



Research Article

Approximate Controllability of Semilinear Heat Equation with Noninstantaneous Impulses, Memory, and Delay

H. Leiva^{1*}, W. Zouhair², M. N. Entekhabi³, E. Lucena⁴

¹Department of Mathematics, School of Mathematical Sciences and Information Technology, Yachay Tech University, Ecuador

²Department of mathematics, Faculty of Applied Sciences Ait Melloul, Ibn Zohr University, Route Nationale N10, Azrou, B.P. 6146, Morocco

³Department of Mathematics, Florida A & M University, Florida 32307, USA

⁴Faculty of Basic Sciences, Department of Mathematics and Statistics, Technical University of Manabi, Portoviejo, Ecuador
Email: hleiva@yachaytech.edu.ec

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Abstract: A semilinear heat equation with noninstantaneous impulses, memory, and delay is studied. Its approximate controllability is obtained by employing a technique that pulls back the control solution into a fixed curve in a short time interval. This technique is a modification of the one used by Bashirov et al. to avoid the use of fixed points, but it only works to prove approximate controllability. This technique can be applied to reaction-diffusion systems that behave like the heat equation, such as the one presented in Oliveira. In the end, we consider some examples to be studied in the future.

Keywords: semilinear heat equation, noninstantaneous impulsive, approximate controllability, evolution equation with memory and delay

MSC: 93B05, 34G20, 35R12

1. Introduction

The theory of impulsive dynamical systems was initiated by [1]. Afterward, it has become an important field of investigation in several areas. They can be found in applications ranging from neural networks to ecology, chemistry, biotechnology, radiophysics, theoretical physics, mathematical economy, and engineering. The interest of this article is the noninstantaneous impulsive semilinear system involving memory and state delay, which is motivated by applications such as species population, nanoscale electronic circuits consisting of single-electron tunneling junctions, and mechanical systems with impacts [2-4]. In general, impulses represent sudden deviations of states at specific times, either by instantaneous jumps or continuous intervals.

In real-life problems, the impulse starts abruptly at a certain moment in time and remains active for a finite time interval. However, the duration of the action is short. Such an impulse is known as a noninstantaneous impulse. This notion appeared for the first time in 2012. After that, it became an area of interest for many researchers. For more, we refer to our readers [5-9].

The phenomenon of impulses implies instantaneous and discontinuous changes at different instants of time, which

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influence the solutions and can lead to the instability (respectively uncontrollability) of the differential equation or, conversely, to its stability (respectively controllability). The evolution of this theory has been rather slow due to the complexity of handling such equations (see [10, 11]). Afterward, many scientists contributed to the enrichment of this theory; they launched different studies on this subject, and a large number of results were established.

Controllability is a mathematical problem that consists of finding a control that steers the system from an arbitrary initial state to a final state in a finite interval of time. The controllability of instantaneous impulsive systems has been extensively studied in the literature; see [12-20]. To the best of our knowledge, there is no work that deals with semilinear heat equations with memory, delay, and noninstantaneous impulsivity simultaneously. Motivated by the above facts, we study the approximate controllability of the following system:

$$\left\{ \begin{array}{l} \frac{\partial \omega}{\partial t} + A\omega = \mathbf{1}_\theta u(t, x) + \int_0^t [M(t, s, x) \\ \quad g(\omega(s-r, x))] ds + f(t, \omega(t-r, x), u(t, x)), \quad \text{in } \bigcup_{i=0}^N (s_i, t_{i+1}] \times [0, \pi], \\ \omega(t, 0) = \omega(t, \pi) = 0, \quad \text{on } (0, T), \\ \omega(s, x) = h(s, x), \quad \text{in } [-r, 0] \times [0, \pi], \\ \omega(t, x) = G_i(t, \omega(t, x), u(t, x)), \quad \text{in } \bigcup_{i=0}^N (t_i, s_i] \times [0, \pi], \end{array} \right. \quad (1)$$

where $0 = s_0 = t_0 < t_1 \leq s_1 < \dots < t_N \leq s_N < t_{N+1} = T$ are fixed real numbers, and the distributed control u belongs to $L^2(0, T; X)$ with $X = L^2[0, \pi]$, $M \in L^\infty([0, T] \times [0, T] \times [0, \pi])$, $f: [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ represents the nonlinear perturbations of the system, $h: [-r, 0] \times [0, \pi] \rightarrow \mathbb{R}$ is a piecewise continuous function. The noninstantaneous impulses are represented by $G_i: (t_i, s_i] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. $A: \mathcal{D}(A) \subset X \rightarrow X$ is the operator $A\psi = -\psi_{xx}$, with domain $\mathcal{D}(A) := \{\psi \in X: \psi, \psi_x \text{ absolutely continuous, } \psi_{xx} \in X, \psi(0) = \psi(\pi) = 0\}$, θ is an open nonempty subset of $[0, \pi]$, and $\mathbf{1}_\theta$ denotes the characteristic function of the set θ .

In general, the effect of such pulses on the behavior of solutions is presented as in Figure 1.

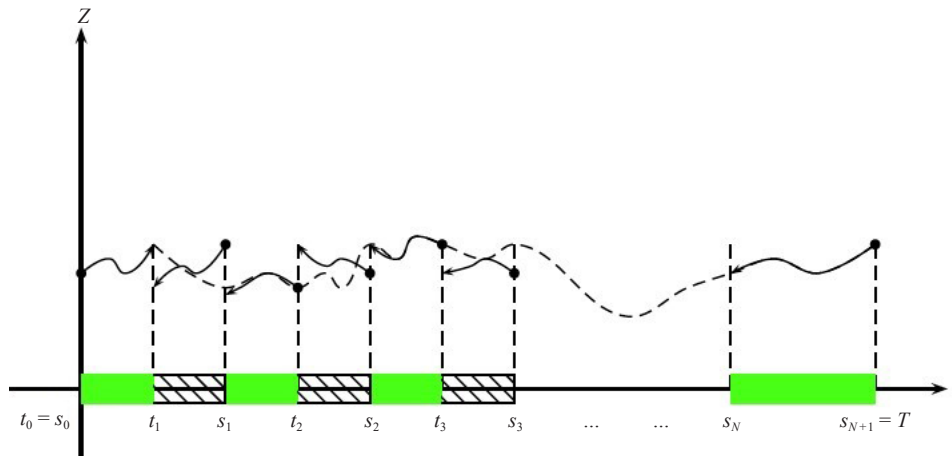


Figure 1. Example of the effect of noninstantaneous pulses

The interval $[t_k, s_k]$ represents the impulsive behavior; it starts at the instant t_k and remains active until the instant s_k .

This paper is organized as follows: in Section 2, we briefly present the problem formulation and related definition. In Sections 3 and 4, we discuss the approximation controllability for the linear and the semilinear systems. The last section is devoted to some related open problems and applications.

2. Abstract formulation of the problem

In this section, we recall some results that will be useful in the sequel. It is well known that $-A: \mathcal{D}(A) \subset \mathcal{X} \rightarrow \mathcal{X}$ is the generator of a strongly continuous analytic semigroup $(S(t))_{t \geq 0}$ on \mathcal{X} . Moreover, the operator A and the semigroup can be represented as follows:

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle \phi_n, \quad x \in \mathcal{X},$$

where $\lambda_n = n^2$, $\phi_n(\xi) = \sin(n\xi)$ and $\langle \cdot, \cdot \rangle$ are the inner products in \mathcal{X} . Also, the strongly continuous semigroup $(S(t))_{t \geq 0}$ generated by $-A$ is compact and presented by

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, \phi_n \rangle \phi_n, \quad x \in \mathcal{X}.$$

Then, we have the following estimation.

$$\|S(t)\| \leq e^{-t}, \quad t \geq 0.$$

Using the above notation, we can rewrite system (1) as the following differential equations with memory as follows for $x \in [0, \pi]$:

$$\left\{ \begin{array}{l} \frac{d\omega}{dt} + A\omega(t) = \mathbf{B}_\theta u(t) + \int_0^t [M^1(t, s) \\ \qquad \qquad \qquad g^1(\omega_s(-r))] ds + f^1(t, \omega_t(-r), u(t)), \quad \text{in } \bigcup_{i=0}^N (s_i, t_{i+1}], \\ \omega(s) = h(s), \quad \text{in } [-r, 0] \\ \omega(t) = G_i^1(t, \omega(t), u(t)), \quad \text{in } \bigcup_{i=1}^N (t_i, s_i], \end{array} \right. \quad (2)$$

where $u \in L^2(0, T; \mathcal{U})$ with $\mathcal{U} = \mathcal{X}$, $B_\theta: \mathcal{U} \rightarrow \mathcal{X}$ is a bounded linear operator defined by $B_\theta u = \mathbf{1}_\theta u$, ω_t stands for the translated function of ω defined by $\omega_t(s) = \omega(t+s)$, with $s \in [-r, 0]$ and the functions $g^1: L^2[0, \pi] \rightarrow L^2[0, \pi]$, $G_i^1: (t_i, s_i] \times \mathcal{X} \times \mathcal{U} \rightarrow L^2[0, \pi]$ and $f^1: [0, T] \times \mathcal{PW} \times \mathcal{U} \rightarrow L^2[0, \pi]$, are defined by

$$M^1(t, s)(x) = M(t, s, x),$$

$$g^1(\omega_t(-r))(x) = g(\omega(t-r, x)),$$

$$f^1(t, \omega_t(-r), u)(x) = f(t, \omega(t-r), u(t, x)),$$

$$G_i^1(t, \omega(t), u(t))(x) = G_i(t, \omega(t, x), u(t, x)) \quad \text{for } i = 0, \dots, N,$$

where \mathcal{PW} is the space of piecewise continuous functions given by

$$\mathcal{PW} = \left\{ h: [-r, 0] \rightarrow \mathcal{X}^{1/2} : h \text{ is piecewise continuous} \right\}.$$

Here, $\mathcal{X}^{1/2}$ is the fractional powered space defined by

$$\mathcal{X}^{1/2} = D(A^{1/2}) = \left\{ x \in \mathcal{X} : \sum_{n=1}^{\infty} \lambda_n \|\langle x, \phi_n \rangle \phi_n\|^2 < \infty \right\},$$

equipped with the norm

$$\|x\|_{\mathcal{X}^{1/2}} = \|A^{1/2}x\|, \quad x \in \mathcal{X}^{1/2}.$$

Next, if the functions g and f are smooth enough, then the fractional powered space $\mathcal{X}^{1/2}$ allows the following Nemytskii operator to be well defined

$$f^2 : [0, T] \times \mathcal{PW} \times \mathcal{U} \rightarrow \mathcal{X},$$

given by

$$f^2(t, \omega, u) = \int_0^t M^1(t, s) \cdot g^1(\omega(-r)) ds + f^1(t, \omega(-r), u(t)).$$

Then, from system (2), we obtain the following nonautonomous evolution equation with noninstantaneous impulses in the Hilbert space \mathcal{X}

$$\begin{cases} \frac{\partial \omega(t)}{\partial t} + A\omega(t) = \mathbf{B}_\theta u(t) + f^2(t, \omega, u) & \text{in } \bigcup_{i=0}^N (s_i, t_{i+1}], \\ \omega(s) = h(s), & \text{in } [-r, 0] \\ \omega(t) = G_i^1(t, \omega(t), u(t)), & \text{in } \bigcup_{i=0}^N (t_i, s_i]. \end{cases} \quad (3)$$

We consider the space $\mathcal{PC}(\mathcal{X})$ of all functions $\varphi : [-r, T] \rightarrow \mathcal{X}^{1/2}$ such that $\varphi(\cdot)$ is piecewise continuous on $[-r, 0]$ and continuous on $[0, T]$ except at points t_i where the side limits $\varphi(t_i^-)$ and $\varphi(t_i^+)$ exist, and $\varphi(t_i^-) = \varphi(t_i)$ for all $i = 1, 2, \dots, N$, endowed with the uniform norm denoted by $\|\cdot\|_{\mathcal{PC}(\mathcal{X})}$.

The problem of the existence of a solution for the semilinear differential system under noninstantaneous impulses and delays has attracted many researchers. For instance, in finite-dimensional Banach space, the existence and uniqueness of solutions for the semilinear differential system under noninstantaneous impulses and delays are obtained by applying Karakostas' fixed point theorem; see [8], and for infinite-dimensional space, we invite interested readers to see [9, 21-23].

To this end, we shall assume that the functions M^1 , f^2 , g^1 , and G_i^1 are smooth enough such that the above system (3) admit only one mild solution on $[-r, T]$ given by:

Definition 2.1 A function $\omega(\cdot) \in \mathcal{PC}(\mathcal{X})$ is called a mild solution for the system (3) if it satisfies the following integral-algebraic equation

$$\omega(t) = \begin{cases} h(t), & t \in [-r, 0], \\ S(t)h(0) + \int_0^t S(t-s)(\mathbf{B}_\theta u(s) + f^2(s, \omega_s, u(s)))ds, & t \in [0, t_1], \\ G_i^1(t, \omega(t), u(t)), & t \in (t_i, s_i], i \in I_N, \\ S(t-s_i)G_i^1(s_i, \omega(s_i), u(s_i)) + \int_{s_i}^t S(t-s)\mathbf{B}_\theta u(s)ds \\ + \int_{s_i}^t S(t-s)f^2(s, \omega_s, u(s))ds. & t \in (s_i, t_{i+1}], i \in I_N, \end{cases} \quad (4)$$

where I_N denote the set $\{1, \dots, N\}$.

In this work, we will assume that the functions appearing in the equation are smooth enough to allow us to prove the existence of mild solutions to system (3). The interested reader can see the work carried out by [8, 21-24], where fixed-point techniques are used to prove the existence of solutions to differential equations with impulses and delay. In this paper, we are interested in studying the controllability of the system (3). In this respect, we assume for the rest of this paper that this system admits a mild solution on $[-r, T]$. As mentioned before, the technique used was based on a fixed-point theorem by transforming the existence of solutions problem into a fixed-point existence problem of a certain operator equation satisfying a specific condition. This led to choosing adequate hypotheses to prove the existence of a fixed point; see [8, 9].

3. Approximate controllability of the linear equation

In this section, we shall present some characterization of the approximate controllability for a general linear system in Hilbert spaces, then we prove, for better understanding of the reader, the approximate controllability of the linear heat equation in any interval $[T-l, T]$, $l > 0$ using the representation of the semigroup $(S(t))_{t \geq 0}$ generated by $-A$, and the fact that $\phi_n(\xi) = \sin n\xi$ are analytic functions. To this end, we note that, for all $\omega_0 \in \mathcal{X}$, $0 \leq t_0 \leq T$ and $u \in L^2(0, T; \mathcal{U})$ the initial value problem

$$\begin{cases} \frac{d\omega}{dt} = -A\omega + \mathbf{B}_\theta u, & \omega \in \mathcal{X}, \\ \omega(t_0) = \omega_0, \end{cases} \quad (5)$$

admits only one mild solution given by

$$\omega(t) = S(t-t_0)\omega_0 + \int_{t_0}^t S(t-s)\mathbf{B}_\theta u(s)ds, \quad t \in [t_0, T].$$

Definition 3.1 The system (5) is said to be approximately controllable on $[t_0, T]$ if for all $\omega_0, \omega^1 \in \mathcal{X}$, $\varepsilon > 0$, there exists $u \in L^2([0, T]; \mathcal{U})$ such that the solution $\omega(t, \omega_0)$ of (5) corresponding to u verifies:

$$\|\omega(T) - \omega^1\|_{\mathcal{X}} < \varepsilon,$$

Definition 3.2 For $l \in [0, T]$, we define the controllability map for the system (5) as follows:

$$G_T : L^2(T-l, T; \mathcal{U}) \rightarrow \mathcal{X}$$

$$G_{Tl}(v) = \int_{T-l}^T S(T-s)B_\theta v(s)ds.$$

Its adjoint operator

$$G_{Tl}^* : \mathcal{X} \rightarrow L^2(T-l, T; \mathcal{U})$$

$$(G_{Tl}^*x)(t) = B_\theta^* S^*(T-t)x, \quad t \in [T-l, T].$$

Therefore, the Grammian operator $Q_{Tl} : \mathcal{X} \rightarrow \mathcal{X}$ is defined by

$$Q_{Tl} = G_{Tl}G_{Tl}^* = \int_{T-l}^T S(T-t)B_\theta B_\theta^* S^*(T-t)dt.$$

The following lemma holds in general for a linear bounded operator $G: W \rightarrow Z$ between Hilbert spaces W and Z .

Lemma 3.1 (see [20, 25-27]). The equation (5) is approximately controllable on $[T-l, T]$ if and only if one of the following statements holds:

- $\text{Rang}(G_{Tl}) = \mathcal{X}$,
- $B_\theta^* S^*(T-t)x = 0, t \in [T-l, T] \Rightarrow x = 0$,
- $\langle Q_{Tl}x, x \rangle > 0, x \neq 0$ in \mathcal{X} ,
- $\lim_{\alpha \rightarrow 0^+} \alpha(\alpha I + Q_{Tl})^{-1}x = 0, \forall x \in \mathcal{X}$.

Remark 3.1 The Lemma 3.1 implies that for all $x \in \mathcal{X}$, we have $G_{Tl}u_\alpha = x - \alpha(\alpha I + Q_{Tl})^{-1}x$, where

$$u_\alpha = G_{Tl}^*(\alpha I + Q_{Tl})^{-1}x, \quad \alpha \in (0, 1].$$

So, $\lim_{\alpha \rightarrow 0} G_{Tl}u_\alpha = x$ and the error $E_{Tl}x$ of this approximation are given by

$$E_{Tl}x = \alpha(\alpha I + Q_{Tl})^{-1}x, \quad \alpha \in (0, 1],$$

and the family of linear operators $\Gamma_{\alpha Tl} : \mathcal{X} \rightarrow L^2(T-l, T; \mathcal{U})$, defined for $0 < \alpha \leq 1$ by

$$\Gamma_{\alpha Tl}x = G_{Tl}^*(\alpha I + Q_{Tl})^{-1}x,$$

satisfies the following limit

$$\lim_{\alpha \rightarrow 0} G_{Tl}\Gamma_{\alpha Tl} = I,$$

in the strong topology.

Lemma 3.2 The linear heat equation (5) is approximately controllable on $[T-l, T]$. Moreover, a sequence of controls steering the system (5) from an initial state y_0 to an ε neighborhood of the final state w^1 at time $T > 0$ is given by $\{u_\alpha^l\}_{0 < \alpha \leq 1} \subset L^2(T-l, T; \mathcal{U})$, where

$$u_\alpha^l = G_{Tl}^*(\alpha I + Q_{\pi l})^{-1}(w^1 - S(l)y_0), \quad \alpha \in (0, 1],$$

and the error of this approximation E_α is given by

$$E_\alpha = \alpha(\alpha I + Q_{Tl})^{-1} (\omega^1 - S(l)y_0).$$

Therefore, the solution $y(t) = y(t, T-l, y_0, u_\alpha^l)$ of the initial value problem

$$\begin{cases} y' = -Ay + B_\theta u_\alpha^l(t), & y \in \mathcal{X}, t > 0, \\ y(T-l) = y_0, \end{cases} \quad (6)$$

satisfies

$$\lim_{\alpha \rightarrow 0^+} y_\alpha^l(T, T-l, y_0, u_\alpha^l) = \omega^1,$$

i.e.,

$$\lim_{\alpha \rightarrow 0^+} y_\alpha^l(T) = \lim_{\alpha \rightarrow 0^+} \left\{ S(l)y_0 + \int_{T-l}^T S(T-s)B_\theta u_\alpha^l(s) ds \right\} = \omega^1.$$

Proof. We shall apply condition (b) from the foregoing lemma. In fact, it is clear that $S^*(t) = S(t)$, $B_\theta^* = B_\theta$. We suppose that $B_\theta^* S^*(\tau-t)\xi = 0$, $t \in [T-l, T]$, which means

$$\sum_{n=1}^{\infty} e^{-n^2(T-t)} \langle \xi, \phi_n \rangle B_\theta \phi_n = 0, \quad t \in [T-l, T].$$

Then,

$$\sum_{n=1}^{\infty} e^{-n^2(T-t)} \langle \xi, \phi_n \rangle \mathbf{1}_\theta \phi_n = 0, \quad t \in [T-l, T].$$

Hence,

$$\sum_{n=1}^{\infty} e^{-n^2(T-t)} \langle \xi, \phi_n \rangle \phi_n(x) = 0, \quad t \in [T-l, T], \quad x \in \theta.$$

So,

$$\sum_{n=1}^{\infty} e^{-n^2 t} \langle \xi, \phi_n \rangle \phi_n(x) = 0, \quad t \in [0, l], \quad x \in \theta.$$

By Lemma 3.14 from [25], we get that

$$\langle \xi, \phi_n \rangle \phi_n(x) = 0, \quad x \in \theta.$$

Now, since $\phi_n(x) = \sin(nx)$ is analytic functions, we get that $\langle \xi, \phi_n \rangle \phi_n(x) = 0$, $\forall x \in [0, \pi]$, $n = 1, 2, \dots$. This implies that

$$\langle \xi, \phi_n \rangle = 0, \quad n = 1, 2, \dots$$

Since $\{\phi_n\}$ is a complete orthonormal set on \mathcal{X} we conclude that $\zeta = 0$. This completes the proof of the approximate controllability of the linear system (5).

4. Approximate controllability of the semilinear system

In this section, we shall prove the main result of this paper, the interior approximate controllability of the noninstantaneous impulsive semilinear heat equation with memory and delay (1). To this end, we will give the definition of approximate controllability for the equivalent evolution of the semilinear system (3).

Definition 4.1 The system (3) is said to be approximately controllable on $[0, T]$, if for all $h \in \mathcal{PW}$, $\omega^1 \in \mathcal{X}$ and $\varepsilon > 0$, there exists $u \in L^2([0, T]; \mathcal{U})$, such that the corresponding solution $\omega(t, 0, h, u)$ of (3) verifies:

$$\|\omega(T) - \omega^1\|_{\mathcal{X}} < \varepsilon.$$

Next, for all $h \in \mathcal{PW}$ and $u \in L^2(0, T; \mathcal{U})$, the initial value problem (3) admit only one mild solution given by (4), and its evaluation in T leads us to the following expression:

$$\begin{aligned} \omega(T) &= S(T - s_N)G_N^1(s_N, \omega(s_N), u(s_N)) \\ &+ \int_{s_N}^T S(T - s) (\mathbf{B}_\theta u(s) + f^2(s, \omega_s, u(s))) ds \\ &= S(T - s_N)G_N^1(s_N, \omega(s_N), u(s_N)) + \int_{s_N}^T S(T - s) \mathbf{B}_\theta u(s) ds \\ &+ \int_{s_N}^T S(T - s) \left(\int_0^s M^1(s, m) g^1(\omega_m(-r)) dm + f^1(s, \omega_s(-r), u(s)) \right) ds. \end{aligned}$$

Now, we are ready to present and prove the main result of this paper.

Theorem 4.1 Assume the existence of a function $\rho \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ such that for all $(t, \Phi, u) \in [0, T] \times \mathcal{PW} \times L^2(0, T; \mathcal{U})$ the following inequality holds:

$$f^1(t, \Phi, u)_{\mathcal{X}} \leq \rho(\|\Phi(-r)\|_{\mathcal{X}^{1/2}}). \tag{7}$$

Then, the noninstantaneous impulsive semilinear heat equation (1) with memory and delay is approximately controllable on $[0, T]$.

Proof. Given $\varepsilon > 0$, $h \in \mathcal{PW}$ and a final state $w^1 \in \mathcal{X}$ we look for a control $u_\alpha^l \in L^2([0, T]; \mathcal{U})$ with $0 < \alpha < 1$ such that the corresponding solution $\omega^{\alpha, l}$ satisfies

$$\|\omega^{\alpha, l}(T) - \omega^1\|_{\mathcal{X}} < \varepsilon.$$

We start by considering $u \in L^2(0, T; \mathcal{U})$ and its corresponding mild solution $\omega(t) = \omega(t, 0, h, u)$ of the initial value problem (3). For $0 < \alpha < 1$ and $0 < l < \min\{T - s_N, r\}$ small enough, we define the control sequence $u_\alpha^l \in L^2(0, T; \mathcal{U})$ as follows:

$$u_\alpha^l(t) = \begin{cases} u(t), & \text{if } 0 \leq t \leq T - l, \\ u_\alpha(t), & \text{if } T - l < t \leq T, \end{cases} \tag{8}$$

where for $T - l < t \leq T$

$$u_\alpha(t) = B_\theta^* S^*(T-t)(\alpha I + Q_{Tl})^{-1} (\omega^l - S(l)\omega(T-l)). \quad (9)$$

The corresponding solution $\omega^{\alpha,l} = \omega(t, u_\alpha^l)$ of the initial value problem (3) at time T can be written as follows:

$$\begin{aligned} \omega^{\alpha,l}(T) &= S(T-s_N)G_N^1(s_N, \omega^{\alpha,l}(s_N), u_\alpha^l(s_N)) + \int_{s_N}^T S(T-s) \left[\mathbf{B}_\theta u_\alpha^l(s) \right. \\ &\quad \left. + \int_0^s M^1(s,m)g^1(\omega_m^{\alpha,l}(-r))dm + f^1(s, \omega_s^{\alpha,l}(-r), u_\alpha^l(s)) \right] ds \\ &= S(l) \left\{ S(T-s_N-l)G_N^1(s_N, \omega^{\alpha,l}(s_N), u_\alpha^l(s_N)) + \int_{s_N}^{T-l} S(T-s-l) \right. \\ &\quad \left. + \left[\mathbf{B}_\theta u_\alpha^l(s) \int_0^s M^1(s,m)g^1(\omega_m^{\alpha,l}(-r))dm + f^1(s, \omega_s^{\alpha,l}(-r), u_\alpha^l(s)) \right] ds \right\} \\ &\quad + \int_{T-l}^T S(T-s) \left[\mathbf{B}_\theta u_\alpha(s) + \int_0^s M^1(s,m)g^1(\omega_m^{\alpha,l}(-r))dm \right. \\ &\quad \left. + f^1(s, \omega_s^{\alpha,l}(-r), u_\alpha(s)) \right] ds. \end{aligned}$$

Then,

$$\begin{aligned} \omega^{\alpha,l}(T) &= S(l)\omega(T-l) + \int_{T-l}^T S(T-s) \left[\mathbf{B}_\theta u_\alpha + \int_0^s (M^1(s,m)g^1 \right. \\ &\quad \left. \omega_m^{\alpha,l}(-r))dm + f^1(s, \omega_s^{\alpha,l}(-r), u_\alpha(s)) \right] ds. \end{aligned}$$

On the other hand, the corresponding solution $y_\alpha^l(t) = y(t, T-l, \omega(T-l), u_\alpha)$ of the linear value problem (6) at time T is given by

$$y_\alpha^l(T) = S(l)\omega(T-l) + \int_{T-l}^T S(T-s)\mathbf{B}_\theta u_\alpha(s)ds. \quad (10)$$

Therefore,

$$\begin{aligned} \omega^{\alpha,l}(T) - y_\alpha^l(T) &= \int_{T-l}^T S(T-s) \left[\int_0^s M^1(s,m)g^1(\omega_m^{\alpha,l}(-r))dm \right. \\ &\quad \left. + f^1(s, \omega_s^{\alpha,l}(-r), u_\alpha(s)) \right] ds, \end{aligned}$$

and by the hypothesis (7), we obtain

$$\begin{aligned} \|\omega^{\alpha,l}(T) - y'_\alpha(T)\| &\leq \int_{T-l}^T \|S(T-s)\| \int_0^s \|M^1(s,m)g^1(\omega_m^{\alpha,l}(-r))\| dm ds \\ &\quad + \int_{T-l}^T \|S(T-s)\| \rho\left(\|\omega^{\alpha,l}(s-r)\|_{\mathcal{X}^{l/2}}\right) ds, \end{aligned}$$

since $0 \leq m \leq s$, $0 < l < r$, and $T-l \leq s \leq T$, then $m-r \leq s-r \leq T-r < T-l$, then

$$\omega^{\alpha,l}(m-r) = \omega(m-r) \text{ and } \omega_s^{\alpha,l}(-r) = \omega(s-r), \quad (11)$$

therefore, for a sufficiently small l we obtain

$$\begin{aligned} \|\omega^{\alpha,l}(T) - y'_\alpha(T)\| &\leq \int_{T-l}^T \|S(T-s)\| \int_0^s \|M^1(s,m)g^1(\omega_m(-r))\| dm ds \\ &\quad + \int_{T-l}^T \|S(T-s)\| \rho\left(\|\omega^{\alpha,l}(s-r)\|_{\mathcal{X}^{l/2}}\right) ds \\ &\leq \frac{\varepsilon}{2}. \end{aligned}$$

Hence, by Lemma 3.2, we can choose $\alpha > 0$, such that

$$\begin{aligned} \|\omega^{\alpha,l}(T) - \omega^1\| &\leq \|\omega^{\alpha,l}(T) - y'_\alpha(T)\| + \|y'_\alpha(T) - \omega^1\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since $\overline{\mathcal{X}^{l/2}} = \mathcal{X}$, the proof of the theorem is complete.

Remark 4.1 The condition (7) allows us to choose unbounded disturbances for the system; for example, we can choose a function with exponential growth that bounds the force term f^l . A particular choice of the function ρ used in the condition (7) is $\rho(\xi) = \exp(\beta\xi) + \eta$, with $\beta \geq 1$.

5. Conclusions and possible extensions

In this work, we have dealt with the approximate controllability of a semilinear heat equation with noninstantaneous impulses, memory, and delay. This is done by employing a technique that avoids fixed-point theorems by pulling back the controlled solution to a fixed curve in a short time interval. This technique is a modification of the one used by [28-30] to avoid the use of fixed points. This proves that the controllability of the linear heat equations is robust under the influence of noninstantaneous impulses, memory, and delay as perturbations if an additional condition is imposed.

To the best of the authors' knowledge, dynamical systems with noninstantaneous impulses have not been studied much in the literature, which opens the doors to many possibilities for dealing with such problems. The technique we used here is simple and can be applied to those control systems governed by diffusion processes like, for example, the Benjamin-Bona-Mohany equation, the strongly damped wave equations, and the beam equations (for more detail about these, see [31]), all of them with noninstantaneous impulses, memory, and delay. We believe that this kind of problem can be formulated with fractional derivatives as well; see, for example [32].

Example 5.1 Our first example is a semilinear, nonautonomous differential equation with noninstantaneous impulses, memory, and delay in a finite dimension

$$\begin{cases} \omega'(t) = A(t)\omega(t) + B(t)u(t) + \int_0^t M(t,s)g(\omega_s)ds \\ \quad + f(t, \omega_t, u(t))ds, & \text{in } \bigcup_{i=0}^N (s_i, t_{i+1}], \\ \omega(s, x) = h(s, x), & \text{in } [-r, 0], \\ \omega(t) = G_i(t, \omega(t), u(t)), & \text{in } \bigcup_{i=1}^N (t_i, s_i], \end{cases}$$

where $A(t), B(t)$ are continuous matrices of dimension $n \times n$ and $n \times m$ respectively, the control function u belongs to $\mathcal{C}(0, T; \mathbb{R}^m)$, $h \in \mathcal{PW}(-r, 0; \mathbb{R}^n)$, $f: [0, T] \times \mathcal{PW}(0, T; \mathbb{R}^n) \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $M \in \mathcal{C}(0, T; \mathbb{R}^n)G_i: (t_i, s_i] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$.

Example 5.2 The second example is about the controllability of a noninstantaneous semilinear beam equation with memory and delay in a bounded domain $\Omega \subseteq \mathbb{R}^N$,

$$\begin{cases} z''(t, x) - 2\gamma\Delta z'(t, x) + \Delta^2 z(t, x) = u(t, x) + f(t, z(t-r), z'(t-r), u) \\ \quad + \int_0^t g(t-s)h(z(s-r, x))ds, & \text{in } \bigcup_{i=0}^N (s_i, t_{i+1}], \\ z(t, x) = \Delta z(t, x) = 0, & \text{on } \partial\Omega, \\ z(s, x) = \varphi_1(s, x), & \text{in } [-r, 0], \\ z'(s, x) = \varphi_2(s, x), & \text{in } [-r, 0], \\ z'(t, x) = \psi_i(t, z(t), z'(t), u(t)), & \text{in } \bigcup_{i=1}^N (t_i, s_i], \end{cases}$$

the damping coefficient $\gamma > 1$, and the real-valued functions $z = z(t, x)$ in $[0, T] \times \Omega$ represent the beam deflection, u in $[0, T] \times \Omega$ is the distributed control, g acts as convolution kernel with respect to the time variable.

Example 5.3 Another example where this technique may be applied is the strongly damped wave equation with Dirichlet boundary conditions with noninstantaneous impulses, memory, and delay in the space $\mathcal{Z}_{1/2} = \mathcal{D}((-\Delta)^{1/2}) \times L^2(\Omega)$,

$$\begin{cases} y'' + \beta(-\Delta)^{1/2}y' + \gamma(-\Delta)y = \mathbf{1}_\theta u + \int_0^t h(s, y(s-r), u(s))ds, & \text{in } \bigcup_{i=0}^N (s_i, t_{i+1}], \\ y = 0, & \text{on } \partial\Omega, \\ y(s) = \phi_1(s), & \text{in } [-r, 0], \\ y'(s) = \phi_2(s), & \text{in } [-r, 0], \\ y'(t) = g_i(t, y(t), y'(t), u(t)), & \text{in } \bigcup_{i=1}^N (t_i, s_i], \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^N , θ is an open nonempty subset of Ω , $\mathbf{1}_\theta$ denotes the characteristic function of the set θ , the distributed control $u \in L^2(0, T; L^2(\Omega))$. ϕ_1, ϕ_2 are piecewise continuous functions.

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Conflict of interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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