

Research Article

The Study of Stability Analysis of Modified Leslie-Gower Herbivore Model with Allee Effect in Plants

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Abstract: A modified Leslie-Gower plant-herbivore model is studied under the Allee effect. Holling-type II functional response is used to modify the model. Delay differential equations play an essential role in making this model more realistic and complicated. The non-trivial equilibrium E^* ($P^* \neq 0, H^* \neq 0$) of the proposed model is calculated. Moreover, the stability and instability of the state variables, which include plant population P and herbivore population H are described graphically. It is shown that the system represents absolute stability when it has no time parameter (τ). When the time parameter is less than the threshold value, then the system exhibits asymptotic stability. In addition, the system surrenders its stability, and Hopf-bifurcation occurs when the time parameter surpasses the threshold value. The time-series graphs are also represented. It is demonstrated that the system becomes more stable with the maximum rate of predation. MATLAB software is used to perform the graphs to justify the theoretical results.

Keywords: Leslie-Gower predator-prey model, delay differential equation, time parameter, Allee effect

MSC: 65P40

1. Introduction

Plant-herbivore models play an essential role in ecology. Many ecologists have analyzed plant-herbivore models with different types of functional responses and Allee effects [1-9]. Functional responses are stated by Holling [10], which describe the rate of consumption of prey by the predator and the size of the prey. The effect that occurs when the density of plants is very small is called the Allee effect. It can be divided into two categories, which include the strong Allee effect and the weak Allee effect [11]. The Allee effect is called strong when the growth rate is non-positive in the low limit of low density and it has a population threshold as well. Conversely, the Allee effect is called weak when the growth rate at zero density is non-negative [12-18]. Kumar and Verma [19] presented the plant-herbivore model incorporated under the Allee effect with Holling-type I functional response. It is shown that the system is stable around the interior point and unstable when the threshold value crosses the delay parameter and Hopf-bifurcation occurs. Dupke et al. [20] applied Holling's concept of Holling-type II functional response and applied it to the habitat selection of herbivores. Liu et al. [21] modified the traditional Holling-type II functional response for the plant-herbivore model and formed a toxin-determined functional response due to chemical defenses so that the herbivores could consume fewer plants. The interpretation of dynamical properties of the stochastic prey-predator model and

non-autonomous deterministic is studied by Sengupta et al. [22] with a sigmoid functional response. Persistence, non-persistence, permanence, and positivity are derived. The interest of the authors is continuously increasing in the study of functional responses, and they are modifying the types of functional responses to make them more difficult to analyze. These types of mathematical models can be utilized to demonstrate the stability behavior of the system along with time series behavior [23-28]. Arancibia-Ibarra et al. [29] studied the Leslie-Gower predator-prey model with Holling-type II functional response under the weak Allee effect and revealed the coexistence and oscillations of both the predator and prey populations. They also proved the bistability of the model for some fixed subset of parameters. Mishra et al. [30] examined a Leslie-Gower predator-prey model and concluded that the prey alone can be responsible for the instability of the model. A prey population can release chemical substances, which can be dangerous for the predator population and may lead to the predator's death. The predation rate may also become lower due to the inability of the predators. Chen et al. [31] studied the Leslie-Gower predator-prey model with the harvesting system. The results due to harvesting declare that the concentration of the predator population is strictly decreasing and the size of prey has no impact under some restrictions. They used Pontryagin's maximal principle to attain the optimal harvesting policy and illustrate a case to represent that the optimal harvesting policy is realizable. Yue [32] examined a prey refuge with a modified Leslie-Gower model. The global attractivity of a positive equilibrium and the stability of the system are studied. Fang et al. [33] analyzed the Leslie-Gower model with a weak Allee effect and concluded that if the intensity of the Allee effect is strong, then both the prey and predator populations will tend to be in abundance. It is also shown that the Allee effect alone can change the stability of the system, and supercritical Hopf-bifurcation occurred. Bedi et al. [34] examined the controllability and stability of Hilfer fractional evolution equations in a Banach space. They investigated the characteristics and uniqueness properties of these equations by employing propagation family theory, non-compactness calculation methods, and the fixed point technique. Bedi et al. [35] investigated the presence of mild solutions and the approximate controllability of Hilfer fractional evolution equations, considering almost sectorial operators and nonlocal conditions. They achieved the existence of solutions by defining Green's function and then determining approximate controllability through the selection of an appropriate control function. Abboubakar et al. [36] examined a Zika disease model incorporating fractional derivatives in the Caputo sense. They demonstrated the existence of equilibrium points and performed stability analysis. Furthermore, they investigated the existence and uniqueness of solutions for the fractional model using the Banach fixed point theory and indicated that the occurrence of the Hopf-bifurcation phenomenon can disappear at a specific value of the fractional order parameter. Yuan et al. [37] introduced a Zika model incorporating three transmission routes and determined that the disease-free equilibrium exhibits global asymptotic stability. They further investigated the individual impacts of the transmission routes on the reproduction number as well as the short-term and long-term host infections.

The motivation for this work lies in the investigation of a crucial ecological problem, namely, the dynamics of a modified Leslie-Gower plant-herbivore model incorporating the Allee effect. Although previous research has extensively investigated different aspects of population dynamics, the application of delay differential equations in combination with the proposed model has yet to be explored. To address this gap, a novel mathematical model is proposed that incorporates delay differential equations to account for past historical interactions between the plant and herbivore populations, with τ as the time parameter. Furthermore, the inclusion of a Holling-type II functional response adds another layer of complexity to the model, allowing for more realistic and intricate ecological interactions. Through this work, the aim is to enhance the understanding of ecological systems and contribute valuable insights to the field of population dynamics with implications for conservation and ecosystem management.

The combination of the Allee effect and the Holling-type II functional response results in increased vulnerability of prey populations to predation at low prey densities. Furthermore, the inclusion of a time delay introduces additional complexity into the dynamics of the system. In general, the combined presence of these factors in a biological system can have significant implications for population dynamics, including aspects such as population persistence and stability, as well as the risk of extinction.

In Section 2, the formulation of the mathematical model with the initial conditions and the description of the state variables and model parameters are given. Section 3 discusses the determination of the non-trivial equilibrium point by equating the derivatives of the model to zero. Section 4 contains the stability analysis of non-trivial equilibrium by considering the Allee effect and Hopf-bifurcation. This section includes the exponential characteristic equation and various lemmas and postulates to explain the nature of the roots using Routh Hurwitz's criteria. Section 5 presents

graphical examples to further illustrate the analytical results. Section 6 focuses on the sensitivity analysis of the state variables with respect to the model parameters (excluding the time parameter). The sensitivity equations are derived through partial derivatives, and the results are depicted graphically. Finally, the summary of the findings and analysis is provided in Section 7 in the form of a conclusion.

2. Mathematical model

2.1 Model formulation

Let $P(t)$ represent the population size of the plants and $H(t)$ be the population size of the herbivores at time t , respectively. Both the plant population and herbivore population grow logistically with internal growth rates r and u respectively. K is known as the carrying capacity of the plant population. The carrying capacity of the herbivore population depends on the plant population and γ is a measure of the quality of the plant as food for the herbivore. The Holling-type II functional response is used where α is called the per capita rate of maximum predation and β is half of the saturated response level. $(P - b)$ is the Allee effect term, where $b > 0$ is the Allee threshold and τ is the time parameter that characterizes past history. A set of non-linear delay differential equations using the above notations is given by:

$$\frac{dP}{dt} = rP \left(1 - \frac{P(t-\tau)}{K} \right) (P-b) - \frac{\alpha PH}{P+\beta}, \quad (1)$$

$$\frac{dH}{dt} = uH \left(1 - \frac{H}{\gamma P} \right). \quad (2)$$

The initial value conditions for the above model are $P(t) > 0$, $H(t) > 0$, and $\forall t \geq 0$. Also, $P(t - \tau) = \varepsilon$, constant $\forall t \in [0, \tau]$.

3. Analysis of the model

3.1 Non-trivial equilibrium $E^* (P^* \neq 0, H^* \neq 0)$

In this section, non-trivial equilibrium is considered, which characterizes the co-existence of all the state variables (P^*, H^*) where none of them is zero because the stability of the system shows no impact of delay on any of the other equilibrium points. The non-trivial equilibrium $E^* (P^* \neq 0, H^* \neq 0)$ is as follows:

Let $P(t - \tau) \equiv P(t)$ and equate the system of equations (1) and (2) to zero:

$$\frac{dP^*}{dt} = 0 \Rightarrow H^* = \frac{r}{\alpha} \left(1 - \frac{P^*}{K} \right) (P^* - b) (P^* + \beta),$$

$$\frac{dH^*}{dt} = 0 \Rightarrow P^* = H^* / \gamma.$$

3.2 Stability analysis of equilibrium $E^* (P^* \neq 0, H^* \neq 0)$

A modified Leslie-Gower plant-herbivore model at the equilibrium E^* is given by the following set of equations:

$$\frac{dP^*}{dt} = rP^* \left(1 - \frac{P^*(t-\tau)}{K} \right) (P^* - b) - \frac{\alpha P^* H^*}{P^* + \beta}, \quad (3)$$

$$\frac{dH^*}{dt} = uH^* \left(1 - \frac{H^*}{\gamma P^*} \right). \quad (4)$$

The exponential characteristic equation for the set of equations (3) and (4) is represented by:

$$\lambda^2 + K_1\lambda + K_2 + (K_3\lambda + K_4)e^{-\lambda\tau} = 0. \quad (5)$$

Here,

$$K_1 = -2rP^* + rb + \frac{\alpha\beta H^*}{(P^* + \beta)^2} - m_4,$$

$$K_2 = m_4 \left[2rP^* - rb - \frac{\alpha\beta H^*}{(P^* + \beta)^2} \right] - m_2 m_3,$$

$$K_3 = \frac{rP^*}{K} (P^* - b),$$

$$K_4 = m_4 \left[\frac{-rP^*}{K} (P^* - b) \right],$$

where

$$m_1 = 2rP^* - rb - \frac{\alpha\beta H^*}{(P^* + \beta)^2} - \frac{rP^*}{K} (P^* - b) e^{-\lambda\tau},$$

$$m_2 = \frac{u}{\gamma} \left(\frac{H^*}{P^*} \right)^2,$$

$$m_3 = -\frac{\alpha P^*}{(P^* + \beta)},$$

$$m_4 = u \left(1 - \frac{2H^*}{\gamma P^*} \right).$$

When $\tau = 0$, the equation (5) is represented as:

$$\lambda^2 + (K_1 + K_3)\lambda + (K_2 + K_4) = 0. \quad (6)$$

Using Hurwitz's criteria, the system shows stability i.e., the roots of equation (6) will contain a real part that is negative if:

$$\begin{aligned} (\mathcal{S}_1) : (K_1 + K_3) &> 0 \\ (\mathcal{S}_2) : (K_2 + K_4) &> 0 \end{aligned}$$

When we change the values of τ , the roots of the negative real parts shift to positive real parts as follows: Let the root of equation (5) be $\lambda = i\omega$, then the equation (5) is given by:

$$\begin{aligned} (i\omega)^2 + K_1(i\omega) + K_2 + (K_3(i\omega) + K_4)e^{-(i\omega)\tau} &= 0. \\ \Rightarrow -\omega^2 + K_1(i\omega) + K_2 + (K_3(i\omega) + K_4)(\cos \omega\tau - i \sin \omega\tau) &= 0. \end{aligned}$$

Segregating real parts from the imaginary parts:

$$-\omega^2 + K_2 = -K_4 \cos \omega\tau - K_3 \omega \sin \omega\tau, \quad (7)$$

$$K_1 \omega = -K_3 \cos \omega\tau + K_4 \sin \omega\tau. \quad (8)$$

$\therefore \omega$ is true for:

$$\omega^4 - (K_3^2 - K_1^2 + 2K_2)\omega^2 + (K_2^2 - K_4^2) = 0. \quad (9)$$

The two roots of equation (9) are:

$$\omega_{1,2}^2 = \frac{(K_3^2 - K_1^2 + 2K_2) \pm \sqrt{(K_3^2 - K_1^2 + 2K_2)^2 - 4(K_2^2 - K_4^2)}}{2}. \quad (10)$$

$\omega_{1,2}^2$ has no positive roots if:

$$(\mathcal{S}_3) : (K_3^2 - K_1^2 + 2K_2) < 0 \text{ and } (K_2^2 - K_4^2) > 0 \text{ or } (K_3^2 - K_1^2 + 2K_2)^2 < 4(K_2^2 - K_4^2).$$

It gives that if the condition (\mathcal{S}_3) holds, equation (10) has no positive roots.

There are following lemmas [39].

Lemma 1. Every root of equation (5) has real parts, which are negative $\forall \tau \geq 0$, if $(\mathcal{S}_1) - (\mathcal{S}_2)$ hold. In contrast, if:

$$(\mathcal{S}_4) : (K_2^2 - K_4^2) < 0 \text{ or } (K_3^2 - K_1^2 + 2K_2) > 0 \text{ and } (K_3^2 - K_1^2 + 2K_2)^2 = 4(K_2^2 - K_4^2).$$

Then, ω_1^2 is the +ve root of equation (7).

Similarly, if:

$$(\mathcal{S}_5) : (K_2^2 - K_4^2) > 0 \text{ or } (K_3^2 - K_1^2 + 2K_2) > 0 \text{ and } (K_3^2 - K_1^2 + 2K_2)^2 > 4(K_2^2 - K_4^2).$$

Then, $\omega_{1,2}^2$ are positive roots of equation (7).

When τ takes certain values, then equation (5) includes purely imaginary roots in both (S_4) and (S_5) . The system of equations (5) and (6) gives the threshold values τ_j^\pm of τ :

$$\tau_j^\pm = \frac{1}{\omega_{1,2}} \cos^{-1} \left[\frac{K_4 (\omega_{1,2}^2 - K_2) - K_1 K_3 \omega_{1,2}^2}{K_3^2 \omega_{1,2}^2 + K_4^2} \right] + \frac{2j\pi}{\omega_{1,2}}, j = 0, 1, 2, \dots \quad (11)$$

The succeeding lemma can condense the above discussion [39].

Lemma 2.

(I) If $(S_1) - (S_2)$ and (S_4) is true, and $\tau = \tau_j^+$, then there are two purely imaginary roots $\pm i\omega_1$ of equation (5).

(II) There are two purely imaginary roots $\pm i\omega_1$ ($\pm i\omega_2$, respectively) of equation (5), if $(S_1) - (S_2)$ and (S_5) is true and $\tau = \tau_j^+$ ($\tau = \tau_j^-$, respectively),

Let $\tau_j^\pm = \mu_j^\pm(\tau) + i\omega_j^\pm(\tau)$; $j = 0, 1, 2, 3, \dots$ to shift the negative real part of some roots to the real part of the equation (5), which is positive when $\tau > \tau_j^+$ and $\tau < \tau_j^-$.

$\mu_j^\pm(\tau_j^\pm) = 0, \omega_j^\pm(\tau_j^\pm) = \omega_{1,2}$ is satisfied by the roots of equation (5).

The following transversality condition holds for the above criteria:

$$\frac{d}{d\tau}(\operatorname{Re} \lambda_j^+(\tau_j^+)) > 0 \text{ and } \frac{d}{d\tau}(\operatorname{Re} \lambda_j^-(\tau_j^-)) < 0.$$

It represents that τ_j^\pm are bifurcating values. The scattering of the zeros of equation (5) can be given by the following postulate [38].

Postulate: Consider, τ_j^+ ($j = 0, 1, 2, 3, \dots$), which is deducted from equation (11).

(I) Every root has -ve real part $\forall \tau \geq 0$ in equation (5), if (S_1) , and (S_2) hold.

(II) Every root has -ve real part in equation (5), if (S_1) , (S_2) and (S_4) hold and when $\tau \in [0, \tau_0^+)$. Equation (5) has two purely imaginary roots $\pm i\omega_1$ when $\tau = \tau_0^+$. There is at least one root with +ve real part in equation (3) when $\tau > \tau_0^+$.

(III) When (S_1) , (S_2) , and (S_5) are true, then there is a positive integer m , such that $0 < \tau_0^+ < \tau_0^- < \tau_1^+ < \tau_1^- \dots < \tau_{m-1}^- < \tau_m^+$, and there are m fluctuations between stability and instability. That is, every root has a negative real part in equation (5) when $\tau \in [0, \tau_0^+), (\tau_0^-, \tau_1^+), \dots, (\tau_{m-1}^-, \tau_m^+)$. There is at least one root with +ve real part in equation (5) if $\tau \in (\tau_0^+, \tau_0^-), (\tau_0^-, \tau_1^-), \dots, (\tau_{m-1}^+, \tau_{m-1}^-)$ and $\tau > \tau_m^+$.

4. Graphical example

The graphical examples are presented by the following parametric values to depict the modified Leslie-Gower plant-herbivore dynamics of a set of equations (1) and (2) (Figures 1-6):

$$r = 0.8, K = 0.4, b = -0.3, \alpha = 0.8, \beta = 0.4, u = 0.2, \gamma = 0.1$$

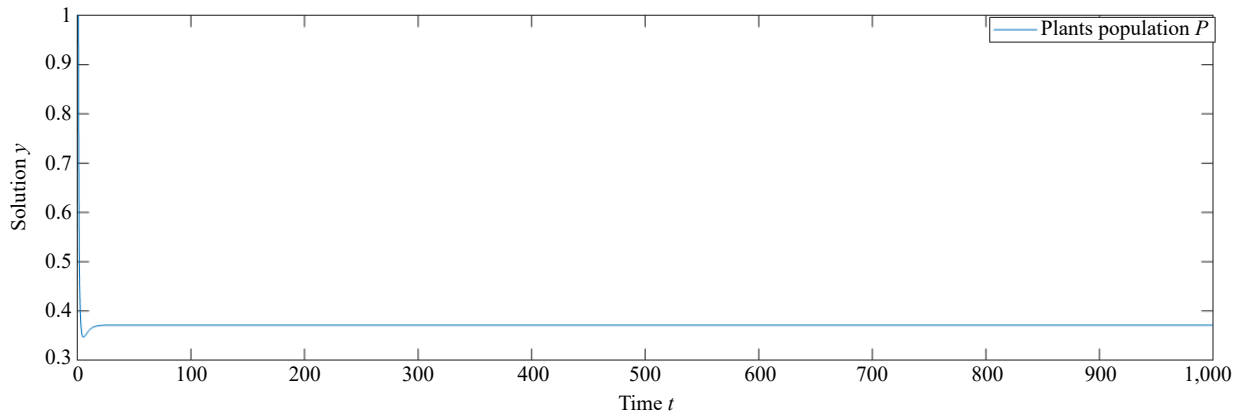


Figure 1. The plant population P is absolutely stable when there is no time parameter, i.e., when $\tau = 0$

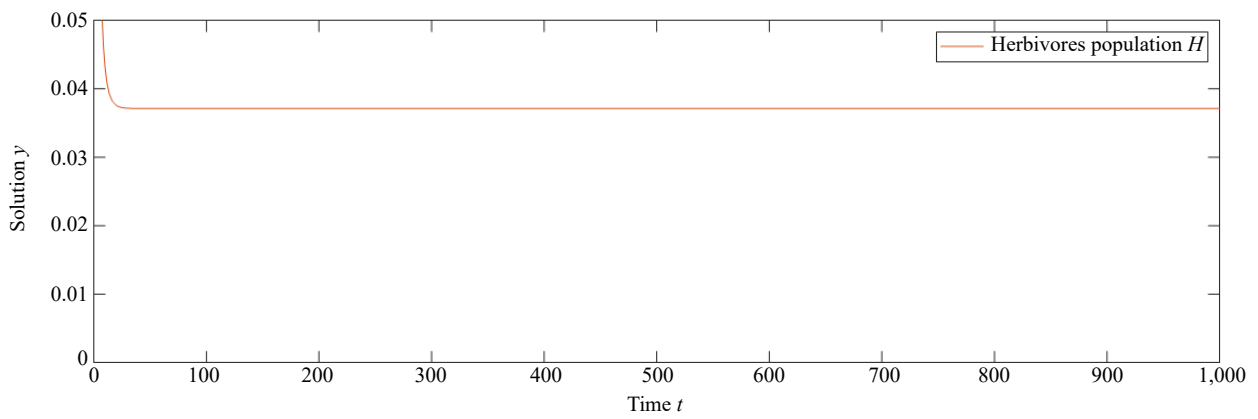


Figure 2. The herbivore population H is absolutely stable when there is no time parameter, i.e., when $\tau = 0$

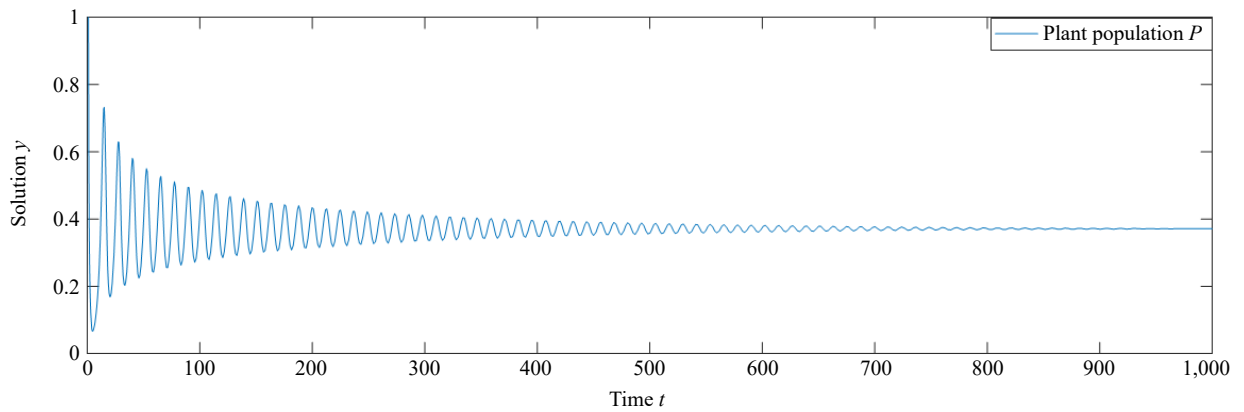


Figure 3. The plant population P is asymptotically stable when the time parameter is less than the threshold value, i.e., when $\tau < 3.2$

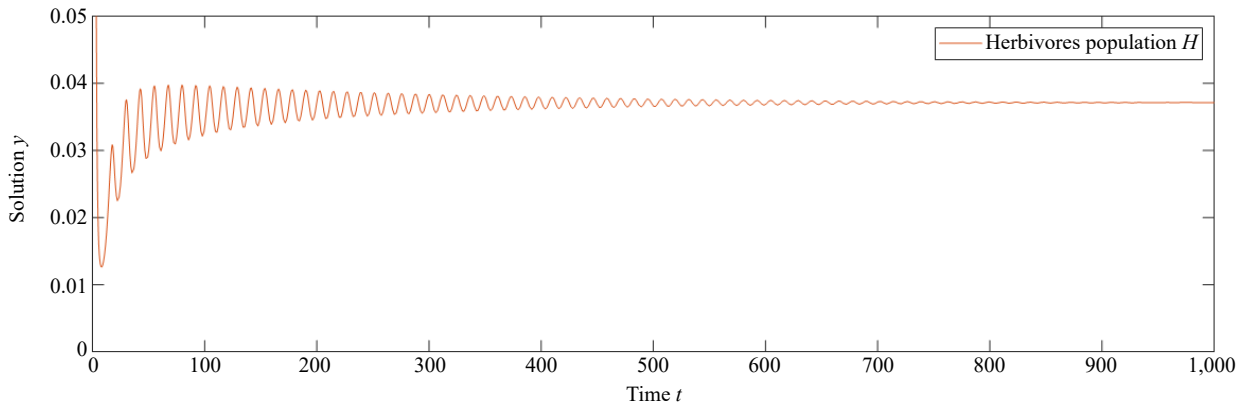


Figure 4. The herbivore population H is asymptotically stable when the time parameter is less than the threshold value, i.e., when $\tau < 3.2$

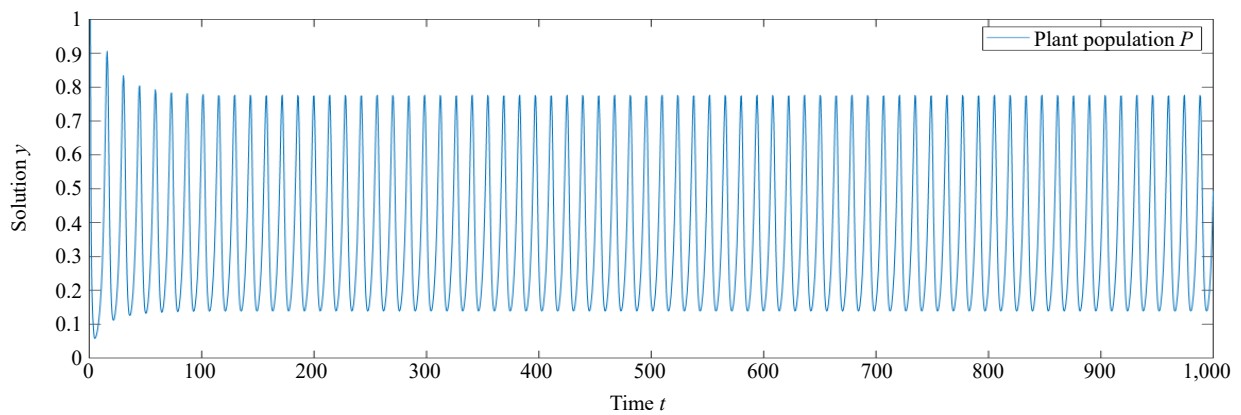


Figure 5. The plant population P represents Hopf-bifurcation when the time parameter surpasses the threshold value, i.e., when $\tau \geq 3.2$

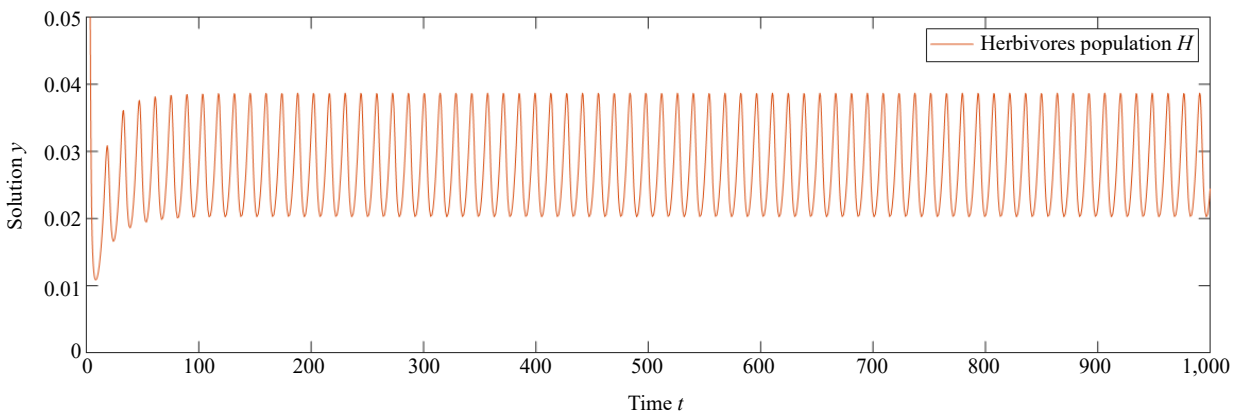


Figure 6. The herbivore population H represents Hopf-bifurcation when the time parameter surpasses the threshold value, i.e., when $\tau \geq 3.2$

5. Sensitivity analysis of state variables with respect to model parameters

Sensitivity analysis is a study that represents the behavior of the stability of the proposed state variables by varying the parameter values and taking the delay parameter (τ) as constant. For instance, the sensitivity equations of the partial derivatives of the solution (P, H) with respect to the per capita rate of maximum predation α are as follows:

$$\frac{dS_1}{dt} = \left[r(2P-b) - \frac{\alpha\beta H}{(P+\beta)^2} + r\left(\frac{b-2P}{K}\right)P(t-\tau) \right] S_1 - \alpha \left[\frac{P^2 + \beta P}{(P+\beta)^2} \right] S_2 + rP \frac{(b-P)}{K} S_1(t-\tau),$$

$$\frac{dS_2}{dt} = \frac{uH^2}{\gamma P^2} S_1 + \left(u - \frac{2uPH}{\gamma P^2} \right) S_2$$

where $S_1 = \frac{\partial P}{\partial \alpha}$ and $S_2 = \frac{\partial H}{\partial \alpha}$.

5.1 Sensitivity of variables to parameter α

When we increase the value of the per capita rate of maximum predation from $\alpha = 0.8$ to $\alpha = 2.3$, the system (P, H) changes its behavior from Hopf-bifurcation to asymptotic stability. After that, when we further increase the per capita rate of maximum predation from $\alpha = 2.3$ to $\alpha = 2.9$, the system (P, H) again shifts its equilibrium from asymptotic stable to absolutely stable as shown in Figures 7 and 8.

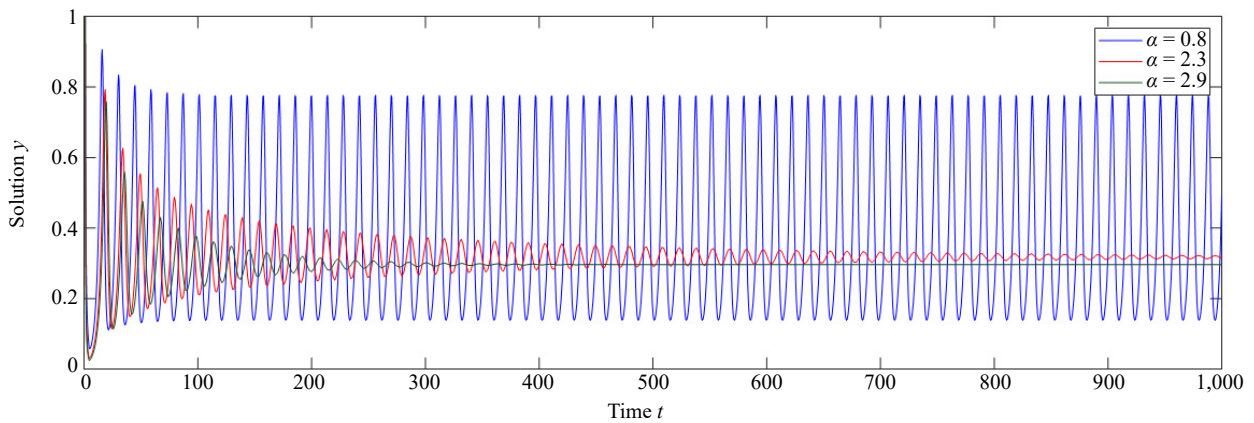


Figure 7. Time series graph of change in density of plant population P with respect to changes in the per capita rate of maximum predation α

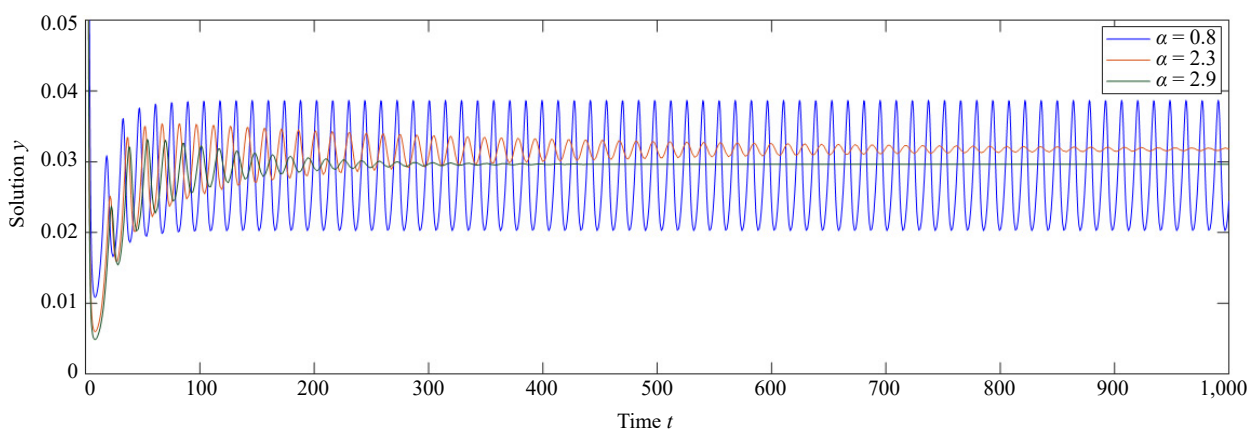


Figure 8. Time series graph of change in density of herbivore population H with respect to changes in the per capita rate of maximum predation α

6. Conclusion

A modified Leslie-Gower plant-herbivore model is taken into consideration by incorporating the Holling-type II functional response under the Allee effect. The time parameter τ is utilized to depict past history. Both populations (P , H) grow logistically at different intrinsic growth rates. The carrying capacity of the herbivore population H depends on the plant population P . The non-trivial equilibrium E^* ($P^* \neq 0$, $H^* \neq 0$) is calculated. The stability behavior of the system and time series graphs are performed graphically. The observation in Figures 1 and 2 is that plant population P and herbivore population H are absolutely stable when the time parameter $\tau = 0$. Both populations are asymptotically stable when the time parameter $\tau < 3.2$. Furthermore, both populations are unstable and demonstrate Hopf-bifurcation when the time parameter $\tau \geq 3.2$.

When we increase the value of the per capita rate of maximum predation α from $\alpha = 0.8$ to $\alpha = 2.3$, the plant population P shifts its behavior from Hopf-bifurcation to asymptotic. And then we further increase the value of the per capita rate of maximum predation α from $\alpha = 2.3$ to $\alpha = 2.9$, the plant population P starts converging to a stable equilibrium point and becomes more and more stable as shown in Figure 7. Similarly, Figure 8 indicates that as we increase the value of the per capita rate of maximum predation α from $\alpha = 0.8$ to $\alpha = 2.3$ and then from $\alpha = 2.3$ to $\alpha = 2.9$, even the herbivore population H become more stable by shifting its equilibrium from Hopf-bifurcation to asymptotic and finally to absolutely stable. It means, if the herbivore population consumes more plants, then the plant population becomes smaller. As a result of this, the herbivore population will start competing with each other, and the size of the herbivore population will also decrease. The system will become more stable due to this scenario.

Conflict of interest

The authors do not have any competing interests to declare that are relevant to the content of this article.

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