Contemporary Mathematics
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UNIVERSAL WISER PUBLISHER

Research Article

Surjective Isometries of Multiplication Operators and Weighted Composition Operators on *n*th Weighted-Type Banach Spaces of Analytic Functions

Shams Alyusof

Al Imam Mohammad Ibn Saud Islamic University (IMSIU) Riyadh, Saudi Arabia
Email: syyusof@imamu.edu.sa

Received: 10 February 2023; **Revised:** 27 March 2023; **Accepted:** 10 April 2023

Abstract: This paper is to characterize isometries of multiplication operators and weighted composition operators on the *n*th weighted-type Banach spaces $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of analytic functions on the open unit disk, of which the Bloch space and the Zygmund space are particular cases at $n = 1, 2$. We give characterizations of the symbols ψ and φ for which the multiplication operator M_ψ and the weighted composition operator $W_{\psi,\phi}$ are surjective isometries. Moreover, we show that generalized weighted composition operators are not isometric on V_n.

Keywords: isometry, multiplication operators, weighted composition operators, *n*th Banach spaces of analytic functions

MSC: 76B15

1. Introduction

Let D represent the open unit disk in the complex plane, $H(D)$ the set of analytic functions on D, and $S(D)$ the set of analytic self maps of D. The *n*th weighted-type Banach spaces $\{\mathcal{V}_n : n \in \mathbb{N}\}$ consist of all $f \in H(D)$, such that

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|f^{(n)}(z)| < \infty,$$

where the norm is defined as follows:

$$\|f\|_{\mathcal{V}_n} = \sum_{j=0}^{n-1} |f^{(j)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f^{(n)}(z)|.$$

In fact, V_1 is the Bloch space, and V_2 is the Zygmund space. Recently, the iterated spaces V_n have been studied in several resources, such as [1] and [2]. In particular, the authors of [1] show some interesting properties of such spaces. Indeed, (V_n) is a nested sequence that is contained in the disk algebra $H^\infty(D) \cap C(\overline{D})$ for all $n \geq 2$. Moreover, for $n \in \mathbb{N}$ and $f \in \mathcal{V}_n$,

DOI: https://doi.org/10.37256/cm.4420232472

$$\| f \|_{\mathcal{V}_{n-1}} \leq \| f \|_{\mathcal{V}_n} .$$

Also, for each $n \geq 3$, they find that V_n is an algebra.

One important application of the nth weighted-type Banach spaces is in the study of approximation theory and numerical analysis. In particular, these spaces can be used to quantify the accuracy of various numerical methods for approximating functions with nth-order derivatives, such as finite difference and finite element methods. Moreover, the nth weighted-type Banach spaces can be used to establish estimates for the rates of convergence of various approximation schemes as well as to obtain error bounds for numerical solutions of differential equations. More details can be found in [3] and [4].

For $\psi \in H(D)$, the multiplication operator on V_n is the linear operator given by

$$M_\psi f = \psi f, \text{ for all } f \in \mathcal{V}_n.$$

And for $\psi \in H(D)$, the composition operator on V_n is the linear operator defined by

$$C_\phi f = f(\phi(z)), \text{ for all } f \in \mathcal{V}_n,$$

the weighted composition operator is the linear operator defined as follows:

$$W_{\psi,\phi} f = (M_\psi C_\phi) f = \psi f(\phi(z)), \text{ for all } f \in \mathcal{V}_n.$$

Furthermore, for $m \in \mathbb{N}$, the generalized weighted composition operator is defined by

$$W_{\psi,\phi}^m f = \psi f^{(m)}(\phi(z)), \text{ for all } f \in \mathcal{V}_n.$$

Given two normed vector spaces X and Y, a linear isometry is a linear map $T: X \rightarrow Y$, which preserves the norms,

$$\| Tx \|_Y = \| x \|_X$$

for all $x \in X$.

Recall that a linear isometry is injective but not necessarily surjective.

Characterizing the isometric multiplication operators and the isometric weighted composition operators on several Banach spaces is an active topic that attracts several researchers. Indeed, in 2008, Bonet et al. [5] characterized the isometric multiplication operators and the isometric weighted composition operators on the weighted Banach space H_v^∞. Moreover, in 2014 and 2018, Le [6] and Zorboska [7] characterized isometric weighted composition operators on Fock and weighted Bergman spaces, respectively.

In 2005, Colonna [8] characterized the isometric composition operators on the Bloch space, which is V_1. This was followed by further characterizations of the isometric multiplication operators on the same space in [9] and [10]. Moreover, in 2014, Li [11] studied isometries among the generalized composition operators on Bloch-type spaces.

Recently, Mas and Vukotić [12] characterized surjective isometric weighted composition operators on a general class of analytic functions, assuming such a class satisfies several axioms, but we have not been able to verify whether Axiom 4 holds for the nth weighted-type Banach spaces V_n for $n \geq 2$. However, we obtain the same conclusions from Theorem 2 of [12] for these spaces.

In this paper, the main goal is to expand the research carried out in [8] and [9] by characterizing the isometric composition operators and the isometric multiplication operators on the spaces V_n for $n \geq 2$. Furthermore, in light of what has been done in [11], we show that generalized weighted composition operators are not isometric on V_n.

The following results are needed throughout this work. They provide significant properties of the spaces V_n and a characterization of the invertible weighted composition operator $W_{\psi,\phi}$ on V_n.

Proposition 1.1. ([1], Proposition 2.1) For each $n \in \mathbb{N}, \mathcal{V}_n \subset \mathcal{V}_{n-1}$. Moreover, the inclusion map from V_n into V_{n-1} is bounded with a norm less than or equal to 1, and for $n \geq 2$, $\| f \|_\infty \leq \| f \|_{\mathcal{V}_n}$ for each $f \in \mathcal{V}_n$.

Theorem 1.2. ([1], Theorem 5.2) For $n \in \mathbb{N}$, if the weighted composition operator $W_{\psi,\phi}$ on V_n is bounded, then it is invertible if ψ is bounded away from zero and φ is automorphism of D. The inverse is also a weighted composition operator on V_n, such that

$$W_{\psi,\phi}^{-1} = W_{\frac{1}{\psi\phi^{-1}},\phi^{-1}}.$$

In particular, the composition operator C_φ is invertible on V_n if φ is an automorphism on D, and the multiplication operator M_ψ on V_n is invertible if ψ is bounded away from zero, such that

$$C_\phi^{-1} = C_{\phi^{-1}}$$

and

$$M_\psi^{-1} = M_{1/\psi}.$$

2. Surjective isometric multiplication operator on V_n

In [9], the authors prove that the multiplication operators on V_1, which is the Bloch space, are isometries if and only if the symbols are constant with modulus one. In this section, we show a similar result for all spaces V_n.

The following finding is needed to express and prove the primary result in this section.

Proposition 2.1. ([1], Proposition 3.1) The multiplication operator M_ψ on V_n is bounded if $\psi \in \mathcal{V}_n$, for all $n \geq 3$.

Similar results are proven for $n = 1, 2$. In particular, M_ψ on V_n is bounded if $\psi \in \mathcal{V}_n \cap \mathcal{B}_{\log}$, for $n = 1, 2$, where $\mathcal{B}_{\log}$ is the logarithmic Bloch space, see ([13], Corollary 2.5) and ([14], Theorem 3.1).

Lemma 2.2. For $n \in \mathbb{N}$, if the bounded surjective multiplication operator M_ψ on V_n is isometric, then ψ does not fix the origin. Moreover, $\|\psi\|_\infty \leq 1$ and $\|\psi\|_{\mathcal{V}_n} = 1$.

Proof. It is clear from Theorem 1.2 that $|\psi(0)| \neq 0$. On the other hand, by Lemma 11 in [15], we have that $\|\psi\|_\infty \leq \|M_\psi\| = 1$, since M_ψ is isometric. Furthermore, take $h = 1$, then since M_ψ is isometric, we get

$$1 = \|h\|_{\mathcal{V}_n} = \|M_\psi(h)\| = \|\psi\|_{\mathcal{V}_n}.$$

Theorem 2.3. For $n \in \mathbb{N}$, the surjective bounded multiplication operator M_ψ acting on V_n is isometric if ψ is a constant of modulus 1.

Proof. It is clear that the multiplication operator M_ψ acting on Vn is isometric if ψ is a constant of modulus 1.

Suppose that the multiplication operator M_ψ is isometric. By Theorem 1.2, $M_{1/\psi}$ is also isometric. Thus, by Lemma 2.2, we have that

$$\|\psi\|_\infty \leq 1, \left\|\frac{1}{\psi}\right\|_\infty \leq 1$$

and

$$\|\psi\|_{\mathcal{V}_n} = \left\|\frac{1}{\psi}\right\|_{\mathcal{V}_n} = 1,$$

which leads to the desired conclusion.

3. Surjective isometric weighted composition operators on V_n

The authors of [1] provide a characterization of bounded weighted composition operators on V_n. Also, they provide a sufficient and necessary condition for isometric composition operators on V_n for $n \geq 2$. In this section, we provide a characterization for the surjective isometric weighted composition operators on V_n for $n \geq 2$. A comparable characterization is provided for the Bloch space V_1 in [12], as it is established that V_1 meets all the necessary axioms for a similar result to be applied.

Theorem 3.1. ([1], Theorem 6.1) For $n \geq 2$, the composition operator C_φ on V_n is isometric if φ is rotation of D.

In the following theorem, we characterize the isometric weighted composition operators on V_n.

Theorem 3.2. For $n \geq 2$, the surjective weighted composition operator $W_{\psi,\phi}$ on V_n is isometric if φ is rotation of D and ψ is a constant of modulus 1.

Proof. It is obvious by Theorems 2.3 and 3.1 that weighted composition operator $W_{\psi,\phi}$ on V_n is isometric if φ is rotation of D and ψ is a constant of modulus 1.

Conversely, suppose that $W_{\psi,\phi}$ is isometric on V_n. Then, from Theorem 1.2, $W_{\frac{1}{\psi\phi^{-1}},\phi^{-1}}$ is also an isometry.

Since $W_{\psi,\phi}$ is isometric, take $f(z) = 1$ as a test function, then

$$\| W_{\psi,\phi}(f) \|_{\mathcal{V}_n} = \| f \|_{\mathcal{V}_n} = \| 1 \|_{\mathcal{V}_n} = 1.$$

Thus,

$$\| \psi \|_{\mathcal{V}_n} = 1.$$

Thus, from Proposition 1.1, we get that $\| \psi \|_\infty \leq 1$.

On the other hand,

$$\left\| W_{\frac{1}{\psi\phi^{-1}},\phi^{-1}}(f) \right\|_{\mathcal{V}_n} = \| f \|_{\mathcal{V}_n} = \| 1 \|_{\mathcal{V}_n} = 1.$$

Thus,

$$\left\| \frac{1}{\psi\phi^{-1}} \right\|_{\mathcal{V}_n} = 1,$$

then $\left\| \frac{1}{\psi} \right\|_{\mathcal{V}_n} = 1,$ since φ is automorphism in Theorem 1.2, and similarly from Proposition 1.1, we get that $\left\| \frac{1}{\psi} \right\|_\infty \leq 1.$

Hence, ψ must be a constant of modulus 1, and from Theorem 3.1, φ should be a rotation of D since $W_{\psi,\varphi}$ could be considered as an isometric composition operator when ψ is a constant of modulus 1.

4. Generalized weighted composition operators on V_n are not isometric

The author of [11] provides characterizations of isometries among the generalized composition operators on Bloch-type spaces. In this section, we show that the generalized weighted composition operators on V_n are not isometric. In particular, the generalized weighted composition operators on V_n are not injective. The following example illustrates this fact.

Recall that for $m \in \mathbb{N}$, the generalized weighted composition operator is defined by

$$W^m_{\psi,\phi} f = \psi f^{(m)}(\phi(z)), \text{ for all } f \in \mathcal{V}_n.$$

Example 4.1. Let $W^3_{\psi,\varphi}$ be acting on V_n for some $n \in \mathbb{N}$, and $P_1(z) = z, P_2(z) = z^2$. Then,

$$W^3_{\psi,\varphi}(P_1(z)) = (\psi(z))P_1^{(3)}(\phi(z)) = (\psi(z))(0) = 0$$

and

$$W^3_{\psi,\varphi}(z) = (\psi)z))P_2^{(3)}(\phi(z)) = (\psi(z))(0) = 0.$$

Hence, $W^3_{\psi,\varphi}$ is not injective since P_1 and P_2 are in the kernel space.

Theorem 4.2. A generalized weighted composition operator $W^m_{\psi,\varphi}$ on V_n is not isometric for all $m \geq 1$.

Proof. It is clear that V_n contains all the polynomials in the open unit disk D.

Let $P(z) = \sum_{j=0}^{k} a_j z^j$, such that $0 \leq k < m$, then

$$W^m_{\psi,\varphi}P(z) = (\psi(z))(P^{(m)} \circ \varphi(z)) = (\psi(z))(0) = 0.$$

Thus, all polynomials of degree less than m are in the kernel space. Hence, $W^m_{\psi,\varphi}$ is not injective, then it is not isometric.

A natural question that arises from this work is whether similar characterizations hold for isometries of multiplication operators and weighted composition operators on the nth weighted-type Banach spaces $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of analytic functions on the open unit ball in $\mathbb{C}^n$.

Acknowledgments

The authors extend their appreciation to the Deputyship for Research & Innovation, Ministry of Education in Saudi Arabia for funding this research through the project number IFP-IMSIU-2023066. The authors also appreciate the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) for supporting and supervising this project.

Conflict of interest

There is no conflict of interest in this study.

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