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Research Article

On the Existence of Solutions to a Fractional Hybrid Thermostat Model

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Abstract: This work is concerned with the existence of solutions to a nonlinear fractional hybrid thermostat model in the settings of Atangana-Baleanu derivatives. We also consider the boundary conditions of this model in the form of hybrid conditions. Imposing some suitable conditions on the given data, we establish the existence result of continuous solutions based on the Dhage fixed-point theorem in Banach algebra. Moreover, an example is constructed to illustrate our theoretical result.

Keywords: Atangana-Baleanu fractional derivatives, fractional differential equations, hybrid thermostat model, Dhage fixed-point theorem

MSC: 26A33, 34A08, 34A38, 34K37

1. Introduction

In this article, we discuss the existence of solutions to the following fractional hybrid problem for the thermostat model (ρ denotes a space variable)

$$\begin{cases} {}^{ABC}\mathbf{D}^{\zeta}_{0+}\left(\frac{y(\rho)}{g(\rho, y(\rho))}\right) + f(\rho, y(\rho)) = 0, \quad \rho \in [0, 1] =: J, \\ \frac{d}{d\rho}\left(\frac{y(\rho)}{g(\rho, y(\rho))}\right)\Big|_{\rho=0} = 0, \quad \lambda\, {}^{ABC}\mathrm{D}^{\zeta-1}_{0+}\left(\frac{y(\rho)}{g(\rho, y(\rho))}\right)\Big|_{\rho=1} + \frac{y(\rho)}{g(\rho, y(\rho))}\Big|_{\rho=0} = 0, \end{cases} \tag{1}$$

where $1 < \zeta < 2$, ${}^{ABC}\mathbf{D}^{\zeta}_{0+}$ and ${}^{ABC}\mathrm{D}^{\zeta-1}_{0+}$ denote the Atangana-Baleanu fractional derivatives in Caputo sense of order ζ and $\zeta - 1$, respectively, $g\colon J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $f\colon J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and $\lambda > 0$ is a real number.

Fractional derivatives serve as a valuable tool to precisely describe numerous physical phenomena in several scientific and engineering fields such as physics, mechanics, economics, social science, bioengineering and biomedical, control theory, environmental science, materials, etc. We refer to the textbook [1] for the theoretical study of fractional calculus.

The quadratic perturbations of nonlinear differential equations, whether of integer or non-integer order, are known as hybrid differential equations, which have received a great deal of attention from many researchers. Hybrid differential

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equations include several dynamical systems as special cases. Dhage et al. [2] studied the existence of solutions as well as several differential inequalities for nonlinear first-order hybrid differential equations. Sun et al. [3] studied the two-point boundary value problem of hybrid differential equations having Riemann-Liouville fractional derivatives via Dhage fixed-point theorem.

A thermostat is a device that systematically detects the temperature of a physical system and implements the requisite actions to maintain the system's temperature at a predetermined set point. Infante et al. [4] studied the thermostat model described by a second-order nonlinear differential equation, incorporating nonlocal boundary conditions. Subsequently, Pimentel et al. [5] discussed the fractional thermostat model in the settings of Caputo derivatives. Recently, Baleanu et al. [6] studied the existence of solutions to the fractional hybrid thermostat model and its inclusion version. Alzabut et al. [7] addressed the discrete fractional-order thermostat model involving Caputo fractional difference operators.

Atangana et al. [8] proposed new fractional derivatives that involve nonsingular kernels and have since become an important tool in fractional calculus, now called the Atangana-Baleanu (AB) fractional derivatives. Later on, the discrete versions of AB fractional derivatives were defined in [9]. Abdeljawad [10] defined the higher-order AB fractional operators and presented several fundamental properties.

The existence of solutions to fractional differential equations (FDEs) involving the AB fractional derivative in Caputo sense has become a hot topic in fractional calculus. The Gronwall inequality in the frame of AB fractional integral is established in [11]. Moreover, they studied the existence and uniqueness of the solution to the nonlinear Cauchy problem with the AB fractional derivative. Fernandez et al. [12] discussed initial value problems of linear FDEs with continuous variable coefficients, and the representation of their derived solutions involves an infinite series of AB fractional operators. Mehmood et al. [13] studied the existence theory and Hyers-Ulam stability for FDEs with nonseparated and integral-type boundary conditions. A class of fractional boundary value problems with Caputo-type AB derivatives was explored in [14]. Kumar et al. [15] investigated initial value problems with delay involving generalized Caputo derivatives. In [16], the authors studied stochastic fractional differential equations and analyzed their Hyers-Ulam stability via the stochastic calculus techniques. However, as far as we know, the thermostat model in the settings of AB fractional derivatives in Caputo sense has not been addressed in the literature yet.

The novelties of this work are as follows. (i) Fully hybrid thermostat model having AB fractional derivatives is considered. (ii) The existence result is established via Dhage fixed-point theorem in Banach algebra. (iii) The constructed theory is validated using an example.

The paper is organized as follows. Some important definitions and lemmas are presented in Section 2. We study the existence of solutions to the fractional hybrid thermostat model (1) in Section 3. Our main result is also illustrated with an example in Section 4. Finally, the conclusions of this work are drawn in Section 5.

2. Preliminaries

First of all, we recall some necessary preliminary concepts of fractional calculus, which are used to conduct this study. Let $C([a, b], \mathbb{R})$, $a < b$, be the Banach space of all continuous functions defined on $[a, b]$ with the norm

$$\|\xi\|_{\mathscr{C}} = \sup_{\rho \in [a, b]} \{|\xi(\rho)|\}.$$

Note that $(C([a, b], \mathbb{R}), \|\cdot\|_{\mathscr{C}})$ is a Banach algebra with respect to the multiplication composition "$\cdot$" defined by

$$(\xi \cdot \zeta)(\rho) = \xi(\rho)\zeta(\rho) \quad \text{for every } \xi, \zeta \in C([a, b], \mathbb{R}).$$

Let $AC([a, b], \mathbb{R})$ be the space of all absolutely continuous functions defined on $[a, b]$, and define

$$AC^n([a, b], \mathbb{R}) = \left\{\xi \mid \xi\colon [a, b] \to \mathbb{R} \text{ and } \xi^{(n-1)} \in AC([a, b], \mathbb{R})\right\}, \ n \in \mathbb{N}.$$

Definition 1 ([1]) The function E_ζ, $\zeta > 0$, defined by

$$E_\zeta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\zeta + 1)} \quad \text{for } z \in \mathbb{C},$$

is called the classical Mittag-Leffler function.

Definition 2 ([1]) The Riemann-Liouville fractional integral of order $\zeta > 0$ for a function $\xi \in L^1((a, b), \mathbb{R})$ is defined as

$$\left(I^\zeta_{a+}\xi\right)(\rho) = \frac{1}{\Gamma(\zeta)} \int_a^\rho (\rho - \tau)^{\zeta-1} \xi(\tau) d\tau \quad \text{for a.e. } \rho \in [a, b].$$

Definition 3 ([8, 12]) Let $0 < \zeta < 1$. The AB fractional derivative in the Caputo sense for a function $\xi \in AC([a, b], \mathbb{R})$ is defined as

$$\left({}^{ABC}\mathrm{D}^\zeta_{a+}\xi\right)(\rho) = \frac{\mathscr{N}(\zeta)}{1-\zeta} \int_a^\rho E_\zeta\left(-\frac{\zeta(\rho-\tau)^\zeta}{1-\zeta}\right) \xi'(\tau) d\tau \quad \text{for all } \rho \in [a, b].$$

The AB fractional derivative in the Riemann-Liouville sense for a function $\xi \in L^1((a, b), \mathbb{R})$ is defined as

$$\left({}^{ABRL}\mathrm{D}^\zeta_{a+}\xi\right)(\rho) = \frac{\mathscr{N}(\zeta)}{1-\zeta} \frac{d}{d\rho} \int_a^\rho E_\zeta\left(-\frac{\zeta(\rho-\tau)^\zeta}{1-\zeta}\right) \xi(\tau) d\tau \quad \text{for a.e. } \rho \in [a, b].$$

The AB fractional integral for a function $\xi \in L^1((a, b), \mathbb{R})$ is defined as

$$\left({}^{AB}\mathrm{I}^\zeta_{a+}\xi\right)(\rho) = \frac{1-\zeta}{\mathscr{N}(\zeta)} \xi(\rho) + \frac{\zeta}{\mathscr{N}(\zeta)} \left(I^\zeta_{a+}\xi\right)(\rho) \quad \text{for a.e. } \rho \in [a, b].$$

The normalization function $\mathscr{N}(\cdot)$ is a real-valued strictly positive function on $[0, 1]$ such that $\mathscr{N}(0) = \mathscr{N}(1) = 1$.

Lemma 1 ([8]) Let $\xi \in AC([a, b], \mathbb{R})$ and $0 < \zeta < 1$. Then

$$\left({}^{ABC}\mathrm{D}^\zeta_{a+}\xi\right)(\rho) = \left({}^{ABRL}\mathrm{D}^\zeta_{a+}\xi\right)(\rho) - \frac{\mathscr{N}(\zeta)}{1-\zeta} \xi(a) E_\zeta\left(-\frac{\zeta\rho^\zeta}{1-\zeta}\right) \quad \text{for all } \rho \in [a, b].$$

Definition 4 ([10]) Let $n < \zeta < n+1$, $n \in \mathbb{N}_0$, and $\alpha = \zeta - n$. The AB fractional derivative in Caputo sense for a function $\xi \in AC^{n+1}([a, b], \mathbb{R})$ is defined as

$$\left({}^{ABC}\mathbf{D}^{\zeta}_{a+}\xi\right)(\rho) := \left({}^{ABC}\mathrm{D}^{\alpha}_{a+}\xi^{(n)}\right)(\rho).$$

The AB fractional derivative in the Riemann-Liouville sense for a function $\xi^{(n)} \in L^1((a,\, b),\, \mathbb{R})$ is defined as

$$\left({}^{ABRL}\mathbf{D}^{\zeta}_{a+}\xi\right)(\rho) := \left({}^{ABRL}\mathrm{D}^{\alpha}_{a+}\xi^{(n)}\right)(\rho).$$

The AB fractional integral for a function $\xi \in L^1((a,\, b),\, \mathbb{R})$ is defined as

$$\left({}^{AB}\mathbf{I}^{\zeta}_{a+}\xi\right)(\rho) := \left(I^{n}_{a+}\, {}^{AB}\mathrm{I}^{\alpha}_{a+}\xi\right)(\rho).$$

Lemma 2 ([10]) Let $\xi \in AC^{n+1}([a,\, b],\, \mathbb{R})$ and $n < \zeta < n+1$ $(n \in \mathbb{N}_0)$. Then

$$\left({}^{AB}\mathbf{I}^{\zeta}_{a+}\, {}^{ABC}\mathbf{D}^{\zeta}_{a+}\xi\right)(\rho) = \xi(\rho) - \sum_{i=0}^{n} \frac{\xi^{(i)}(a)}{i!}(\rho - a)^i \quad \text{for all } \rho \in [a,\, b].$$

Lemma 3 ([14]) Let $\xi \in AC([a,\, b],\, \mathbb{R})$ and $n < \zeta < n+1$ $(n \in \mathbb{N}_0)$. Then

$$\left({}^{ABC}\mathbf{D}^{\zeta}_{a+}\, {}^{AB}\mathbf{I}^{\zeta}_{a+}\xi\right)(\rho) = \xi(\rho) - \xi(a) E_{\zeta-n}\left(-\frac{\zeta - n}{n+1-\zeta}\rho^{\zeta-n}\right) \quad \text{for all } \rho \in [a,\, b].$$

Theorem 1 Dhage fixed-point theorem [17]. Let $\mathscr{O}$ be a nonempty, closed, convex and bounded subset of a Banach algebra $\mathscr{X}$, and let $\mathscr{S}\colon \mathscr{X} \to \mathscr{X}$ and $\mathscr{T}\colon \mathscr{O} \to \mathscr{X}$ be two operators such that

(a) $\mathscr{T}$ is completely continuous,

(b) $\mathscr{S}$ is Lipschitzian with a Lipschitz constant l, say, and

(c) $\xi = \mathscr{S}\xi\mathscr{T}\zeta$ for all $\zeta \in \mathscr{O} \Rightarrow \xi \in \mathscr{O}$.

Then the operator equation $\mathscr{S}\xi\mathscr{T}\xi = \xi$ has a solution in $\mathscr{O}$, whenever $\mathscr{M}l < 1$, where $\mathscr{M} = \sup\{\|\mathscr{T}\xi\|_{\mathscr{X}} \colon \xi \in \mathscr{O}\}$.

Dhage fixed-point theorem is a hybrid fixed-point theorem, blending the ideas of the Banach contraction principle with the Leray-Schauder alternative theorem. It asserts the existence of a fixed-point, without uniqueness, of the product of two operators in a Banach algebra. This theorem is particularly useful for studying nonlinear perturbed operator equations.

The following lemma plays an essential role in our existence theorem.

Lemma 4 Let $1 < \zeta < 2$ and $\Omega \subset \mathbb{R}$ be a bounded set. Let $g\colon J \times \Omega \to \mathbb{R} \setminus \{0\}$ is continuous with $\rho \mapsto g(\rho,\, y(\rho)) \in AC(J,\, \mathbb{R})$ for $y \in AC(J,\, \mathbb{R})$, and let $\phi \in AC(J,\, \mathbb{R})$ with $\phi(0) = 0$. Then the solution to the fractional hybrid problem

$$\begin{cases} {}^{ABC}\mathbf{D}^{\zeta}_{0+}\left(\frac{y(\rho)}{g(\rho,\, y(\rho))}\right) + \phi(\rho) = 0, \qquad \rho \in [0,\, 1], \\[2ex] \frac{d}{d\rho}\left(\frac{y(\rho)}{g(\rho,\, y(\rho))}\right)\Big|_{\rho=0} = 0, \quad \lambda\, {}^{ABC}\mathrm{D}^{\zeta-1}_{0+}\left(\frac{y(\rho)}{g(\rho,\, y(\rho))}\right)\Big|_{\rho=1} + \frac{y(\rho)}{g(\rho,\, y(\rho))}\Big|_{\rho=0} = 0, \quad \lambda > 0 \end{cases} \tag{2}$$

is

$$y(\rho) = g(\rho, y(\rho))\left[\lambda \int_0^1 \phi(\tau)d\tau - \frac{2-\zeta}{\mathscr{N}(\zeta-1)}\int_0^\rho \phi(\tau)d\tau \right.$$

$$\left. - \frac{1}{\mathscr{N}(\zeta-1)\Gamma(\zeta-1)}\int_0^\rho (\rho-\tau)^{\zeta-1}\phi(\tau)d\tau\right] \quad \text{for all } \rho \in [0, 1]. \tag{3}$$

Proof. Applying ${}^{AB}\mathbf{I}_{0+}^{\zeta}$ to both sides of the FDE mentioned in (2), we find, in the light of Lemma 2, that

$$\frac{y(\rho)}{g(\rho, y(\rho))} = c_1 + c_{t2}\rho - \frac{2-\zeta}{\mathscr{N}(\zeta-1)}\int_0^\rho \phi(\tau)d\tau - \frac{1}{\mathscr{N}(\zeta-1)\Gamma(\zeta-1)}\int_0^\rho (\rho-\tau)^{\zeta-1}\phi(\tau)d\tau \tag{4}$$

for all $\rho \in J$, where c_1 and c_2 are real constants. Now

$$\frac{d}{d\rho}\left(\frac{y(\rho)}{g(\rho, y(\rho))}\right) = c_2 - \frac{2-\zeta}{\mathscr{N}(\zeta-1)}\phi(\rho) - \frac{\zeta-1}{\mathscr{N}(\zeta-1)\Gamma(\zeta-1)}\int_0^\rho (\rho-\tau)^{\zeta-2}\phi(\tau)d\tau \quad \text{for all } \rho \in J.$$

The condition $\frac{d}{d\rho}\left(\frac{y(\rho)}{g(\rho, y(\rho))}\right)\Big|_{\rho=0} = 0$ yields $c_2 = 0$ since $\phi(0) = 0$ and $\phi \in AC(J, \mathbb{R})$. Then, by Definition 4, we have

$$\frac{y(\rho)}{g(\rho, y(\rho))} = c_1 - I_{0+}{}^{AB}\mathbf{I}_{0+}^{\zeta-1}\phi(\rho) \quad \text{for all } \rho \in J.$$

In view of Lemma 3, we have

$${}^{ABC}D_{0+}^{\zeta-1}\left(\frac{y(\rho)}{g(\rho, y(\rho))}\right) = -\int_0^\rho \phi(\tau)d\tau \quad \text{for all } \rho \in J.$$

Applying the boundary condition $\lambda {}^{ABC}D_{0+}^{\zeta-1}\left(\frac{y(\rho)}{g(\rho, y(\rho))}\right)\Big|_{\rho=1} + \frac{y(\rho)}{g(\rho, y(\rho))}\Big|_{\rho=0} = 0$, it follows that

$$c_1 = \lambda \int_0^1 \phi(\tau)d\tau.$$

Substituting the values of c_1 and c_2 into (4), we obtain the required solution given in (3). □

3. Existence of solutions

We now present the existence theorem based on Theorem 1 for the hybrid thermostat model (1).

Theorem 2 Let $1 < \zeta < 2$ and $\Omega \subset \mathbb{R}$ be a bounded set. Assume the following

(H1) $g\colon J \times \Omega \to \mathbb{R} \setminus \{0\}$ is continuous with $\rho \mapsto g(\rho, y(\rho)) \in AC(J, \mathbb{R})$ for $y \in AC(J, \mathbb{R})$, and there exists a bounded function $\psi\colon J \to [0, \infty)$ such that

$$|g(\rho, y) - g(\rho, \tilde{y})| \leq \psi(\rho)|y - \tilde{y}| \quad \text{for all } \rho \in J \text{ and } y, \tilde{y} \in \Omega.$$

(H2) $f: J \times \Omega \to \mathbb{R}$ is continuous with $\rho \mapsto f(\rho, y(\rho)) \in AC(J, \mathbb{R})$ for $y \in AC(J, \mathbb{R})$, and there exists a function $\gamma \in L^\infty([0, 1], [0, \infty))$ such that

$$|f(\rho, y)| \leq \gamma(\rho) \quad \text{for } a.e.\ \rho \in J \text{ and all } y \in \Omega.$$

Then the hybrid thermostat model (1) has at least one solution in $C(J, \mathbb{R})$, provided $f(0, y(0)) = 0$ and

$$\Lambda \psi^* \|\gamma\|_{L^\infty} < 1, \tag{5}$$

where

$$\Lambda = \lambda + \frac{2-\zeta}{\mathscr{N}(\zeta-1)} + \frac{1}{\zeta \mathscr{N}(\zeta-1)\Gamma(\zeta-1)} \quad \text{and} \quad \psi^* = \sup_{\rho \in J}\{|\psi(\rho)|\}.$$

Proof. Let us first set $\mathscr{X} = C(J, \mathbb{R})$ and define a nonempty, closed, convex, and bounded subset $\mathscr{O}$ of $\mathscr{X}$ as

$$\mathscr{O} = \{y \in \mathscr{X}: \|y\|_{\mathscr{C}} \leq R\},$$

where

$$\rho \geq R = \frac{g_0 \Lambda \|\gamma\|_{L^\infty}}{1 - \Lambda \psi^* \|\gamma\|_{L^\infty}} \quad \text{with} \quad g_0 = \sup_{\rho \in J}\{|g(\rho, 0)|\}.$$

By Lemma 4, we define two operators $\mathscr{S}: \mathscr{X} \to \mathscr{X}$ and $\mathscr{T}: \mathscr{O} \to \mathscr{X}$ as follows

$$(\mathscr{S}y)(\rho) := g(\rho, y(\rho)) \quad \text{for each } \rho \in J$$

and

$$\begin{aligned}(\mathscr{T}y)(\rho) :=& \lambda \int_0^1 f(\tau, y(\tau))d\tau - \frac{2-\zeta}{\mathscr{N}(\zeta-1)} \int_0^\rho f(\tau, y(\tau))d\tau \\ &- \frac{1}{\mathscr{N}(\zeta-1)\Gamma(\zeta-1)} \int_0^\rho (\rho-\tau)^{\zeta-1} f(\tau, y(\tau))d\tau \quad \text{for each } \rho \in J.\end{aligned}$$

Therefore, the operator equation $(\mathscr{S}y)(\rho)(\mathscr{T}y)(\rho) = y(\rho)$ for each $\rho \in J$ is the integral equation of (1).

We show that $\mathscr{S}$ is a Lipschitzian operator. Let $y, \tilde{y} \in \mathscr{X}$. By (H1), for each $\rho \in J$, we have

$$|(\mathscr{S}y)(\rho) - (\mathscr{S}\tilde{y})(\rho)| = |g(\rho, y(\rho)) - g(\rho, \tilde{y}(\rho))| \leq \psi(\rho)|y(\rho) - \tilde{y}(\rho)| \leq \psi^* \|y - \tilde{y}\|_{\mathscr{C}}.$$

Taking supremum over ρ, we get

$$\|\mathscr{S}y - \mathscr{S}\tilde{y}\|_{\mathscr{C}} \leq \psi^* \|y - \tilde{y}\|_{\mathscr{C}} \qquad \text{for all } y, \tilde{y} \in \mathscr{X},$$

which shows that $\mathscr{S}$ is a Lipschitzian operator with the Lipschitz constant ψ^*.

Next, we prove that $\mathscr{T}$ is a completely continuous operator.

In order to show the continuity of the operator $\mathscr{T}$, let $\{y_m\}_{m=1}^{\infty} \subset \mathscr{O}$ be a sequence converging to a point $y \in \mathscr{O}$. Since the function f is continuous, we have $\lim_{m\to\infty} f(\rho, y_m(\rho)) = f(\rho, y(\rho))$. By the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
\lim_{m\to\infty} (\mathscr{T}y_m)(\rho) =& \lambda \int_0^1 \lim_{m\to\infty} f(\tau, y_m(\tau))d\tau - \frac{2-\zeta}{\mathscr{N}(\zeta-1)} \int_0^\rho \lim_{m\to\infty} f(\tau, y_m(\tau))d\tau \\
& - \frac{1}{\Gamma(\zeta-1)\mathscr{N}(\zeta-1)} \int_0^\rho (\rho-\tau)^{\zeta-1} \lim_{m\to\infty} f(\tau, y_m(\tau))d\tau \\
=& \lambda \int_0^1 f(\tau, y(\tau))d\tau - \frac{2-\zeta}{\mathscr{N}(\zeta-1)} \int_0^\rho f(\tau, y(\tau))d\tau \\
& - \frac{1}{\Gamma(\zeta-1)\mathscr{N}(\zeta-1)} \int_0^\rho (\rho-\tau)^{\zeta-1} f(\tau, y(\tau))d\tau \\
=& (\mathscr{T}y)(\rho) \quad \text{for each } \rho \in J,
\end{aligned}$$

which implies that the operator $\mathscr{T}$ is continuous on $\mathscr{O}$.

In order to show the boundedness of $\mathscr{T}(\mathscr{O})$, let $y \in \mathscr{O}$ be arbitrary. By (H2), for each $\rho \in J$, we have

$$
\begin{aligned}
|(\mathscr{T}y)(\rho)| \leq & \lambda \int_0^1 |f(\tau, y(\tau))|d\tau + \frac{2-\zeta}{\mathscr{N}(\zeta-1)} \int_0^\rho |f(\tau, y(\tau))|d\tau \\
& + \frac{1}{\Gamma(\zeta-1)\mathscr{N}(\zeta-1)} \int_0^\rho (\rho-\tau)^{\zeta-1}|f(\tau, y(\tau))|d\tau \\
\leq & \lambda \int_0^1 \gamma(\tau)d\tau + \frac{2-\zeta}{\mathscr{N}(\zeta-1)} \int_0^\rho \gamma(\tau)d\tau + \frac{1}{\Gamma(\zeta-1)\mathscr{N}(\zeta-1)} \int_0^\rho (\rho-\tau)^{\zeta-1}\gamma(\tau)d\tau \\
\leq & \left[\lambda + \frac{2-\zeta}{\mathscr{N}(\zeta-1)}\rho + \frac{1}{\zeta\mathscr{N}(\zeta-1)\Gamma(\zeta-1)}\rho^\zeta\right] \|\gamma\|_{L^\infty} \\
\leq & \Lambda\|\gamma\|_{L^\infty},
\end{aligned}
$$

which implies that $\mathscr{T}(\mathscr{O})$ is a bounded set.

In order to show the equicontinuity of $\mathscr{T}(\mathscr{O})$, let $\rho_1, \rho_2 \in J$ with $\rho_1 < \rho_2$. By (H2), for every $y \in \mathscr{O}$, we have

$$
\begin{aligned}
& |(\mathscr{T}y)(\rho_2) - (\mathscr{T}y)(\rho_1)| \\
\leq & \frac{2-\zeta}{\mathscr{N}(\zeta-1)} \int_{\rho_1}^{\rho_2} \gamma(\tau)d\tau + \frac{1}{\mathscr{N}(\zeta-1)\Gamma(\zeta-1)} \left[\int_{\rho_1}^{\rho_2} (\rho_2-\tau)^{\zeta-1}\gamma(\tau)d\tau\right. \\
& \left. + \int_0^{\rho_1} \left[(\rho_2-\tau)^{\zeta-1} - (\rho_1-\tau)^{\zeta-1}\right] \gamma(\tau)d\tau\right] \\
\leq & \left[\frac{2-\zeta}{\mathscr{N}(\zeta-1)}(\rho_2-\rho_1) + \frac{1}{\zeta\mathscr{N}(\zeta-1)\Gamma(\zeta-1)}(\rho_2^\zeta - \rho_1^\zeta)\right] \|\gamma\|_{L^\infty}.
\end{aligned}
$$

Since the function ρ^ζ, $1 < \zeta < 2$, is uniformly continuous on J, the set $\mathscr{T}(\mathscr{O})$ is equicontinuous. Therefore, in view of Arzelà-Ascoli theorem, $\mathscr{T}(\mathscr{O})$ is a relatively compact subset of $\mathscr{X}$. Hence, we conclude that $\mathscr{T}$ is completely continuous.

Let $y \in \mathscr{X}$ be an arbitrary element such that $y = \mathscr{S}y\mathscr{T}\tilde{y}$ for every $\tilde{y} \in \mathscr{O}$. By (H1) and (H2), for each $\rho \in J$, we have

$$
\begin{aligned}
|y(\rho)| \leq & |\mathscr{S}y(\rho)||\mathscr{T}\tilde{y}(\rho)| \\
& \leq (|g(\rho, y(\rho)) - g(\rho, 0)| + |g(\rho, 0)|) \left[\lambda \int_0^1 |f(\tau, \tilde{y}(\tau))|d\tau + \frac{2-\zeta}{\mathscr{N}(\zeta-1)} \int_0^\rho |f(\tau, \tilde{y}(\tau))|d\tau \right. \\
& \quad \left. + \frac{1}{\Gamma(\zeta-1)\mathscr{N}(\zeta-1)} \int_0^\rho (\rho-\tau)^{\zeta-1} |f(\tau, \tilde{y}(\tau))|d\tau \right] \\
& \leq (\psi(\rho)|y(\rho)| + g_0) \left[\lambda \int_0^1 \gamma(\tau)d\tau + \frac{2-\zeta}{\mathscr{N}(\zeta-1)} \int_0^\rho \gamma(\tau)d\tau \right. \\
& \quad \left. + \frac{1}{\mathscr{N}(\zeta-1)\Gamma(\zeta-1)} \int_0^\rho (\rho-\tau)^{\zeta-1} \gamma(\tau)d\tau \right] \\
& \leq (\psi^*|y(\rho)| + g_0)\Lambda\|\gamma\|_{L^\infty},
\end{aligned}
$$

which implies that

$$
|y(\rho)| \leq \frac{g_0\Lambda\|\gamma\|_{L^\infty}}{1-\Lambda\psi^*\|\gamma\|_{L^\infty}}.
$$

Taking supremum over ρ, one has

$$
\|y\|_{\mathscr{C}} \leq \frac{g_0\Lambda\|\gamma\|_{L^\infty}}{1-\Lambda\psi^*\|\gamma\|_{L^\infty}} = R.
$$

This implies that $y \in \mathscr{O}$. Finally, one has

$$
\mathscr{M} = \sup\{\|\mathscr{T}\xi\|_{\mathscr{C}} \colon \xi \in \mathscr{O}\} \leq \Lambda\|\gamma\|_{L^\infty},
$$

and therefore we get from (5) that

$$
\mathscr{M}\psi^* \leq \Lambda\psi^*\|\gamma\|_{L^\infty} < 1.
$$

By Theorem 1, the operator equation $\mathscr{S}y\mathscr{T}y = y$ has a solution in $\mathscr{O}$, which implies that the hybrid thermostat model (1) indeed has a solution in $C(J, \mathbb{R})$. This completes the proof of the theorem. □

Remark 1 For a particular case $g(\rho,\ y(\rho)) \equiv 1$ for $\rho \in J$, the problem (1) is readily reduced to the fractional thermostat model. So we can easily obtain sufficient conditions to admit the existence of solutions to the reduced thermostat model. It is also a new finding in the realm of fractional calculus.

4. Example

Motivated by the previous works presented in [3, 6], we have constructed an example in this section to support our main result. We first choose the normalization function $\mathcal{N}(\sigma) = 1 - \sigma + \frac{\sigma}{\Gamma(\sigma)}$ for $\sigma \in [0, 1]$. We consider the following problem

$$\begin{cases} {}^{ABC}\mathbf{D}_{0+}^{3/2}\left(\dfrac{y(\rho)}{\frac{e^{-\rho}}{\sqrt{2}+|y(\rho)|}}\right) + \rho^2 \sin y(\rho) = 0, \qquad \rho \in [0, 1], \\ \dfrac{d}{d\rho}\left(\dfrac{y(\rho)}{\frac{e^{-\rho}}{\sqrt{2}+|y(\rho)|}}\right)\Bigg|_{\rho=0} = 0, \quad \frac{1}{2}{}^{ABC}\mathrm{D}_{0+}^{1/2}\left(\dfrac{y(\rho)}{\frac{e^{-\rho}}{\sqrt{2}+|y(\rho)|}}\right)\Bigg|_{\rho=1} + \dfrac{y(\rho)}{\frac{e^{-\rho}}{\sqrt{2}+|y(\rho)|}}\Bigg|_{\rho=0} = 0, \end{cases} \tag{6}$$

where $\zeta = 3/2$ and $\lambda = 1/2$. (6) can be regarded as the form of (1), where

$$g(\rho, y) = \frac{e^{-\rho}}{\sqrt{2} + |y|} \quad \text{and} \quad f(\rho, y) = \rho^2 \sin y.$$

Clearly,

$$|g(\rho, y) - g(\rho, \tilde{y})| \le \frac{e^{-\rho}}{2}|y - \tilde{y}| \;\text{ and }\; |f(\rho, y)| \le \rho^2.$$

Here, $\psi(\rho) = e^{-\rho}/2$ and $\gamma(\rho) = \rho^2$. Then $\Lambda \approx 1.6202$ and $\Lambda\psi^*\|\gamma\|_{L^\infty} \approx 0.8101 < 1$. Thus, by Theorem (2), the problem (6) has a solution in $C(J, \mathbb{R})$.

5. Conclusions

In this work, we conducted a comprehensive study on the fractional hybrid thermostat model with the AB derivative in Caputo sense. The proof of our existence result relied on the Dhage fixed-point theorem in Banach algebra. An illustrative example was also presented to support our main theory. The results derived in this paper have filled (at least partially) the research gap regarding the existence of solutions for the thermostat model involving nonsingular derivatives and also directly enriched the literature on the Atangana-Baleanu fractional operators. As an interesting future study, we suggest that the reader may consider the inclusion version of our studied model (1).

Conflict of interest

The authors declare that they have no conflict of interest.

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