \phi_k < \zeta \leqq \varphi_k \text{ and } |\delta(f_\zeta;\, \dot{\mathcal{P}}) - f| \geqq \epsilon\},$$

we have

$$
\begin{aligned}
|\mathcal{E}_k - f| &= \left| \frac{1}{(\varphi_k - \phi_k)} \sum_{\lambda=\phi_k+1}^{\varphi_k} \delta(f_\lambda;\, \dot{\mathcal{P}}) - f \right| \\
&\leqq \left| \frac{1}{(\varphi_k - \phi_k)} \sum_{\lambda=\phi_k+1}^{\varphi_k} \left[\delta(f_\lambda;\, \dot{\mathcal{P}}) - f\right] \right| + \left| \frac{1}{(\varphi_k - \phi_k)} \sum_{\lambda=\phi_k+1}^{\varphi_k} f - f \right| \\
&\leqq \frac{1}{(\varphi_k - \phi_k)} \sum_{\substack{\lambda=\phi_k+1 \\ (\zeta \in \mathcal{R}_\epsilon)}}^{\varphi_k} \left|\delta(f_\lambda;\, \dot{\mathcal{P}}) - f\right| + \frac{1}{(\varphi_k - \phi_k)} \sum_{\substack{\lambda=\phi_k+1 \\ (\zeta \in \mathcal{R}_\epsilon^c)}}^{\varphi_k} \left|\delta(f_\varrho;\, \dot{\mathcal{P}}) - f\right| \\
&\quad + |f| \left| \frac{1}{(\varphi_k - \phi_k)} \sum_{\lambda=\phi_k+1}^{\varphi_k} - 1 \right| \\
&\leqq \frac{1}{(\varphi_k - \phi_k)} |\mathcal{R}_\epsilon| + \frac{1}{(\varphi_k - \phi_k)} |\mathcal{R}_\epsilon^c| = 0.
\end{aligned}
$$

This implies that

$$
|\mathcal{E}_k - f| < \epsilon.
$$

Hence, (f_k) is $(\text{stat}_{\text{DCRS}})$ summable to f in $\mathbb{R}^n$.

Next, for the converse statement, not to valid we present the example as below:

Example 1 Let $\phi_k = 2k$, $\varphi_k = 4k$, and let $f_k : I \subseteq \mathbb{R}^n \to \mathbb{R}$ be a measurable functions sequence such taht

$$
f_k(x) = \begin{cases} -1 & (x \in \mathbb{Q} \cap I;\ \ k = \text{even}) \\ \\ 1 & (x \in \mathbb{R} - \mathbb{Q} \cap I;\ \ k = \text{odd}). \end{cases} \tag{8}
$$

The given measurable functions (f_k) trivially reveals that, this is neither Riemann-Stieltjes integrable nor $(\text{DCRS}_{\text{stat}})$ integrable. But, based on our mean as mentioned in (7), it is quite easy to observe that, the measurable functions (f_k) has deferred Cesàro Riemann-Stieltjes sum $\frac{1}{2}$ allied with the tagged partition $\dot{\mathcal{P}}$. Hence, the measurable functions (f_k) is $(\text{stat}_{\text{DCRS}})$ summable to $\frac{1}{2}$ over I while it is not $(\text{DCRS}_{\text{stat}})$ integrable. □

Similarly, we easily make the two definitions for a sequence of distribution functions via the deferred Cesàro mean.

Definition 9 Let $(F_n(x))$ be a sequence of distribution function, and let $E(X_n)$ be the expectation of (X_n). Then, for each $\varepsilon > 0$, $E(X_n)$ is deferred Cesàro statistically Riemann-Stieltjes $(\text{DCERS}_{\text{stat}})$ integrable to f, if

$$
\{\zeta : \phi_k < \zeta \leqq \varphi_k \ \text{ and } \ |\delta(E(X_\zeta;\ \zeta);\ \dot{\mathcal{P}}) - f| \geqq \epsilon\}
$$

ensures natural (asymptotic) density zero. That is,

$$\lim_{k\to\infty} \frac{|\{\zeta : \phi_k < \zeta \leqq \varphi_k \text{ and } |\delta(E(X_\zeta;\ \zeta);\ \dot{\mathcal{P}}) - f| \geqq \epsilon\}|}{(\varphi_k - \phi_k)} = 0.$$

We write

$$\text{DCERS}_{\text{stat}} \lim_{k\to\infty} \delta(E(X_k;\ k);\ \dot{\mathcal{P}}) = f.$$

Definition 10 Let $(F_n(x))$ be a sequence of distribution function, and let $E(X_n)$ be the expectation of (X_n). Then, for each $\varepsilon > 0$, $E(X_n)$ is statistically deferred Cesàro Riemann-Stieltjes $(\text{stat}_{\text{DCERS}})$ summable to f, if

$$\{\zeta : \phi_k < \zeta \leqq \varphi_k \text{ and } |\mathcal{E}_n(E(X_\zeta;\ \zeta) - f| \geqq \epsilon\}$$

ensures natural (asymptotic) density zero. That is,

$$\lim_{k\to\infty} \frac{|\{\zeta : \phi_k < \zeta \leqq \varphi_k \text{ and } |\mathcal{E}_\zeta(E(X_\zeta;\ \zeta) - f| \geqq \epsilon\}|}{(\varphi_k - \phi_k)} = 0.$$

We write

$$\text{stat}_{\text{DCERS}} \lim_{k\to\infty} \mathcal{E}_k(E(X_k;\ k) = f.$$

Now, we wish to connect the above two notions via the following theorem.

Theorem 4 If $(F_n(x))$ be the distribution functions, and let $E(X_n)$ be the expectation of (X_n), then $E(X_n)$ is deferred Cesàro statistically Riemann-Stieltjes $(\text{DCERS}_{\text{stat}})$ integrable to f in $\mathbb{R}^n$ implies, it is statistically deferred Cesàro Riemann-Stieltjes $(\text{stat}_{\text{DCERS}})$ summable to the same function f in $\mathbb{R}^n$, but the converse statement is not generally true.

Proof. As the proof of Theorem 4 can be done in the similar lines of our above proved Theorem 3, so, we plump for omitting the details. □

4. Concluding remarks

Through this study, we have preluded the conception of statistical Riemann-Stieltjes sum on the sequence space via the deferred Cesàro summability mean and established some fundamental limit theorems. Next, considering the probability space, we also established some basic new results based on the Riemann-Stieltjes integral for the sequence of distribution functions. Finally, over both the spaces we established some inclusion theorems via our proposed deferred Cesàro summability means associated with statistical Riemann-Stieltjes integral for the sequence of functions as well as the sequence of distribution functions.

Many researchers have considered different summability means on the sequence spaces to prove several approximation results. A list of some articles has been mentioned in the references. Further, combining the existing ideas and direction of the sequence spaces associated with our proposed mean, many new Korvokin-type approximation theorems can be proved under different settings of algebraic and trigonometric functions.

Influenced by a recently published articles by Jena et al. [25], we extract the cognizance of the interested learner's concerning the possibilities of establishing some Korvokin-type approximation theorems over the sequence space as well as the probability space. Also, in view of a latest result of Baliarsingh [15] and Hazarika et al. [26] the consciousness of the curious readers is drawn out for future researches pertaining to fuzzy approximation theorems.

Authors contributions

All the authors contributed equally and significantly in writing this paper.

Conflict of interest

The authors declare that they have no competing interests.

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