Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Face Bimagic Mean Labeling of Herschel Graphs and Its Operations

S. Vani Shree , S. Dhanalakshmi*

Department of Mathematics, Faculty of Engineering and Technology, SRM Institute Of Science And Technology, Ramapuram, Chennai, Tamil Nadu 600089, India
Email: dhanalas1@srmist.edu.in

Received: 13 February 2023; **Revised:** 20 July 2023; **Accepted:** 20 July 2023

Abstract: A bijection ϕ: $V(G) \to \{1, 2, \ldots, |V(G)|\}$ is called a (1,0,0)-F-face bimagic mean labeling (FBMML) of G if the induced face labeling $\phi^*(f_i) = \left\lfloor \frac{\text{sum of the label of the vertices in the boundary of } f_i}{\text{sum of vertices of } f_i} \right\rfloor = \left\lfloor \frac{\sum_{v_j \grave{o} f_i} \phi(v_j)}{\sum v_j \in f} \right\rfloor = k_1$ or k_2, constants. This work investigates the possibility of face bimagic mean labeling of a Herschel graph. Also, a study of face bimagic mean labeling during the process of fusion of vertices of degree 3 of a Herschel graph and the duplication of vertices by the edge of a Herschel graph is provided in this research paper.

***Keywords*:** Herschel graph, bimagic labeling, fusion, duplication, mean labeling, face bimagic mean labeling

MSC: 05C78

1. Introduction

Graph labeling is a dynamic and emerging topic in graph theory due to its widespread applications across all branches of research and industry. Graph labeling problems have become sources of inspiration for many graph theory enthusiasts. Particularly, it offers a variety of computer science applications, including data mining, software testing, image processing, communication networks, and information security. A thorough examination of the various types of graph labeling can be found in [1]. Harary [2] has been cited for common terms and notations used in graph theory. A graph label is a map that connects the nodes of a graph to a collection of numbers, often a collection of positive or non-negative integers. Total labeling is the term used to describe labeling where labels are applied to both vertices and edges. A discussion of magic labeling and standard terminology is referred to in [3]. Somasundaram and Ponraj [4] developed the conception of mean graph labeling.

Chemistry, electrical and electronic engineering, games, and contemporary art are just a few of the fields that require knowledge of polyhedral. A polyhedral graph is an undirected graph built from the vertices and edges of a convex polyhedron. The smallest polyhedral network without a Hamiltonian cycle is called the Herschel graph, as British astronomer Alexander Stewart Herschel dealt with this concept. The Herschel graph is a planar graph, and it can be drawn in the plane with none of its edges crossing. It is also 3-vertex-connected; the removal of any two of its

DOI: https://doi.org/10.37256/cm.5120242489

vertices leaves a connected subgraph. The Herschel graph is also a bipartite graph, and its vertices can be separated into two subsets of five and six vertices, respectively. Every edge has an endpoint in each subset. Hence, a bipartite graph is extensively applied in modern coding theory, concurrent systems, projective geometry, distributed systems, and so on.

Data fusion is the collaborative analysis of various interconnected datasets that offer different perspectives on the same phenomenon. In general, more precise inferences can be drawn from the correlation and fusion of data from several sources than from the investigation of a single dataset. Data fusion is required for integrating information obtained from various sources (sensors, databases, information gathered by humans, etc.). Similar to the concept of data fusion in a network system, there is an approach in graph operations that involves the fusion of vertices, as stated below:

Let u and v be two distinct vertices of a graph G. A new graph G_1 is constructed by fusing two vertices u and i by a single vertex x in G_1, which has every edge that was incident with either u or v in G now incident with x in G_1.

Making similar copies of data without making any changes is known as data duplication, which is frequently carried out for data distribution across various systems or for backup purposes. In the same manner, in graphs, the duplication of a vertex also takes place, which is defined below:

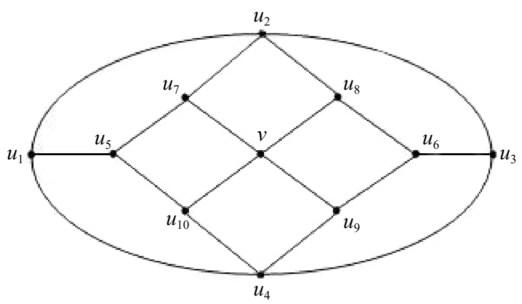
Duplication of a vertex v of graph G produces a new graph $G^{'}$ by adding a new vertex $v^{'}$ such that $N(v^{'}) = N(v)$. In other words, a vertex $v^{'}$ is said to be the duplication of v if all the vertices that are adjacent to v in G are also adjacent to $v^{'}$ in $G^{'}$. The objective of this study is to focus on vertex duplication and vertex fusion of a Herschel graph and their significant applications in the handling of data obtained from a large data set.

Ganesan and Balamurugan [5] have examined the prime labeling of the Herschel graph. (1,1,0)-F-face magic mean labeling (FMML) of $P_n + K_1$, latitude graph, cyclic ladder, and some graph duplication were tested in [6]. Adding up, Sugumaran and Rajesh [7] verified the cordial labeling of Herschel graph. In [8], face bimagic labeling has been proven for the double duplication of some graphs. Arockiaraj and Meena Kumari [9, 10] introduced the concept of F-FMML and further verified it for the square of a path graph, a globe graph, and for duplication of graphs. The concepts of F-FMML of the P_n graph, butterfly graph, and duplication of graphs were investigated in [11, 12]. The encryption of strong-face bimagic labeling has been authenticated in [13].

Vani Shree and Dhanalakshmi [14] have observed new results on face magic mean labeling of graphs. Vani Shree and Dhanalakshmi [15] initiated face bimagic mean labelling, which has been demonstrated for the duplication of a path graph. Vani Shree et.al. [16] explored the application of edge bimagic mean labeling in data security. The significant aim of this paper is to prove (1,0,0), (0,1,0), (1,0,1), (0,1,1), (1,1,0)-face bimagic mean labeling of a Herschel graph, (1,0,0), (1,0,1), (0,1,0), (1,1,0)-face bimagic mean labeling when the vertices of degree 3 of a Herschel graph are fused, and (1,0,0), (1,0,1)-face bimagic mean labeling when the vertex by an edge of a Herschel graph is duplicated.

2. Basic definitions

A Herschel graph H_s is a bipartite undirected graph with 11 vertices and 18 edges. In this paper, the position of vertices v, u_1, u_2, ..., u_{10} of H_s is fixed as indicated in Figure 1, unless otherwise specified.



Note: As this small polyhedron has symmetric features, the labeling of the fusion of some vertices of degree 3 follows the same labeling pattern as the similar structure of vertices of degree 3. Hence, the labeling of the fusion of u_1 and u_5; u_7 and u_5; u_5 and u_{10} is similar to that of u_6 and u_3; u_6 and u_8; u_6 and u_9. Here, the face bimagic mean labeling pattern for fusion of u_6 and u_3; u_6 and u_8 are provided with suitable illustrations.

Figure 1. Herschel graph H_s

A bimagic mean labeling on graph G is a one-one map f from $V(G) \cup E(G)$ the integers 1,2,…$|V + E|$, where $V = |V(G)|$ and $E = |E(G)|$ with the property that has given any edge (u, v)

$$\left\lfloor \frac{f(u)+f(v)}{2} \right\rfloor = k_1 \text{ or } k_2.$$

A bijection ϕ: $E(G) \to \{1, 2, ...|E(G)|\}$ is called a (0,1,0)-F-face bimagic mean labeling (F-FBMML) of G if the induced face labeling

$$\phi^*(f_i) = \left\lfloor \frac{\text{sum of the label of the edges in the boundary of } f_i}{\text{sum of edges of } f_i} \right\rfloor$$
$$= \left\lfloor \frac{\sum_{e_j \epsilon f_i} \phi(f_j)}{\sum e_j \in f_i} \right\rfloor = k_1 \text{ or } k_2, \text{ constants.}$$

A bijection ϕ: $V(G) \cup F(G) \to \{1, 2, ..., |V(G)| + |E(G)|\}$ is called a (1,1,0)-F-face bimagic mean labeling of G if the induced face labeling

$$\phi^*(f_i) = \left\lfloor \frac{\text{sum of the label of the vertices and edges in the boundary of } f_i}{\text{sum of vertices and edges of } f_i} \right\rfloor$$
$$= \left\lfloor \frac{\sum_{v_j \epsilon f_i} \phi(v_j) + \sum_{e_j \epsilon f_i} \phi(e_j)}{\sum v_j \in f_i + \sum e_j \in f_i} \right\rfloor = k_1 \text{ or } k_2, \text{ constants.}$$

A bijection ϕ: $V(G) \cup F(G) \to \{1, 2, ..., |V(G)| + |F(G)|\}$ is called a (1,0,1)-F-face bimagic mean labeling of G if the induced face labeling

$$\phi^*(f_i) = \left\lfloor \frac{\text{sum of the label of the vertices and faces in the boundary of } f_i}{\text{sum of vertices and faces of } f_i} \right\rfloor$$
$$= \left\lfloor \frac{\sum_{v_j \epsilon f_i} \phi(v_j) + \sum_{f_j \epsilon f_i} \phi(f_j)}{\sum v_j \in f_i + \sum f_j \in f_i} \right\rfloor = k_1 \text{ or } k_2, \text{ constants.}$$

A bijection ϕ: $E(G) \cup F(G) \to \{1, 2, ..., |E(G)| + |F(G)|\}$ is called a (0,1,1)-F-face bimagic mean labeling of G if the induced face labeling

$$\phi^*(f_i) = \left\lfloor \frac{\text{sum of the label of the edges and faces in the boundary of } f_i}{\text{sum of edges and faces of } f_i} \right\rfloor$$
$$= \left\lfloor \frac{\sum_{e_j \epsilon f_i} \phi(e_j) + \sum_{f_j \epsilon f_i} \phi(f_j)}{\sum e_j \in f_i + \sum f_j \in f_i} \right\rfloor = k_1 \text{ or } k_2, \text{ constants.}$$

3. Results and discussions

Theorem 3.1. The Herschel graph H_s admits face bimagic mean labeling of a graph of types (1,0,0).

Proof. Let $G(V, E, F)$ be a Herschel graph H_s for mean bimagic labeling of a graph that contains 11 vertices, 18 edges and eight interior faces, and one exterior face. Let the set of vertex be V, the set of edges be E, and the face set be F, whose interior face and exterior face of G are denoted as f_i and f_0, respectively.

(1,0,0) F-FBMML

Consider a mapping ψ: $V(G) \rightarrow \{1, 2, 3 \ldots |V(G)|\}$ as

$$\psi(v) = 3$$
$$\psi(u_{2i-1}) = i;\ i = 1, 2$$
$$\psi(u_{2i}) = i + 7;\ i = 1, 2$$
$$\psi(u_i) = 16 - i;\ i = 5, 6$$
$$\psi(u_{2i-1}) = 2i;\ i = 3, 4$$
$$\psi(u_{2i}) = 2i;\ i = 2, 3$$

Hence, the induced face mean bimagic labeling in G is obtained as

$$\psi^*(f_{2i-1}) = \frac{24-i}{4} = \lfloor 5.75 \rfloor \text{ or } \lfloor 5.5 \rfloor = 5;\ i = 1,2$$
$$\psi^*(f_{2i}) = \frac{23+i}{4} = \lfloor 6 \rfloor \text{ or } \lfloor 6.25 \rfloor = 6;\ i = 1,2$$
$$\psi^*(f_i) = \frac{27}{4} = \lfloor 6.75 \rfloor = 6;\ i = 5,7$$
$$\psi^*(f_i) = \frac{25}{4} = \lfloor 6.25 \rfloor = 5;\ i = 6,8$$
$$\psi^*(f_0) = \frac{20}{4} = \lfloor 5 \rfloor = 5;\ i = 1,2.$$

The obtained constants k_1 and k_2 are 5 and 6, respectively. Labeling of vertices followed by the given pattern is shown in Figure 2.

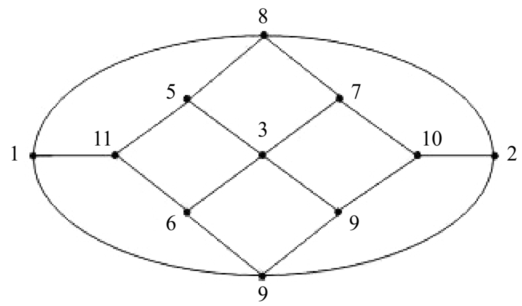


Figure 2. (1,0,0)-FBMML of H_s

Theorem 3.2. The Herschel graph, H_s, admits face bimagic mean labeling of a graph of type (0,1,0).

Proof. Let $G(V, E, F)$ be a Herschel graph, H_s, for mean bimagic labeling of a graph that contains 11 vertices, 18 edges and eight interior faces, and one exterior face. Let the set of vertex be V, the set of edges be E, and the face set be F, whose interior face and exterior face of G are denoted as f_i and f_0, respectively.

(0, 1, 0) F-FBMML

Consider a mapping ψ: $E(G) \rightarrow \{1, 2, 3\ldots|E(G)|\}$ as

$$\psi\left(vu_{2i-1}\right) = 2i-;\ i = 3,4$$
$$\psi\left(vu_{2i}\right) = 2i;\ i = 3,4$$
$$\psi\left(u_{i+4}u_{12-2i}\right) = i+1;\ i = 1,2$$
$$\psi\left(u_{i+4}u_{2i+5}\right) = 17-i;\ i = 1,2$$
$$\psi\left(u_{\frac{i+1}{2}}u_{4+i}\right) = \frac{21+i}{2};\ i = 1,3$$
$$\psi\left(u_{i}u_{4i}\right) = 16+i;\ i = 1,2$$
$$\psi\left(u_{i+1}u_{3i}\right) = 11-i;\ i = 1,2$$
$$\psi\left(u_{i+1}u_{i+1}\right) = i;\ i = 1$$
$$\psi\left(u_{i+2}u_{2i+2}\right) = 14;\ i = 1$$
$$\psi\left(u_{2i+2}u_{8+i}\right) = 2i+2;\ i = 1$$
$$\psi\left(u_{2i+2}u_{9+i}\right) = 13;\ i = 1$$

The induced face bimagic mean labeling

$$\psi^{*}\left(f_i\right) = \frac{42}{4}\lfloor 10.5 \rfloor = 10;\ i = 0,1,7$$
$$\psi^{*}\left(f_i\right) = \frac{40}{4}\lfloor 10 \rfloor = 10;\ i = 5,6$$
$$\psi^{*}\left(f_i\right) = \frac{31}{4}\lfloor 7.75 \rfloor = 7;\ i = 2,3,4$$
$$\psi^{*}\left(f_8\right) = \frac{43}{4}\lfloor 10.75 \rfloor = 10$$

The constants k_1 and k_2 are 7 and 10, respectively. Labeling of edges followed by the above pattern is shown in Figure 3.

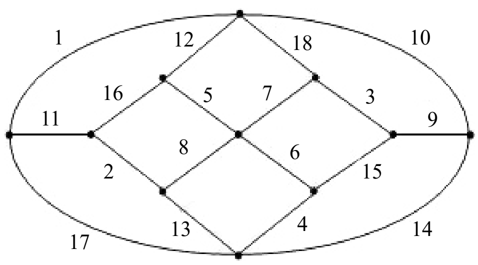


Figure 3. (0,1,0)-FBMML of H_s

Theorem 3.3. The Herschel graph, H_s, admits face bimagic mean labeling of a graph of type (0,1,1).

Proof. Let $G(V, E, F)$ be a Herschel graph, H_s, for mean bimagic labeling of graph which contains 11 vertices, 18 edges and eight interior faces, and one exterior face. Let the set of vertex be V, the set of edges be E, and the face set be F, whose interior face and exterior face of G are denoted as f_i and f_0, respectively.

(0,1,1) F-FBMML

Consider a mapping ψ: $E(G) \cup F(G) \rightarrow \{1, 2, 3\ldots|E(G) + |F(G)|\}$ as

$$\psi\left(vu_{2i-1}\right)=2i-1;\ i=3,4$$
$$\psi\left(vu_{2i}\right)=2i;\ i=3,4$$
$$\psi\left(u_{i+4}u_{12-2i}\right)=i+1;\ i=1,2$$
$$\psi\left(u_{i+4}u_{2i+5}\right)=17-i;\ i=1,2$$
$$\psi\left(u_{\frac{i+1}{2}}u_{4+i}\right)=\frac{21+i}{2};\ i=1,3$$
$$\psi\left(u_iu_{4i}\right)=16+i;\ i=1,2$$
$$\psi\left(u_{i+1}u_{3i}\right)=11-i;\ i=1,2$$
$$\psi\left(u_iu_{i+1}\right)=i;\ i=1$$
$$\psi\left(u_{i+2}u_{2i+2}\right)=14;\ i=1$$
$$\psi\left(u_{2i+2}u_{8+i}\right)=2i+2;\ i=1$$
$$\psi\left(u_{2i+2}u_{9+i}\right)=13;\ i=1$$
$$\psi\left(f_1\right)=21$$
$$\psi\left(f_{i+1}\right)=28-i;\ i=1,2,3$$
$$\psi\left(f_{4+i}\right)=22+i;\ i=1,2$$
$$\psi\left(f_{6+i}\right)=21-i;\ i=1,2$$
$$\psi\left(f_0\right)=22$$

The induced face bimagic mean labeling

$$\psi^*\left(f_i\right)=\frac{60-i}{5}=\lfloor 11.6\rfloor,\ \lfloor 11.4\rfloor,\text{ and }\lfloor 11.2\rfloor=11;\ i=2,3,4$$
$$\psi^*\left(f_i\right)=\frac{62}{5}=\lfloor 12.4\rfloor=12;\ i=7,8$$
$$\psi^*\left(f_i\right)=\frac{63}{5}=\lfloor 12.6\rfloor=12;\ i=1,5$$
$$\psi^*\left(f_i\right)=\frac{64}{5}=\lfloor 12.8\rfloor=12;\ i=0,6$$

The constants k_1 and k_2 are 11 and 12, respectively. Labeling of edges and faces followed by the above pattern is shown in Figure 4.

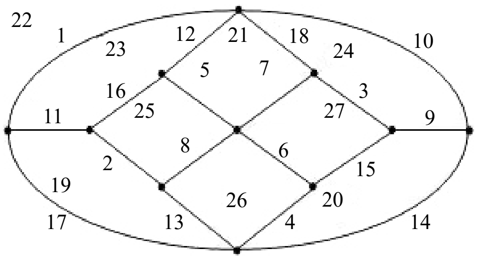


Figure 4. (0,1,1)-FBMML of H_s

Theorem 3.4. The Herschel graph H_s admits face bimagic mean labeling of a graph of type (1,0,1).

Proof. Let $G(V, E, F)$ be a Herschel graph H_s for mean bimagic labeling of graph which contains 11 vertices, 18 edges and eight interior faces, and one exterior face. Let the set of vertex be V, the set of edges be E, and the face set be F, whose interior face and exterior face of G are denoted as f_i and f_0, respectively.

(1,0,1) F-FBMML

Consider a mapping ψ: $V(G) \cup F(G) \to \{1, 2, 3\ldots|V(G)| + |F(G)|\}$ as

$$\psi(v) = 3$$
$$\psi(u_{2i-1}) = i;\ i = 1,2$$
$$\psi(u_{2i}) = i + 7;\ i = 1,2$$
$$\psi(u_i) = 16 - i;\ i = 5,6$$
$$\psi(u_{2i-1}) = 2i;\ i = 2,3$$
$$\psi(u_{2i}) = 2i;\ i = 2,4$$
$$\psi(f_{2i-1}) = 17 + i;\ i = 1,2$$
$$\psi(f_{2i}) = 18 - i;\ i = 1,2$$
$$\psi(f_{2i-1}) = 11 + i;\ i = 3,4$$
$$\psi(f_{2i}) = 9 + i;\ i = 3,4$$
$$\psi(f_0) = 20$$

Hence, the induced face mean bimagic labeling in G is obtained as

$$\psi^*(f_i) = \frac{41}{5} = \lfloor 8.2 \rfloor = 8;\ i = 1,2,3,4$$
$$\psi^*(f_i) = \frac{39}{5} = \lfloor 7.8 \rfloor = 7;\ i = 5,6$$
$$\psi^*(f_i) = \frac{38}{5} = \lfloor 7.6 \rfloor = 7;\ i = 8$$
$$\psi^*(f_i) = \frac{42}{5} = \lfloor 8.4 \rfloor = 8;\ i = 7$$
$$\psi^*(f_0) = \frac{40}{5} = \lfloor 8 \rfloor = 8;\ i = 0$$

The obtained constants k_1 and k_2 are 7 and 8. Labeling of vertices and faces followed by the above pattern is shown in Figure 5.

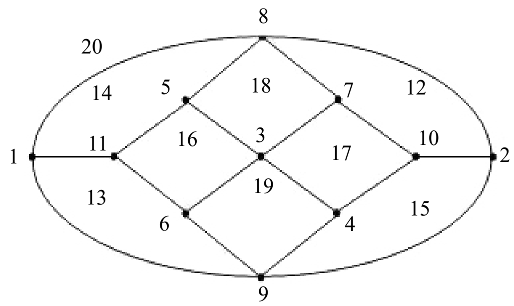


Figure 5. (1,0,1)-FBMML of H_s

Theorem 3.5. The Herschel graph, H_s, admits face bimagic mean labeling of a graph of type (1,1,0).

Proof. Let $G(V, E, F)$ be a Herschel graph, H_s, for mean bimagic labeling of graph which contains 11 vertices, 18 edges and eight interior faces, and one exterior face. Let the set of vertex be V, set of edges be E, and the face set be F, whose interior face and exterior face of G are denoted as f_i and f_0, respectively.

(1,1,0) F-FBMML

Consider a mapping ψ: $V(G) \cup E(G) \rightarrow \{1, 2, 3\ldots|V(G)| + |E(G)|\}$ as

$$\psi(v) = 5$$
$$\psi(u_7) = 15$$
$$\psi(u_{2i-1}) = 12 - i;\ i = 1,2$$
$$\psi(u_{i+1}) = 10 - i;\ i = 1,3$$
$$\psi(u_{i+3}) = 2i + 2;\ i = 2,3$$
$$\psi(u_{i+7}) = 11 + i;\ i = 1,2,3$$
$$\psi(vu_{i+6}) = 23 + i;\ i = 1,3$$
$$\psi(vu_{i+6}) = 29 - i;\ i = 2,4$$
$$\psi(u_{2i-1}u_{2i}) = 17 + i;\ i = 1,2$$
$$\psi(u_i u_{5-i}) = 18 - i;\ i = 1,2$$
$$\psi(u_{2i-1}u_{4+i}) = 30 - i;\ i = 1,2$$
$$\psi(u_{i+1}u_{6+i}) = \frac{i+1}{2};\ i = 1,3$$
$$\psi(u_{10-i}u_{2i}) = i - 1;\ i = 4,5$$
$$\psi(u_{6-2i}u_{12-2i}) = 24 - i;\ i = 1,2$$
$$\psi(u_6 u_9) = 21$$

Hence, the induced face mean bimagic labeling in G is obtained as

$$\psi^*(f_i) = \frac{115}{8} = \lfloor 14.375 \rfloor = 14;\ i = 1,2,3$$
$$\psi^*(f_4) = \frac{115}{8} = \lfloor 14.125 \rfloor = 14$$
$$\psi^*(f_i) = \frac{109}{8} = \lfloor 13.625 \rfloor = 13;\ i = 5,6$$
$$\psi^*(f_8) = \frac{110}{8} = \lfloor 13.75 \rfloor = 13$$
$$\psi^*(f_7) = \frac{108}{8} = \lfloor 13.5 \rfloor = 13$$
$$\psi^*(f_0) = \frac{107}{8} = \lfloor 13.375 \rfloor = 13$$

The derived constants k_1 and k_2 are 13 and 14. Figure 6 displays the relevant labeling pattern and related constants.

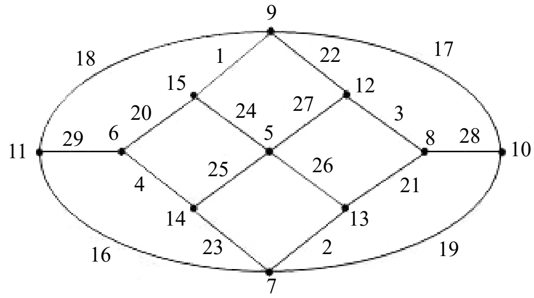


Figure 6. (1,1,0)-FBMML of H_s

Theorem 3.6. The vertex by the edge duplication of Herschel graph $D_v(H_s)$ admits face bimagic mean labeling of a graph of type (1,0,0).

Proof. Let $G(V, E, F)$ be a vertex by the edge duplication of Herschel graph, H_s, for mean bimagic labeling of graph which contains 33 vertices, 51 edges and 19 interior faces, and one exterior face. Let the set of vertices be V, the set of edges be E, and the face set be F, whose interior face and exterior face of G are denoted as f_i and f_0, respectively.

(1,0,0) F-FBMML

Consider a mapping ψ: $V(G) \rightarrow \{1, 2, 3\ldots |V(G)|\}$ as

$$\psi(v) = i + 1;\ i = 1$$
$$\psi(x) = 18$$
$$\psi(y) = 32$$
$$\psi(u_i) = 11 + i;\ i = 1,3$$
$$\psi\left(\frac{u_{6+i}}{2}\right) = 11 + i;\ i = 2,4$$
$$\psi(u_i) = 11 + i;\ i = 6,8$$
$$\psi(u_{i+1}) = 20 + i;\ i = 1$$
$$\psi(u_{3i+1}) = 20 + i;\ i = 2$$
$$\psi\left(\frac{u_{16+i}}{2}\right) = 24 + i;\ i = 2,4$$
$$\psi(x_{2+i}) = 8 + i;\ i = 1,2$$
$$\psi(x_{4+i}) = 2i - 1;\ i = 1,2$$
$$\psi(x_{5+i}) = i;\ i = 4,5$$
$$\psi(x_i) = i - 1;\ i = 7,8$$
$$\psi(x_i) = 10 + i;\ i = 1$$
$$\psi(x_{i+1}) = 7 + i;\ i = 1$$
$$\psi(y_{2+i}) = 33 - 3i;\ i = 1,2$$
$$\psi\left(\frac{y_{9+i}}{2}\right) = 34 + i;\ i = 1,3$$
$$\psi\left(\frac{y_{13+i}}{2}\right) = 22 + i;\ i = 1,3$$
$$\psi(y_i) = 28 + i;\ i = 1$$

$$\psi(y_i) = 22 + i;\ i = 2$$
$$\psi(y_{i+1}) = 12 + i;\ i = 8$$
$$\psi(y_{i+1}) = 7 + i;\ i = 9$$

Hence, the induced face mean bimagic labeling in G is obtained as

$$\psi^*(f_1) = \frac{64}{4} = \lfloor 16 \rfloor = 16$$
$$\psi^*(f_i) = \frac{70 - i}{4} = \lfloor 17 \rfloor \text{ and } \lfloor 17.25 \rfloor = 17;\ i = 1, 2$$
$$\psi^*(f_{4+i}) = \frac{66 + 2i}{4} = \lfloor 17 \rfloor \text{ and } \lfloor 17.5 \rfloor = 17;\ i = 1, 2$$
$$\psi^*(f_{6+i}) = \frac{72 - i}{4} = \lfloor 17.75 \rfloor \text{ and } \lfloor 17.5 \rfloor = 17;\ i = 1, 2$$
$$\psi^*(f_0) = \frac{208}{12} = \lfloor 17.3 \rfloor = 17$$
$$\psi^*(f_i') = \frac{51 + i}{3};\ i = 1, 2$$
$$\psi^*(f_i') = \frac{54 - i}{3};\ i = 4, 5$$
$$\psi^*(f_i') = \frac{51}{3};\ i = 6, 7, 8$$
$$\psi^*(f_i') = \frac{59 - i}{3};\ i = 9, 10$$
$$\psi^*(f_3') = \frac{53}{3}$$
$$\psi^*(f_3') = \frac{52}{3}$$

The obtained constants k_1 and k_2 are 16 and 17. Labeling of vertices followed by the above pattern is shown in Figure 7.

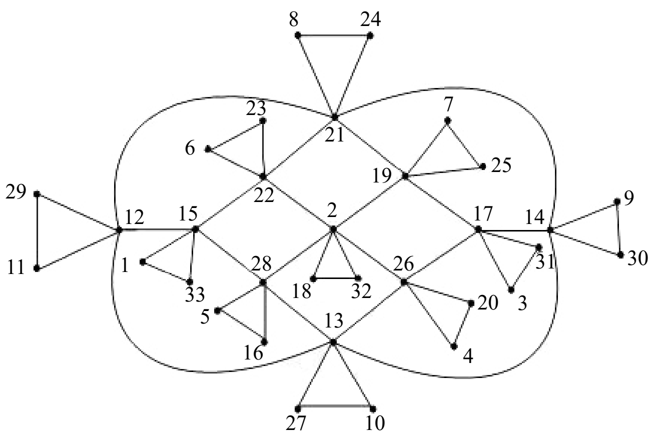


Figure 7. (1,0,0)-D_v of FBMML of Herschel graph

Theorem 3.7. The vertex by the edge duplication of Herschel graph $D_v(H_s)$ admits face bimagic mean labeling of a graph of type (1,0,1).

Proof. Let $G(V, E, F)$ be a vertex by edge duplication of Herschel graph H_s for mean bimagic labeling of graph which contains 33 vertices, 51 edges and 19 interior faces, and one exterior face. Let the set of vertex be V, the set of edges be E and the face set be F whose interior face and exterior face of G are denoted as f_i and f_0, respectively.

(1,0,1) F-FBMML

Consider a mapping ψ: $V(G) \cup F(G) \rightarrow \{1, 2, 3 \ldots |V(G)| + |F(G)|\}$ as

$$\psi(v) = 2$$
$$\psi(x) = 13$$
$$\psi(y) = 33$$
$$\psi(u_{2i-1}) = 18 + i;\ i = 1, 3$$
$$\psi(u_{2i}) = 14 + i;\ i = 1, 3$$
$$\psi\left(\frac{u_{8+i}}{2}\right) = 16 - i;\ i = 2, 4$$
$$\psi(u_{6+i}) = 28 + i;\ i = 1, 2, 3, 4$$
$$\psi(x_i) = 19 + i;\ i = 1, 3$$
$$\psi(x_i) = 24 - i;\ i = 2, 4$$
$$\psi(x_{4+i}) = 2i - i;\ i = 1, 2$$
$$\psi(x_i) = i - 1;\ i = 7, 8$$
$$\psi(x_{5+i}) = i;\ i = 4, 5$$
$$\psi(y_i) = 27 - i;\ i = 1, 3$$
$$\psi(y_i) = 21 + i;\ i = 2, 4$$
$$\psi(y_{4+i}) = 26 + i;\ i = 1, 2$$
$$\psi(y_{6+i}) = 10 - i;\ i = 1, 2$$
$$\psi(y_{8+i}) = 12 - i;\ i = 1, 2$$
$$\psi(f_i) = 39 + i;\ i = 1, 2$$
$$\psi(f_3) = 34$$
$$\psi(f_i) = 43 - i;\ i = 4, 5, 6, 7, 8$$
$$\psi(f_0) = 53$$
$$\psi(f_1^{'}) = \frac{42}{4}$$
$$\psi(f_1^{'}) = \frac{47 - i}{4};\ i = 2, 3, 4$$
$$\psi(f_1^{'}) = \frac{57 - i}{4};\ i = 5, 6, 7, 8, 9, 10, 11$$

Hence, the induced face mean bimagic labeling in G is obtained as

$$\psi^*(f_i) = \frac{116}{5} = \lfloor 23.2 \rfloor = 23;\ i = 1,2,3,4$$

$$\psi^*(f_i) = \frac{115}{5} = \lfloor 23 \rfloor = 23;\ i = 5,6$$

$$\psi^*(f_i) = \frac{117}{5} = \lfloor 23.4 \rfloor = 23;\ i = 7,8$$

$$\psi^*(f_0) = \frac{299}{13} = \lfloor 23 \rfloor = 23$$

$$\psi^*(f_i^{'}) = \frac{105}{4} = \lfloor 26.25 \rfloor = 26;\ i = 1,2,3,4$$

$$\psi^*(f_i^{'}) = \frac{94}{4} = \lfloor 26.25 \rfloor = 26;\ i = 5,6,7,8,9,10,11$$

The constants k_1 and k_2 are 23 and 26, respectively. Figure 8 shows the corresponding labeling pattern and related constants.

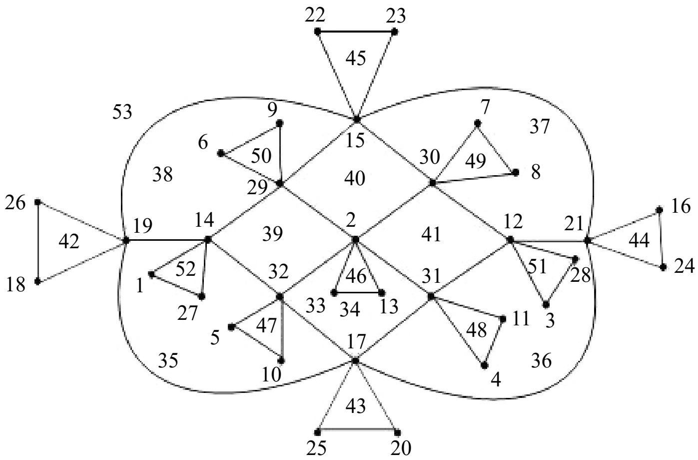


Figure 8. (1,0,1)-D_v of FBMML of Herschel graph

Theorem 3.8. The fusion of any two adjacent vertices of degree 3 of a Herschel graph H_s admits face bimagic mean labeling of a graph of type (1,0,0).

Proof. Let $G(V, E, F)$ be a fusion of a Herschel graph H_s for the mean bimagic labeling of graph that contains $|V(G)| = 10$ vertices , $|E(G)| = 17$ edges and $|F(G)| = 9$ faces, which has eight interior faces and one exterior face. Let the set of vertex be V, the set of edges be E and the face set be F, whose interior face and exterior face of G are denoted as f_i and f_0, respectively.

Fusion of u_6 and u_3. Consider that u_6 and u_3 are fused together as a single vertex u.

(1,0,0) F-FBMML

Consider a mapping ψ: $V(G) \cup F(G) \rightarrow \{1, 2, 3\ldots |V(G)|\}$ as

$$\psi(v) = 2$$
$$\psi(u) = 4$$
$$\psi\left(u_i\right) = i;\ i = 1,8$$
$$\psi\left(u_2\right) = 7$$
$$\psi\left(u_{10}\right) = 3$$
$$\psi\left(u_{i+1}\right) = 6 + i;\ i = 3,4$$
$$\psi\left(u_{2i+1}\right) = 2 + i;\ i = 3,4$$

Hence, the induced face mean bimagic labeling in G is obtained as

$$\psi^*(f_i) = \frac{21+i}{4} = \lfloor 5.25 \rfloor \text{ or } \lfloor 5.5 \rfloor = 5;\ i = 0,1$$
$$\psi^*(f_i) = \frac{23}{4} = \lfloor 5.75 \rfloor = 5;\ i = 5,8$$
$$\psi^*(f_i) = \frac{20}{4} = \lfloor 5 \rfloor = 5;\ i = 2,3,4$$
$$\psi^*(f_i) = \frac{19}{3} = \lfloor 6.3 \rfloor = 6;\ i = 6,7$$

The constants k_1 and k_2 are 5 and 6, respectively. Labeling of vertices followed by the above pattern is shown in Figure 9.

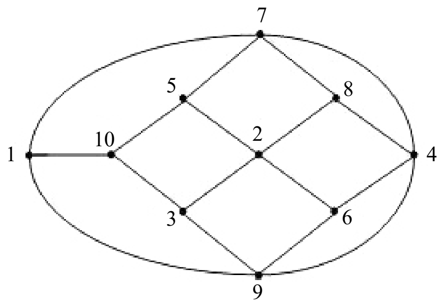


Figure 9. (1,0,0)-FBMML fusion of u_6 and u_3 in H_s

Fusion of u_6 and u_8. Consider that u_6 and u_8 are fused together as a single vertex u.

(1,0,0) F-FBMML

Consider a mapping f: $V(G) \rightarrow \{1, 2, 3\ldots|V(G)|$ such that

$$\psi(u) = 7$$
$$\psi(v) = 3$$
$$\psi\left(u_{2i}\right) = 7 + i;\ i = 1,2$$
$$\psi\left(u_i\right) = i;\ i = 1$$
$$\psi\left(u_i\right) = i + 1;\ i = 3$$
$$\psi\left(u_i\right) = i - 4;\ i = 9,10$$
$$\psi\left(u_i\right) = 2i;\ i = 5$$
$$\psi\left(u_7\right) = 2$$

Hence, the induced face mean bimagic labeling in G is obtained as

$$\psi^*(f_1) = \frac{22}{4} = \lfloor 5.5 \rfloor = 5$$
$$\psi^*(f_1) = \frac{20}{4} = \lfloor 5 \rfloor = 5$$
$$\psi^*(f_2) = \frac{15}{3} = \lfloor 5 \rfloor = 5$$
$$\psi^*(f_6) = \frac{19}{3} = \lfloor 6.3 \rfloor = 6$$
$$\psi^*(f_8) = \frac{26}{4} = \lfloor 6.5 \rfloor = 6$$
$$\psi^*(f_i) = \frac{21}{4} = \lfloor 5.25 \rfloor = 5;\ i = 4,5$$
$$\psi^*(f_i) = \frac{23}{4} = \lfloor 5.75 \rfloor = 5;\ i = 3,7$$

The obtained constants k_1 and k_2 are 5 and 6.

Hence, the fusion of any two adjacent vertices of degree 3 of a Herschel graph H_s admits face bimagic mean labeling of a graph of type (1,0,0).

Theorem 3.9. The fusion of any two adjacent vertices of degree 3 of a Herschel graph H_s admits face bimagic mean labeling of a graph of type (0,1,0).

Proof. Let $G(V, E, F)$ be a fusion of a Herschel graph H_s for the mean bimagic labeling of graph which contains $|V(G)| = 10$ vertices, $|E(G)| = 17$ edges, and $|F(G)| = 9$ faces, which has eight interior faces and one exterior face. Let the set of vertex be V, set of edges be E, and the face set be F, whose interior face and exterior face of G are denoted as f_i and f_0, respectively.

Fusion of u_6 and u_3. Consider that u_6 and u_3 are fused together as a single vertex u.

(0,1,0) F-FBMML

Consider a mapping ψ: $E(G) \rightarrow \{1, 2, 3\ldots |E(G)|\}$ as

$$\psi(vu_{2i+1}) = 2+i;\ i = 3,4$$
$$\psi(vu_{2i}) = 3+i;\ i = 4,5$$
$$\psi(u_1u_{2i}) = 11-i;\ i = 1,2$$
$$\psi(uu_{i+1}) = 10+i;\ i = 1,2$$
$$\psi(u_{i-1}u_{i-6}) = i-7;\ i = 10,11$$
$$\psi(u_{6+i}u_{1+i}) = i;\ i = 1$$
$$\psi(u_{6+i}u) = i;\ i = 2$$
$$\psi(u_{6+i}u) = 12+i;\ i = 3$$
$$\psi(u_{2i}u_{2i+6}) = 20-3i;\ i = 1,2$$
$$\psi(u_1u_5) = 16$$
$$\psi(u_5u_7) = 13$$

Hence, the induced face mean bimagic labeling in G is obtained as

$$\psi^*(f_i) = \frac{30}{4} = \lfloor 7.5 \rfloor = 7;\ i = 1,2,4$$
$$\psi^*(f_3) = \frac{31}{4} = \lfloor 7.75 \rfloor = 7$$
$$\psi^*(f_5) = \frac{40}{4} = \lfloor 10 \rfloor = 10$$
$$\psi^*(f_8) = \frac{43}{4} = \lfloor 10.75 \rfloor = 10$$
$$\psi^*(f_0) = \frac{42}{4} = \lfloor 10.5 \rfloor = 10$$
$$\psi^*(f_i) = \frac{30}{3} = \lfloor 10 \rfloor = 10;\ i = 6,7$$

The obtained constants k_1 and k_2 are 7 and 10. Labeling of edges followed by the above pattern is shown in Figure 10.

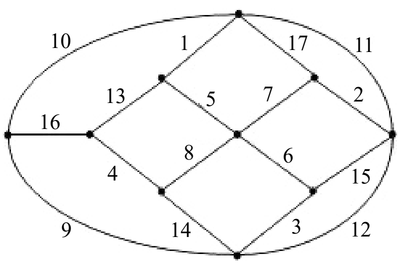


Figure 10. (0,1,0)-FBMML of fusion of u_6 and u_3 in H_s

Fusion of u_6 and u_8. Consider that u_6 and u_8 are fused together as a single vertex u.

(0,1,0) F-FBMML

Consider a mapping ψ: $E(G) \rightarrow \{1, 2, 3\ldots|E(G)|\}$ as

$$\psi\left(vu_{i+6}\right)=6-i;\ i=1$$
$$\psi\left(vu_{2i}\right)=112-i;\ i=4,5$$
$$\psi\left(vu_{8+i}\right)=5-i;\ i=1$$
$$\psi\left(u_{i+1}u_{i+6}\right)=i;\ i=3,4$$
$$\psi\left(uu_{i+1}\right)=13-i;\ i=1,2,3$$
$$\psi\left(u_{4+i}u_{i+6}\right)=12+i;\ i=1$$
$$\psi\left(u_1u_{4+i}\right)=14+i;\ i=1$$
$$\psi\left(u_{i+1}u_{i+6}\right)=i;\ i=1$$
$$\psi\left(u_4u_{10}\right)=14$$
$$\psi\left(uu_2\right)=17$$
$$\psi\left(uu_3\right)=2$$
$$\psi\left(uu_9\right)=16$$

Hence, the induced face mean bimagic labeling in G is obtained as

$$\psi^*(f_i)=\frac{42}{4}=\lfloor 10.5\rfloor=10;\ i=0,8$$
$$\psi^*(f_i)=\frac{31}{4}=\lfloor 7.75\rfloor=7;\ i=1,7$$
$$\psi^*(f_i)=\frac{33-i}{4}=\lfloor 7.25\rfloor \text{ or } \lfloor 7.5\rfloor=7;\ i=3,4$$
$$\psi^*(f_5)=\frac{41}{4}=\lfloor 10.25\rfloor=10$$
$$\psi^*(f_0)=\frac{42}{4}=\lfloor 10.5\rfloor=10$$
$$\psi^*(f_i)=\frac{30}{4}=\lfloor 10\rfloor=10;\ i=2,6$$

The derived constants k_1 and k_2 are 7 and 10

Hence, the fusion of any two adjacent vertices of degree 3 of a Herschel graph H_s admits face bimagic mean labeling of a graph of type (0,1,0).

Theorem 3.10. The fusion of any two adjacent vertices of degree 3 of a Herschel graph H_s admits face bimagic mean labeling of a graph of type (1,0,1).

Proof. Let $G(V, E, F)$ be a fusion of a Herschel graph H_s for mean bimagic labeling of graph which contains $|V(G)| = 10$ vertices , $|E(G)| = 17$ edges, and $|F(G)| = 9$ faces, which has eight interior faces and one exterior face. Let the set of vertex be V, set of edges be E, and the face set be F, whose interior face and exterior face of G are denoted as f_i and f_0, respectively.

Fusion of u_6 and u_3. Consider that u_6 and u_3 are fused together as a single vertex u.

(1,0,1) F-FBMML

Consider a mapping ψ: $V(G) \cup F(G) \rightarrow \{1, 2, 3\ldots|V(G)| + |F(G)|\}$ as

$$\psi(v) = 2$$
$$\psi(u) = 4$$
$$\psi(u_i) = i;\ i = 1,8$$
$$\psi(u_2) = 7$$
$$\psi(u_{10}) = 3$$
$$\psi(u_{i+1}) = 6 + i;\ i = 3,4$$
$$\psi(u_{2i+1}) = 2 + i;\ i = 3,4$$
$$\psi(f_i) = 19 - i;\ i = 1,2,3,4$$
$$\psi(f_0) = 19$$
$$\psi(f_{i+2}) = 2i + 5;\ i = 3,4$$
$$\psi(f_i) = 2i - 2;\ i = 7,8$$

Hence, the induced face mean bimagic labeling in G is obtained as

$$\psi^*(f_i) = \frac{38 - i}{5} = \lfloor 7.4 \rfloor \text{ or } \lfloor 7.2 \rfloor \text{ or } \lfloor 7 \rfloor = 7;\ i = 1,2,3$$
$$\psi^*(f_i) = \frac{40}{5} = \lfloor 8 \rfloor = 8;\ i = 0,1$$
$$\psi^*(f_{5+i}) = \frac{29 + i}{4} = \lfloor 7 \rfloor \text{ or } \lfloor 7.75 \rfloor = 7;\ i = 1,2$$
$$\psi^*(f_5) = \frac{36}{5} = \lfloor 7.2 \rfloor = 7$$
$$\psi^*(f_7) = \frac{37}{5} = \lfloor 7.4 \rfloor = 7$$

The obtained constants k_1 and k_2 are 7 and 8.

Fusion of u_6 and u_8. Consider that u_6 and u_8 are fused together as a single vertex u.

(1,0,1) F-FBMML

Consider a mapping ψ: $V(G) \cup F(G) \rightarrow \{1, 2, 3 \ldots |V(G)| + |F(G)|\}$ as

$$\psi(v) = 3$$
$$\psi(u) = 7$$
$$\psi(u_{2i}) = 7 + i;\ i = 1,2$$
$$\psi(u_i) = i;\ i = 1$$
$$\psi(u_i) = i + 1;\ i = 3$$
$$\psi(u_i) = i - 4;\ i = 9,10$$
$$\psi(u_i) = 2i;\ i = 5$$
$$\psi(u_7) = 2$$
$$\psi\left(f_{\frac{i+1}{2}}\right) = 20 - i;\ i = 1,3$$
$$\psi(f_{4+i}) = 14 + i;\ i = 1,2$$

$$\psi(f_{6+i}) = 13 - i;\ i = 1,2$$
$$\psi(f_0) = 14$$
$$\psi(f_3) = 13$$
$$\psi(f_4) = 18$$

Hence, the induced face mean bimagic labeling in G is obtained as

$$\psi^*(f_i) = \frac{37}{5} = \lfloor 7.4 \rfloor = 7;\ i = 4,7,8$$
$$\psi^*(f_i) = \frac{36}{5} = \lfloor 7.2 \rfloor = 7;\ i = 0,3,5$$
$$\psi^*(f_1) = \frac{39}{5} = \lfloor 7.8 \rfloor = 7$$
$$\psi^*(f_2) = \frac{32}{4} = \lfloor 8 \rfloor = 8$$
$$\psi^*(f_6) = \frac{35}{4} = \lfloor 8.75 \rfloor = 8$$

The derived constants k_1 and k_2 are 7 and 8. Labeling of vertices and faces followed by the above pattern is shown in Figure 11.

Hence, the fusion of any two adjacent vertices of degree 3 of a Herschel graph H_s admits face bimagic mean labeling of a graph of type (1,0,1).

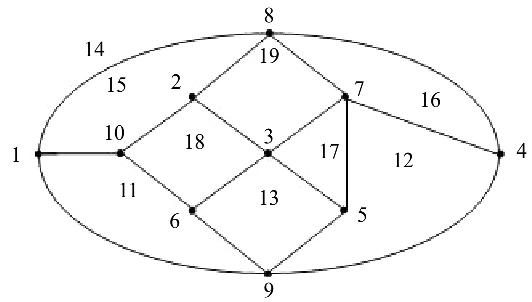


Figure 11. (1,0,1)-FBMML fusion of u_6 and u_8 in H_s

Theorem 3.11. The fusion of any two adjacent vertices of degree 3 of a Herschel graph H_s admits face bimagic mean labeling of a graph of type (1,1,0).

Proof. Let $G(V, E, F)$ be a fusion of a Herschel graph H_s for mean bimagic labeling of graph which contains $|V(G)| = 10$ vertices , $|E(G)| = 17$ edges, and $|F(G)| = 9$ faces, which has eight interior faces and one exterior face. Let the set of vertex be V, set of edges be E, and the face set be F, whose interior face and exterior face of G are denoted as f_i and f_0, respectively.

Fusion of u_6 and u_3. Consider that u_6 and u_3 are fused together as a single vertex u.

(1,1,0) F-FBMML

Consider a mapping ψ: $V(G) \cup E(G) \rightarrow \{1, 2, 3\ldots|V(G)| + |E(G)|\}$ as

$$\psi(u) = 7$$
$$\psi(v) = 5$$
$$\psi\left(u_i\right) = 11 - i;\ i = 1, 2$$
$$\psi\left(u_{7-i}\right) = 2i + 2;\ i = 2, 3$$
$$\psi\left(u_i\right) = 2i;\ i = 7, 9, 10$$
$$\psi\left(u_8\right) = 22$$
$$\psi\left(vu_i\right) = 16 + i;\ i = 8, 9, 10$$
$$\psi\left(u_i\right) = 20 + i;\ i = 7$$
$$\psi\left(u_{2i}u_{2i+6}\right) = 14 - i;\ i = 1, 2$$
$$\psi\left(uu_i\right) = 19 + i;\ i = 4$$
$$\psi\left(uu_i\right) = i - 5;\ i = 8$$
$$\psi\left(uu_i\right) = i + 2;\ i = 9$$
$$\psi\left(uu_{i+1}\right) = 16 + i;\ i = 1$$
$$\psi\left(uu_{i+6}\right) = 12 + i;\ i = 3$$
$$\psi\left(u_iu_{i+1}\right) = 18 + i;\ i = 1$$
$$\psi\left(u_2u_7\right) = 1$$
$$\psi\left(u_1u_5\right) = 21$$
$$\psi\left(u_1u_4\right) = 16$$
$$\psi\left(u_2u_7\right) = 15$$

Hence, the induced face mean bimagic labeling in G is obtained as

$$\psi^*(f_i) = \frac{115}{8} = \lfloor 14.325 \rfloor = 14;\ i = 1, 2$$
$$\psi^*(f_i) = \frac{95}{8} = \lfloor 11.875 \rfloor = 11;\ i = 5, 8$$
$$\psi^*(f_{i+5}) = \frac{71}{6} = \lfloor 11.83 \rfloor = 11;\ i = 1, 2$$
$$\psi^*(f_3) = \frac{118}{6} = \lfloor 14.725 \rfloor = 14$$
$$\psi^*(f_4) = \frac{113}{8} = \lfloor 14.125 \rfloor = 14$$
$$\psi^*(f_0) = \frac{119}{8} = \lfloor 14.825 \rfloor = 14$$

The obtained constants k_1 and k_2 are 11 and 14.

Fusion of u_6 and u_3. Consider that u_6 and u_3 are fused together as a single vertex u.

(1,1,0) F-FBMML

Consider a mapping ψ: $V(G) \cup E(G) \rightarrow \{1, 2, 3 \ldots |V(G)| + |E(G)|\}$ as

$$\psi(v)=5$$
$$\psi(u)=15$$
$$\psi(u_i)=8+i;\ i=1,2$$
$$\psi(u_{i+1})=5+i;\ i=2,3$$
$$\psi(u_{8+i})=18-i;\ i=1,2$$
$$\psi(u_i)=i+1;\ i=5$$
$$\psi(u_7)=18$$
$$\psi(vu_{8+i})=10+2i;\ i=1,2$$
$$\psi(u_{i+1}u_{i+2})=23-i;\ i=1,2$$
$$\psi(u_iu_{i+5})=7-i;\ i=4,5$$
$$\psi(u_1u_{i+5})=18+i;\ i=1,2$$
$$\psi(uu_{i+1})=25+2;\ i=2$$
$$\psi(uu_{i+1})=i+2;\ i=1$$
$$\psi(u_2u_7)=1$$
$$\psi(u_4u_{10})=24$$
$$\psi(u_1u_5)=27$$
$$\psi(u_7u_5)=25$$
$$\psi(vu_7)=11$$
$$\psi(uu_9)=23$$
$$\psi(uv)=13$$

Hence, the induced face mean bimagic labeling in G is obtained as

$$\psi^*(f_i)=\frac{99}{8}=\lfloor 12.375\rfloor=12;\ i=1,3$$
$$\psi^*(f_6)=\frac{84}{6}=\lfloor 14\rfloor=14$$
$$\psi^*(f_3)=\frac{85}{6}=\lfloor 14.16\rfloor=14$$
$$\psi^*(f_5)=\frac{115}{8}=\lfloor 14.325\rfloor=14$$
$$\psi^*(f_4)=\frac{97}{8}=\lfloor 12.125\rfloor=12$$
$$\psi^*(f_7)=\frac{98}{8}=\lfloor 12.25\rfloor=12$$
$$\psi^*(f_8)=\frac{112}{8}=\lfloor 14\rfloor=14$$
$$\psi^*(f_0)=\frac{116}{8}=\lfloor 14.5\rfloor=14$$

The derived constants k_1 and k_2 are 12 and 14.

Hence, the fusion of any two adjacent vertices of degree 3 of a Herschel graph H_s admits face bimagic mean labeling of a graph of type (1,1,0).

4. Conclusion

This research work has analyzed the following results namely (1,0,0),(0,1,0),(1,0,1),(0,1,1),(1,1,0)-face bimagic mean labeling of a Herschel graph, (1,0,0),(1,0,1),(0,1,0),(1,1,0)-face bimagic mean labeling during the fusion of vertices of degree 3 of a Herschel graph and (1,0,0), (1,0,1)-face bimagic mean labeling during the process of duplication of a vertex by the edge of a Herschel graph. In the future, this labeling technique will be applied in the encryption process as well as the decryption process.

Conflict of interest

There is no conflict of interest in this study.

References

[1] Gallian JA. A dynamic survey of graph labeling. *The Electronic Journal of Combinatorics*. 2022.

[2] Harary F. *Graph theory*. Philipines: Addision-Wesley Publishing Company; 1969.

[3] Wallis WD. *Magic graphs*. Birkäuser Boston; 2001.

[4] Somasundaram S, Ponraj R. Mean labeling of graphs. *National Academy Science Letters*. 2003; 26: 210-213.

[5] Ganesan V, Balamurugan K. On prime labeling of Herschel graph. *International Journal of Engineering Research and Modern Education*. 2016; 1(2): 33-42. Available from: https://doi.org/10.5281/zenodo.61834.

[6] Meena Kumari A, Arockiaraj S. On (1,1,0)-F-face magic mean labeling of some graphs. *Utilitas Mathematica*. 2017; 103: 139-159.

[7] Sugumaran A, Rajesh K. Sum divisor cordial labeling of Herschel graph. *Annals of Pure and Applied Mathematics*. 2017; 14(3): 465-472. Available from: http://dx.doi.org/10.22457/apam.v14n3a14.

[8] Roopa B, Shobana L. Face bimagic labeling of some graphs. *Turkish Journal of Computer and Mathematics Education*. 2021; 12(6): 101-107. Available from: https://doi.org/10.17762/turcomat.v12i6.1273.

[9] Arockiaraj S, Meena Kumari A. On F-face magic mean labeling of some planar graphs. *International Journal of Pure and Applied Mathematics.* 2017; 117(6): 1-8.

[10] Meena Kumari A, Arockiaraj S. Further results on F-face magic mean graphs. *International Journal of Advanced Science and Technology*. 2020; 29(7): 1129-1138.

[11] Arockiaraj S, Meena Kumari A. On F-face magic mean labeling of some duplicated graphs. *International Journal of Applied Mathematics*. 2018; 31(2): 231-240. Available from: https://doi.org/10.12732/ijam.v31i2.5.

[12] Meena Kumari A, Arockiaraj S. On (1,0,0)-F-face magic mean labeling of some graphs. *Journal of Physics: Conference Series*. 2018; 1139: 012049. Available from: https://doi.org/10.1088/1742-6596/1139/1/012049.

[13] Ahmed MA, Babbujee JB. Encryption through labeled graphs using strong face bimagic labeling. *International Mathematical Forum.* 2017; 12(4): 151-158. Available from: https://doi.org/10.12988/imf.2017.612167.

[14] Vani Shree S, Dhanalakshmi S. On face bimagic mean labeling of duplication of a path graph. *Telematique*. 2022; 21(1).

[15] Vani Shree S, Dhanalakshmi S. New results on face magic mean labeling of graphs. *Mathematics and Statistics.* 2023; 11(1): 199-205. Available from: https://doi.org/10.13189/ms.2023.110123.

[16] Vani Shree S, Dhanalakshmi S. Application of edge bimagic mean labeling in data security. *Asia Pacific Journal of Mathematics*. 2023; 10: 41. Available from: https://doi.org/10.28924/APJM/10-41.