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Research Article

Semi-analytical Approach to Nonlinear Partial Differential Equations Using Homotopy Analysis Technique (HAM)

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Abstract: This work considers a novel semi-analytical method named the homotopy analysis method (HAM) to study the nonlinear gas dynamic equation. The obtained HAM solution is validated by comparing it with the exact available solution and compared with the (Adomian decomposition method) ADM solution and numerical solution to test the efficiency of the proposed method. The efficiency of the proposed approach can be demonstrated numerically and graphically, and it is found to be in excellent agreement with the current approach.

***Keywords*:** semi-analytical method, gas dynamic equation, nonlinear partial differential equation, homotopy analysis method (HAM)

MSC: 35-00, 35F25, 76N15

1. Introduction

Almost all problems encountered in physics, chemistry, biology, and engineering science are practically nonlinear and are explained by nonlinear partial differential equations. The nonlinear partial differential equation is particularly challenging to accurately solve. There are numerous numerical methods, such as the Euler method [1], the Runge-Kutta method [2], and analytical techniques, such as the variational iteration method [3], Fourier analysis [4], perturbation method [5], and the Adomian decomposition method (ADM), available for solving nonlinear partial differential equations. The analytical method, which is more time-consuming and occasionally impractical, provides precise answers. While numerical approaches offer an approximation of the answer with a tolerance that is tolerable, take less time, and are generally feasible, but numerical methods fail to provide solution curves and give discrete points.

Liao [6, 7] of Shanghai Jiaotong University introduced a new semi-analytical technique, homotopy analysis method (HAM), to solve nonlinear partial and ordinary differential equations in his PhD dissertation in 1992 and added a non-zero auxiliary parameter, h, referred to as the convergence control parameter, in 1997. It is applicable for both strongly and weakly nonlinear problems. The concept of homotopy from topology is used in the homotopy analysis approach to produce a convergent series solution for nonlinear systems. No matter if a given nonlinear problem's governing equations, boundary conditions, and initial conditions contain small or large quantities, the homotopy analysis approach may be used to solve them. HAM is widely used for solving many partial differential equations such as the Benjamin-Bona-Mahony-Burgers (BBMB) equation [8], the Jaffery-Hamel fluid model [9], the Cauchy reaction-diffusion problem

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[10], the Klein-Gordon equation [11], the heat transfer problem [12], etc. [13, 14]. The method is very useful to find the solution while dealing with nonlinearity.

It is well recognized that the conservation rules found in engineering procedures, such as the conservation of mass, conservation of momentum, conservation of energy, etc., are mathematically expressed in the equation of gas dynamics. The three different types of nonlinear waves, such as shock fronts, rarefactions, and contact discontinuities, can all be described by the nonlinear equations of ideal gas dynamics. The nonlinear flux vectors of the inviscid gas dynamic equation are one-degree homogeneous functions, which allows for the dividing of flux vectors into sub-vectors by similarity transformations. Steger and Warming [15] addressed this remarkable property in 1981. Many numerical and analytical methods, such as the differential transform method [16], ADM [17], etc., have been employed to solve the gas dynamic equation.

The objective of the current study is to use a novel semi-analytic approach, i.e., HAM, to study the nonlinear gas dynamic equation and test the proposed method's efficiency by comparing it with the available numerical solutions. The efficiency of the proposed approach can be demonstrated with the help of the convergence control parameter, which gives a better solution than other analytical/numerical techniques. Finally, the obtained numerical solutions are found to be in good agreement with the current process that has been demonstrated numerically and graphically.

2. Analysis of HAM

Let us consider the following nonlinear equations:

$$\mathcal{N}_i[\omega(\varsigma,\kappa)] = 0, \;\; i = 1, 2, \ldots, n, \tag{1}$$

where $\mathcal{N}_i$ is a nonlinear operator that represents the whole system of equations, ς is a spatial variable, independent variables are denoted by κ, $\omega_i(\varsigma,\kappa)$, are unknown functions.

According to the concept of homotopy, the so-called zero-order deformation equation is

$$(1-\alpha)\mathcal{L}[\Psi_i(\varsigma,\kappa;\alpha) - \omega_{i,0}(\varsigma,\kappa)] = h_i\alpha\mathcal{N}_i[\Psi_i(\varsigma,\kappa;\alpha)], \tag{2}$$

where $h_i \neq 0$ are an auxiliary parameter, $\mathcal{L}$ is an auxiliary linear operator, $\alpha \in [0,1]$ is the embedding parameter, $\Psi_i(\varsigma,\kappa;\alpha)$ are unknown function, and $\omega_{i,0}(\varsigma,\kappa)$ is an initial approximation of $\omega_i(\varsigma,\kappa)$.

Since $\alpha = 0$ and $\alpha = 1$, it holds

$$\Psi_i(\varsigma,\kappa;0) = \omega_{i,0}(\varsigma,\kappa), \;\; \Psi_i(\varsigma,\kappa;1) = \omega_i(\varsigma,\kappa), \tag{3}$$

respectively. As a result, α changes from 0 to 1, $\Psi_i(\varsigma,\kappa;\alpha)$ changes from the initial approximation $\omega_{i,0}(\varsigma,\kappa)$ to the solution $\omega_i(\varsigma,\kappa)$. Now, expand $\Psi_i(\varsigma,\kappa;\alpha)$ in Taylor series with respect to the parameter α, one has

$$\Psi_i(\varsigma,\kappa;\alpha) = \omega_{i,0}(\varsigma,\kappa) + \sum_{m=1}^{\infty}\omega_{i,m}(\varsigma,\kappa)\alpha^m, \tag{4}$$

where

$$\omega_{i,m}(\varsigma,\kappa) = \frac{1}{m!}\frac{\partial^m\Psi_i(\varsigma,\kappa;\alpha)}{\partial\alpha^m}\bigg|_{\alpha=0}. \tag{5}$$

The above series will converge at $\alpha = 1$ if the auxiliary linear operator, initial approximation $\omega_i(\varsigma,\kappa)$, auxiliary parameter h_i, and auxiliary function are all suitably selected, and one has

$$\omega_i(\varsigma,\kappa)=\omega_{i,0}(\varsigma,\kappa)+\sum_{m=1}^{\infty}\omega_{i,m}(\varsigma,\kappa). \tag{6}$$

It must be one of the solutions of original nonlinear equations. Define vector,

$$\bar{\omega}(\varsigma,\kappa)=\{\omega_{i,0}(\varsigma,\kappa),\omega_{i,1}(\varsigma,\kappa),\omega_{i,2}(\varsigma,\kappa),...,\omega_{i,m}(\varsigma,\kappa)\}. \tag{7}$$

By placing $\alpha=0$ and dividing with $m!$, the following mth-order deformation equation can be constructed by differentiating the zero-order deformation equation m-times with regard to the embedding parameter q.

$$\mathcal{L}[\omega_{i,m}(\varsigma,\kappa)-\lambda_m\omega_{i,m-1}(\varsigma,\kappa)]=h_i\Re_{i,m}(\bar{\omega}_{i,m-1}), \tag{8}$$

where $\Re_{i,m}(\bar{\omega}_{i,m-1})$ stands for

$$\Re_{i,m}(\bar{\omega}_{i,m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}_i[\Psi_i(\varsigma,\kappa;\alpha)]}{\partial\alpha^{m-1}}\bigg|_{\alpha=0} \tag{9}$$

and

$$\lambda_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases} \tag{10}$$

3. Numerical application

3.1 *Homogeneous gas dynamics equation*

Consider the homogeneous gas dynamics equation [17]

$$\frac{\partial \mathrm{u}(\varsigma,\kappa)}{\partial\kappa}+\mathrm{u}\frac{\partial \mathrm{u}(\varsigma,\kappa)}{\partial\varsigma}-\mathrm{u}(\varsigma,\kappa)\{1-\mathrm{u}(\varsigma,\kappa)\}=0, \tag{11}$$

with initial condition

$$\mathrm{u}(\varsigma,0)=e^{-\varsigma}. \tag{12}$$

$\mathrm{u}(\varsigma,\kappa)=e^{\kappa-\varsigma}$ is exact solution for this problem equation (11).

The initial approximation for the HAM to solve equations (11) and (12) is

$$\mathrm{u}_0(\varsigma,\kappa)=\mathrm{u}(\varsigma,0)=e^{-\varsigma}. \tag{13}$$

For equation (11), the nonlinear operator can be defined as

$$\mathcal{N}[\Psi(\varsigma,\kappa;\alpha)]=\frac{\partial\Psi(\varsigma,\kappa;\alpha)}{\partial\kappa}+\Psi(\varsigma,\kappa;\alpha)\frac{\partial\Psi(\varsigma,\kappa;\alpha)}{\partial\varsigma}-\Psi(\varsigma,\kappa;\alpha)\{1-\Psi(\varsigma,\kappa;\alpha)\}. \tag{14}$$

Constructing zero-order deformation equation

$$(1-\alpha)\mathcal{L}[\Psi(\varsigma,\kappa;\alpha)-\mathrm{u}_0(\varsigma,\kappa)]=h\alpha\mathcal{N}[\Psi(\varsigma,\kappa;\alpha)]. \tag{15}$$

Here, $\mathcal{L}$ is a linear operator, defined as

$$\mathcal{L}[\Psi(\varsigma,\kappa;\alpha)] = \frac{\partial \Psi(\varsigma,\kappa;\alpha)}{\partial \kappa}, \tag{16}$$

$$\mathcal{L}[c] = 0, \tag{17}$$

where, c is integral constant.

When $\alpha = 0$ and $\alpha = 1$,

$$\Psi(\varsigma,\kappa;0) = \mathrm{u}_0(\varsigma,\kappa),\ \ \Psi(\varsigma,\kappa;1) = \mathrm{u}(\varsigma,\kappa). \tag{18}$$

As a result, as α changes from 0 to 1, $\Psi(\varsigma,\kappa;\alpha)$ varies from initial approximation $\mathrm{u}_0(\varsigma,\kappa) = e^{-\varsigma}$ to the solution $\mathrm{u}(\varsigma,\kappa)$. Now, expand $\Psi(\varsigma,\kappa;\alpha)$ with respect to parameter α in Taylor series, then one has

$$\Psi(\varsigma,\kappa;\alpha) = \mathrm{u}_0(\varsigma,\kappa) + \sum_{m=1}^{\infty} \mathrm{u}_m(\varsigma,\kappa)\alpha^m, \tag{19}$$

where

$$\mathrm{u}_m(\varsigma,\kappa) = \frac{1}{m!}\frac{\partial^m \Psi(\varsigma,\kappa;\alpha)}{\partial \alpha^m}\bigg|_{\alpha=0}. \tag{20}$$

The above series converges at $\alpha = 1$ when the initial approximation $\Psi(\varsigma,\kappa;\alpha)$ and the auxiliary parameter h are carefully chosen,

$$\Psi(\varsigma,\kappa;1) = \mathrm{u}(\varsigma,\kappa) = \mathrm{u}_0(\varsigma,\kappa) + \sum_{m=1}^{\infty} \mathrm{u}_m(\varsigma,\kappa). \tag{21}$$

Above equation (21) is one of the solutions of our differential equation. Now, defining the vectors

$$\overline{\mathrm{u}}(\varsigma,\kappa) = \{\mathrm{u}_0(\varsigma,\kappa), \mathrm{u}_1(\varsigma,\kappa), \mathrm{u}_2(\varsigma,\kappa), \ldots, \mathrm{u}_m(\varsigma,\kappa)\}. \tag{22}$$

By placing α = 0 and dividing with m!, the following mth-order deformation equation can be constructed by differentiating equation (15) m-times with regard to the embedding parameter α.

$$\mathcal{L}[\mathrm{u}_m(\varsigma,\kappa) - \lambda_m \mathrm{u}_{m-1}(\varsigma,\kappa)] = h\Re_m(\overline{\mathrm{u}}_{m-1}), \tag{23}$$

where

$$\Re_m(\overline{\mathrm{u}}_{m-1}) = \frac{\partial \mathrm{u}_{m-1}}{\partial \kappa} + \sum_{i=0}^{m-1} \mathrm{u}_i \frac{\partial \mathrm{u}_{m-1-i}}{\partial \varsigma} - \mathrm{u}_{m-1} + \sum_{i=0}^{m-1} \mathrm{u}_i \mathrm{u}_{m-1-i}. \tag{24}$$

The solution of equation (23) becomes

$$\mathrm{u}_m(\varsigma,\kappa) = \lambda_m \mathrm{u}_{m-1}(\varsigma,\kappa) + h\mathcal{L}^{-1}[\Re_m(\overline{\mathrm{u}}_{m-1})]. \tag{25}$$

As $\mathrm{u}_0(\varsigma,\kappa) = e^{-\varsigma}$,we get

$$\mathrm{u}_1(\varsigma,\kappa) \ = -h\kappa e^{-\varsigma} \tag{26}$$

$$\mathrm{u}_2(\varsigma,\kappa) \ = -h\kappa e^{-\varsigma} - h^2\kappa e^{-\varsigma} + \frac{1}{2}h^2\kappa^2 e^{-\varsigma} \tag{27}$$

$$u_3(\varsigma,\kappa) = -h\kappa e^{-\varsigma} - h^2\kappa e^{-\varsigma} + \frac{1}{2}h^2\kappa^2 e^{-\varsigma} - \frac{1}{6}(h^2\kappa e^{-\varsigma}(6h - 3\kappa - 6h\kappa + h\kappa^2 + 6)) \tag{28}$$

$$\begin{aligned} u_4(\varsigma,\kappa) &= -h\kappa e^{-\varsigma} - h^2\kappa e^{-\varsigma} + \frac{1}{2}h^2\kappa^2 e^{-\varsigma} - \frac{1}{6}(h^2\kappa e^{-\varsigma}(6h - 3\kappa - 6h\kappa + h\kappa^2 + 6)) \\ &- \frac{1}{24}(h^2\kappa e^{-\varsigma}(-h^2\kappa^3 + 12h^2\kappa^2 - 36h^2\kappa + 24h^2 + 8h\kappa^2 - 48h\kappa + 48h - 12\kappa + 24)) \end{aligned} \tag{29}$$

Series solution expression can be written as

$$\begin{aligned} &u(\varsigma,\kappa) = \sum_{m=0}^{\infty} u_m(\varsigma,\kappa) \\ &= e^{-\varsigma} - h\kappa e^{-\varsigma} - h\kappa e^{-\varsigma} - h^2\kappa e^{-\varsigma} + \frac{1}{2}h^2\kappa^2 e^{-\varsigma} - h\kappa e^{-\varsigma} - h^2\kappa e^{-\varsigma} + \frac{1}{2}h^2\kappa^2 e^{-\varsigma} \\ &- \frac{1}{6}(h^2\kappa e^{-\varsigma}(6h - 3\kappa - 6h\kappa + h\kappa^2 + 6)) - h\kappa e^{-\varsigma} - h^2\kappa e^{-\varsigma} + \frac{1}{2}h^2\kappa^2 e^{-\varsigma} \\ &- \frac{1}{6}(h^2\kappa e^{-\varsigma}(6h - 3\kappa - 6h\kappa + h\kappa^2 + 6)) \\ &- \frac{1}{24}(h^2\kappa e^{-\varsigma}(-h^2\kappa^3 + 12h^2\kappa^2 - 36h^2\kappa + 24h^2 + 8h\kappa^2 - 48h\kappa + 48h - 12\kappa + 24)) \ldots\ldots \end{aligned} \tag{30}$$

This is approximate series solution of homogeneous gas dynamic equation by HAM.

Putting $h = -1$, we can construct following series as $m \to \infty$:

$$\begin{aligned} u(\varsigma,\kappa) &= e^{-\varsigma}[1 + 2\kappa - \kappa + \frac{\kappa^2}{2} + \frac{\kappa^3}{6} + \frac{\kappa^4}{24} + \ldots\ldots] \\ &= e^{-\varsigma}[1 + \kappa + \frac{\kappa^2}{2!} + \frac{\kappa^3}{3!} + \frac{\kappa^4}{4!} + \ldots\ldots] \\ &= e^{\kappa-\varsigma}, \end{aligned}$$

which is the exact solution of homogeneous gas dynamics equation.

3.2 *Non-homogeneous gas dynamics equation*

Consider non-homogeneous gas dynamics equation [17]

$$\frac{\partial u(\varsigma,\kappa)}{\partial \kappa} + u\frac{\partial u(\varsigma,\kappa)}{\partial \varsigma} - u(\varsigma,\kappa)\{1 - u(\varsigma,\kappa)\} = -e^{\kappa-\varsigma} \tag{31}$$

with initial condition

$$u(\varsigma,0) = 1 - e^{-\varsigma}. \tag{32}$$

$u(\varsigma,\kappa) = 1 - e^{\kappa-\varsigma}$ is the exact solution for this problem equation (31).

The initial approximation for the HAM to solve equations (31) and (32) is

$$u_0(\varsigma,\kappa) = u(\varsigma,0) = 1 - e^{-\varsigma}. \tag{33}$$

For equation (31), nonlinear operator can be defined as

$$\mathcal{N}[\Psi(\varsigma,\kappa;\alpha)] = \frac{\partial \Psi(\varsigma,\kappa;\alpha)}{\partial \kappa} + \Psi(\varsigma,\kappa;\alpha)\frac{\partial \Psi(\varsigma,\kappa;\alpha)}{\partial \varsigma} - \Psi(\varsigma,\kappa;\alpha)\{1-\Psi(\varsigma,\kappa;\alpha)\} + e^{\kappa-\varsigma}. \tag{34}$$

Constructing zero-order deformation equation

$$(1-\alpha)\mathcal{L}[\Psi(\varsigma,\kappa;\alpha) - \mathrm{u}_0(\varsigma,\kappa)] = h\alpha\mathcal{N}[\Psi(\varsigma,\kappa;\alpha)]. \tag{35}$$

Here, $\mathcal{L}$ is a linear operator, defined as

$$\mathcal{L}[\Psi(\varsigma,\kappa;\alpha)] = \frac{\partial \Psi(\varsigma,\kappa;\alpha)}{\partial \kappa}, \tag{36}$$

with the condition

$$\mathcal{L}[c] = 0, \tag{37}$$

where, c is an integral constant.

When $\alpha = 0$ and $\alpha = 1$,

$$\Psi(\varsigma,\kappa;0) = \mathrm{u}_0(\varsigma,\kappa), \;\; \Psi(\varsigma,\kappa;1) = \mathrm{u}(\varsigma,\kappa).$$

As a result, α changes from 0 to 1, $\Psi(\varsigma,\kappa;\alpha)$ changes from initial approximation $\mathrm{u}_0(\varsigma,\kappa) = 1 - e^{-\varsigma}$ to the solution $\mathrm{u}(\varsigma,\kappa)$. Now, expand $\Psi(\varsigma,\kappa;\alpha)$ with respect to parameter α in Taylor series, then one has

$$\Psi(\varsigma,\kappa;\alpha) = \mathrm{u}_0(\varsigma,\kappa) + \sum_{m=1}^{\infty} \mathrm{u}_m(\varsigma,\kappa)\alpha^m, \tag{39}$$

where

$$\mathrm{u}_m(\varsigma,\kappa) = \frac{1}{m!}\frac{\partial^m \Psi(\varsigma,\kappa;\alpha)}{\partial \alpha^m}\bigg|_{\alpha=0}. \tag{40}$$

The above series will converge at $\alpha = 1$ if the auxiliary linear operator, initial approximation $\mathrm{u}_0(\varsigma,\kappa)$, auxiliary parameter h_i, and auxiliary function are all suitably selected,

$$\Psi(\varsigma,\kappa;1) = \mathrm{u}(\varsigma,\kappa) = \mathrm{u}_0(\varsigma,\kappa) + \sum_{m=1}^{\infty} \mathrm{u}_m(\varsigma,\kappa). \tag{41}$$

Above equation (41) is one of the solutions of our differential equation. Now, defining the vectors

$$\overline{\mathrm{u}}(\varsigma,\kappa) = \{\mathrm{u}_0(\varsigma,\kappa), \mathrm{u}_1(\varsigma,\kappa), \mathrm{u}_2(\varsigma,\kappa), \ldots, \mathrm{u}_m(\varsigma,\kappa)\}. \tag{42}$$

By placing $\alpha = 0$ and dividing with $m!$, the following mth-order deformation equation can be constructed by differentiating equation (35) m-times with regard to the embedding parameter α.

$$\mathcal{L}[u_m(\varsigma,\kappa) - \lambda_m \mathrm{u}_{m-1}(\varsigma,\kappa)] = h\Re_m(\overline{\mathrm{u}}_{m-1}), \tag{43}$$

where

$$\Re_m(\overline{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial \kappa} + \sum_{i=0}^{m-1} u_i \frac{\partial u_{m-1-i}}{\partial \varsigma} - u_{m-1} + \sum_{i=0}^{m-1} u_i u_{m-1-i} + e^{\kappa-\varsigma}. \tag{44}$$

The solution of equation (43) becomes

$$u_m(\varsigma,\kappa) = \lambda_m u_{m-1}(\varsigma,\kappa) + h\mathcal{L}^{-1}[\Re_m(\overline{u}_{m-1})]. \tag{45}$$

As $u_0(\varsigma,\kappa) = 1 - e^{-\varsigma}$, we get

$$\begin{aligned}
u_1(\varsigma,\kappa) &= (e^{\kappa-\varsigma} - e^{-\varsigma})(h),\\
u_2(\varsigma,\kappa) &= (e^{\kappa-\varsigma} - e^{-\varsigma})(h(h+1)),\\
u_3(\varsigma,\kappa) &= (e^{\kappa-\varsigma} - e^{-\varsigma})(h(h+1)(h+2)),\\
u_4(\varsigma,\kappa) &= (e^{\kappa-\varsigma} - e^{-\varsigma})(h(h+1)(h+2)(h+3)).
\end{aligned}$$

Series solution expression can be written as

$$\begin{aligned}
u(\varsigma,\kappa) = \sum_{m=0}^{\infty} u_m(\varsigma,\kappa) &= 1 - e^{-\varsigma} + (e^{\kappa-\varsigma} - e^{-\varsigma})(h) + (e^{\kappa-\varsigma} - e^{-\varsigma})(h(h+1))\\
&+ (e^{\kappa-\varsigma} - e^{-\varsigma})(h(h+1)(h+2)) + (e^{\kappa-\varsigma} - e^{-\varsigma})(h(h+1)(h+2)(h+3)).....
\end{aligned} \tag{46}$$

This is approximate series solution of non-homogeneous gas dynamic equation by HAM.

Putting $h = -1$, we can construct following

$$\begin{aligned}
u(\varsigma,\kappa) &= 1 - e^{-\varsigma} + (e^{\kappa-\varsigma} - e^{-\varsigma})(-1)\\
&= 1 - e^{\kappa-\varsigma},
\end{aligned} \tag{47}$$

which is exact solution of non-homogeneous gas dynamics equation.

4. Convergence analysis

In this section, the proof of the convergence of the HAM-derived series solution to the exact solution of the equations (11) and (31).

Theorem: The solution of equations (11) and (31) must hold as long as the series (21) and (41) is convergent, where is governed by the m-th order deformation equation (8) under equation (9) and (10).

Proof: Let the series of system

$$u(\varsigma,\kappa) = u_0(\varsigma,\kappa) + \sum_{m=1}^{\infty} u_m(\varsigma,\kappa), \tag{48}$$

be convergent. Then,

$$\lim_{m\to\infty} u_m(\varsigma,\kappa) = 0. \tag{49}$$

Now, we have

$$\sum_{m=1}^{n}[\mathrm{u}_m(\varsigma,\kappa)-\lambda_m \mathrm{u}_{m-1}(\varsigma,\kappa)]=\mathrm{u}_1+(\mathrm{u}_2-\mathrm{u}_1)+(\mathrm{u}_3-\mathrm{u}_2)+\ldots+(\mathrm{u}_{n-1}-\mathrm{u}_{n-2})+(\mathrm{u}_n-\mathrm{u}_{n-1})=\mathrm{u}_n. \tag{50}$$

From (49), as $n\to\infty$, we get that

$$\sum_{m=1}^{\infty}[\mathrm{u}_m(\varsigma,\kappa)-\lambda_m \mathrm{u}_{m-1}(\varsigma,\kappa)]=\lim_{m\to\infty}\mathrm{u}_m(\varsigma,\kappa)=0. \tag{51}$$

According to the definition of the linear operator $\mathcal{L}$, we obtain

$$\sum_{m=1}^{\infty}\mathcal{L}[\mathrm{u}_m(\varsigma,\kappa)-\lambda_m \mathrm{u}_{m-1}(\varsigma,\kappa)]=\mathcal{L}\sum_{m=1}^{\infty}[\mathrm{u}_m(\varsigma,\kappa)-\lambda_m \mathrm{u}_{m-1}(\varsigma,\kappa)]=0. \tag{52}$$

From the above expressions and equation (8), we have

$$\sum_{m=1}^{\infty}\mathcal{L}[\mathrm{u}_m(\varsigma,\kappa)-\lambda_m \mathrm{u}_{m-1}(\varsigma,\kappa)]=h\sum_{m=0}^{\infty}\Re_m(\overline{\mathrm{u}}_{m-1}), \tag{53}$$

as $h\neq 0$, we obtain

$$\sum_{m=0}^{\infty}\Re_m(\overline{\mathrm{u}}_{m-1})=0. \tag{54}$$

From equations (24) and (44), it holds

$$\sum_{m=0}^{\infty}\Re_m(\overline{\mathrm{u}}_{m-1})=\sum_{m=1}^{\infty}\frac{\partial \mathrm{u}_{m-1}}{\partial\kappa}+\sum_{m=1}^{\infty}\sum_{i=0}^{m-1}\mathrm{u}_i\frac{\partial \mathrm{u}_{m-1-i}}{\partial\varsigma}-\sum_{m=1}^{\infty}\mathrm{u}_{m-1}+\sum_{m=1}^{\infty}\sum_{i=0}^{m-1}\mathrm{u}_i\mathrm{u}_{m-1-i}+f(\varsigma,\kappa), \tag{55}$$

where $f(\varsigma,\kappa)=0$ for Section 3.1 and $f(\varsigma,\kappa)=e^{\kappa-\varsigma}$ for Section 3.2.

From equations (54) and (55), we get that

$$\frac{\partial \mathrm{u}(\varsigma,\kappa)}{\partial\kappa}+\mathrm{u}\frac{\partial \mathrm{u}(\varsigma,\kappa)}{\partial\varsigma}-\mathrm{u}(\varsigma,\kappa)\{1-\mathrm{u}(\varsigma,\kappa)\}=f(\varsigma,\kappa). \tag{56}$$

5. Results and discussion

Here, the nonlinear gas dynamic equations are solved by the HAM. The obtained results of the problems by HAM and ADM [17] up to fifth term approximation and the fourth order Runge-Kutta (RK-4) method have been shown in Table 1 for Example 1. It is observable that the solution is quite similar to the exact solution and previously available result. Table 2 presents the comparative study of the solution obtained by HAM up to the fifth term approximation for h = –1 and the ADM solution and the RK-4 method. The h-curves for Sections 3.1 and 3.2 are given in Figures 1 and 2, respectively. From both h-curves, it is visible that the convergent solution is very close to h = –1. Solution obtained by HAM for different values of h is compared with the exact solution in Figure 3 and Figure 4 for Sections 3.1 and 3.2, respectively.

Table 1. Solution by ADM [11], HAM, and RK-4 method for Section 3.1 and comparison of the absolute error

ς	κ	ADM	HAM	RK-4	$\lvert u_{ADM} - u_{Exact} \rvert$	$\lvert u_{HAM} - u_{Exact} \rvert$	$\lvert u_{RK-4} - u_{Exact} \rvert$
-0.8	0.1	2.45961	2.45961	2.714310	1.8×10^{-7}	1.8×10^{-7}	2.5×10^{-1}
-0.6	0.2	2.22553	2.22553	2.634804	5.0×10^{-6}	5.0×10^{-6}	4.0×10^{-1}
-0.4	0.3	2.01372	2.01372	2.476599	3.1×10^{-5}	3.1×10^{-5}	4.6×10^{-1}
-0.2	0.4	1.82200	1.82200	2.250750	1.1×10^{-4}	1.1×10^{-4}	4.2×10^{-1}
0	0.5	1.64843	1.64843	1.980000	2.8×10^{-4}	2.8×10^{-4}	3.3×10^{-1}
0.2	0.6	1.49123	1.49123	1.691819	5.8×10^{-4}	5.8×10^{-4}	1.9×10^{-1}
0.4	0.7	1.34879	1.34879	1.410906	1.0×10^{-3}	1.0×10^{-3}	6.1×10^{-2}
0.6	0.8	1.21967	1.21967	1.545870	1.7×10^{-3}	1.7×10^{-3}	3.2×10^{-1}
0.8	0.9	1.10258	1.10258	0.931909	2.5×10^{-3}	2.5×10^{-3}	1.7×10^{-1}
1	1	0.99634	0.99634	0.745215	3.6×10^{-3}	3.6×10^{-3}	3.5×10^{-1}

Table 2. Solution by ADM [11], HAM, and RK-4 method for Section 3.2

ς	κ	ADM	HAM	RK-4	$\lvert u_{RK-4} - u_{Exact} \rvert$
-0.8	0.1	-1.45960	-1.45960	-1.365496	9.4×10^{-2}
-0.6	0.2	-1.22554	-1.22554	-1.130039	9.5×10^{-2}
-0.4	0.3	-1.01375	-1.01375	-0.960993	5.2×10^{-2}
-0.2	0.4	-0.82211	-0.82211	-0.832083	9.9×10^{-3}
0	0.5	-0.64872	-0.64872	-0.728768	8.0×10^{-2}
0.2	0.6	-0.49182	-0.49182	-0.642556	1.5×10^{-1}
0.4	0.7	-0.34985	-0.34985	-0.568274	2.1×10^{-1}
0.6	0.8	-0.22140	-0.22140	-0.502657	2.8×10^{-1}
0.8	0.9	-0.10517	-0.10517	-0.443578	3.3×10^{-1}
1	1	0	0	-0.389608	3.8×10^{-1}

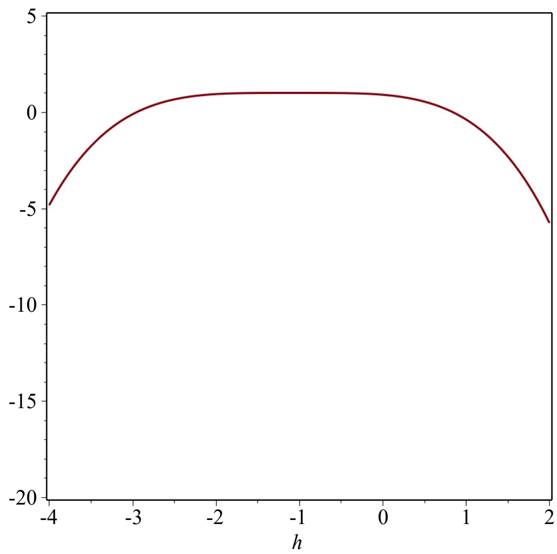

Figure 1. *h*-curve for Section 3.1

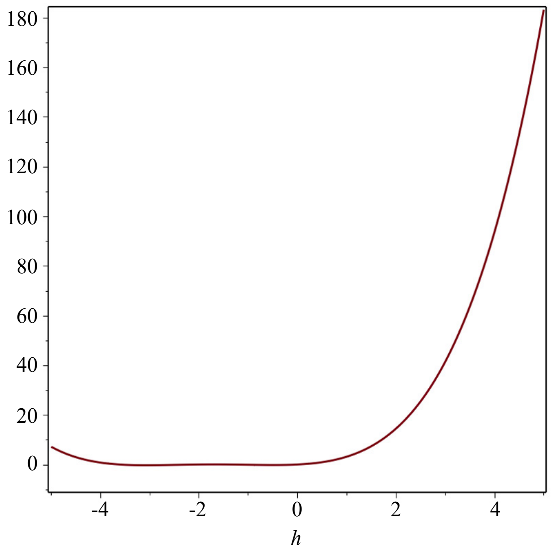

Figure 2. *h*-curve for Section 3.2

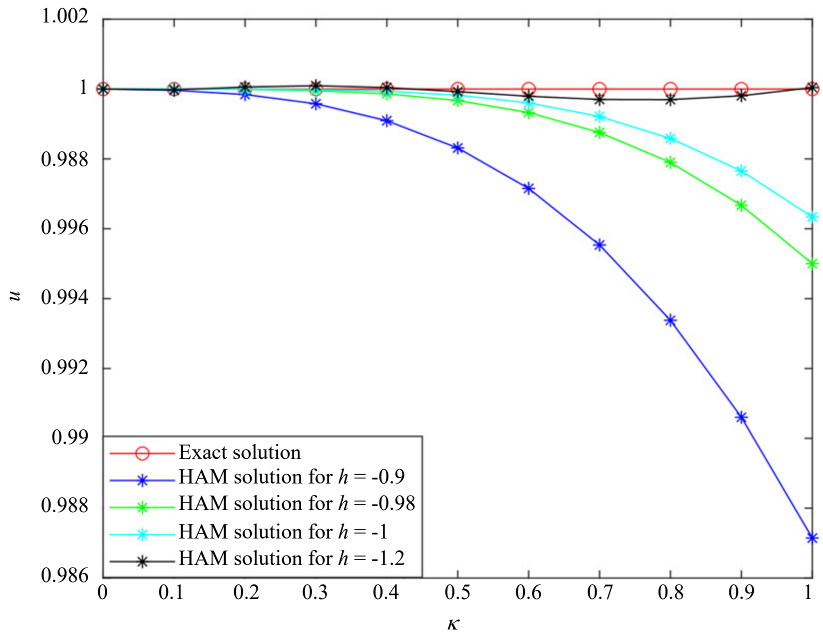

Figure 3. Plot of $u(\varsigma, \kappa)$ for Section 3.1 at different *h*

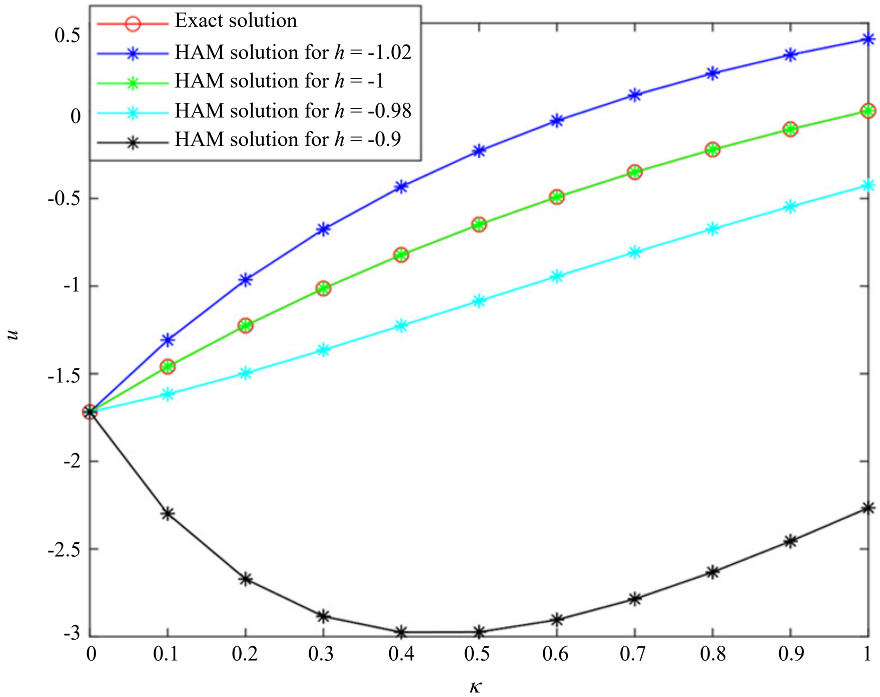

Figure 4. Plot of $u(\varsigma, \kappa)$ for Section 3.1 at different h

6. Conclusion

In this work, a semi-analytical method has been employed to study the gas dynamic equations with initial conditions. Upon comparing the obtained solution with the exact solution and the available numerical solution, it is observed that the HAM effectively finds the solution to a nonlinear partial differential equation and demonstrates the proposed method's efficacy as the homotopy approach helps to normalize the nonlinear problem very easily. It can be concluded that the auxiliary parameter, h, may set a limit on the convergence region, and the extra parameter, h, provides an additional means of conveniently adjusting and controlling the convergence region.

Conflict of interest

The authors declare no conflict of interests.

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