Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

On Two Simple, Three-Parameter, Three-Dimensional, Non-Exchangeable Copulas

Christophe Chesneau

Department of Mathematics, LMNO, University of Caen, 14032 Caen, France
Email: chesneau.christophe@gmail.com

Received: 28 February 2023; **Revised:** 28 March 2023; **Accepted:** 21 April 2023

Abstract: In this paper, we use the notion of a copula to provide theoretical contributions to the development of three-dimensional dependence models. In particular, we suggest two new three-dimensional copulas whose forms are simple and adaptable; they are based on polynomials, power functions, and three tuning parameters. In order to rely on the existing literature, we mention that the second copula can be viewed as a generalization of the three-dimensional Farlie-Gumbel-Morgenstern copula. Both copulas have the feature of being non-exchangeable (for most of the parameter values). Theoretical results are demonstrated, including wide admissible sets of values for the parameters and closed-form expressions for the medial correlation and Spearman's rho. By using our methodology, the limitations imposed by the exchangeable property, which are typical of traditional three-dimensional copulas in the literature, are thus overcome, and new approaches to dependence modeling are opened up.

***Keywords*:** three-dimensional copulas, dependence models, multivariate analysis, correlation measures

MSC: 60E15, 62H99

1. Introduction

Copulas are of great interest because they might provide practical tools for developing flexible dependence models for multivariate random vectors. They can be defined as functions linking the marginal distribution functions to the parent joint distribution function (see [1, 2]). A lot of modern copula theory and practice can be found in [3-6]. Among the standard high-dimensional copulas, we may mention the normal, lognormal, student, Farlie-Gumbel-Morgenstern (FGM), and Archimedean copulas. Current developments include the D-vine and R-vine copulas (see [7, 8]). As a matter of fact, original functional constructions of copulas in high dimensions are still uncommon, despite an expanded selection of potential two-dimensional (2D) models being available today. This is especially true for three-dimensional (3D) copulas, which have recently attracted new interest. Some of the most well-known 3D copulas, in particular, have been used in a wide range of important applications. We may refer to the 3D copula-based method for the statistical dependence in wireless sensor networks developed in [9], the 3D copula-based model for evaluating flooding events in Ravenna (Italy) in [10], the 3D copula-based method elaborated in [11] for the impact and risk assessment of droughts, the 3D copula-based stochastic frontier model with sample selection created in [12], and the 3D copula-based approach used in [13] to design coastal structures, among others.

The main drawback of the most well-known 3D copulas, and the 3D Archimedean copulas in particular, is the restriction on the number of parameters. These copulas are easy to compute and are implemented in almost all software, but they cannot cover all practical dependency scenarios. They are sometimes abusively used because of their accessibility. Creating novel types of copulas with appropriate numbers of parameters and desirable properties for the tractability of computations and simulations is a crucial area of research. Indeed, when real data are taken into account, it becomes necessary to fit asymmetric tail behavior and a wide range of dependence structures. New 3D copulas with two or three parameters and non-exchangeability properties have been developed to overcome these constraints. On this topic, the major references are [14-18]. In particular, in [18, Proposition 7], the following 3D copula is presented:

$$C(x,y,z) = xyz + \lambda xy(1-x)(1-y)z^{\theta}, \ (x,y,z) \in [0,1]^3, \tag{1}$$

with $\theta \geq 1$ and $|\lambda| \leq 1/\theta$. The function $P(x,y,z) = \lambda xy(1-x)(1-y)z^{\theta}$, thus perturbs the 3D independence copula, i.e., $\Pi(x,y,z) = xyz$. The copula in equation (1) has the originality of being in three dimensions, simple, and not exchangeable. As a result, it provides a new and manageable alternative to the numerous copulas found in the literature. It is separable with respect to x, y, and z since we can write $P(x, y, z)$ as $P(x,y,z) = \phi(x)\psi(y)\varphi(z)$, with $\phi(x) = \lambda x(1-x), \psi(y) = y(1-y)$, and $\varphi(z) = z^{\theta}$. The direction of choosing new perturbation functions in this setting, particularly those with the non-separable property, is yet substantially unexplored.

The purpose of this paper is to modestly complete this research gap in the area of new, original, and simple constructions for 3D copulas with non-separable perturbation functions. More precisely, we propose two new 3D copulas based on polynomials, power functions, and three tuning parameters. Their forms are very flexible and straightforward to comprehend. A generalization of the 3D FGM copula can be seen in one of them. Note that, except for particular parameter values, none of the proposed copulas is exchangeable, which is a sufficiently interesting property to be communicated. It is established that the parameters have a wide range of acceptable sets of values. Differentiation and inequality techniques are used as the foundation for the proofs. The primary functions, including the survival copula and density copula, are then revealed. We investigate different correlation properties and provide closed-form expressions for the 3D medial correlation and Spearman's rho. Thus, our method circumvents the drawbacks posed by the exchangeable properties typical of standard 3D copulas in the literature.

The rest of this paper is organized as follows: Section 2 introduces the notion of multidimensional copulas and describes the first new 3D copula along with its main characteristics. Section 3 does the same thing, but with the other new 3D copula. Section 4 discusses our findings and future work.

2. First 3D copula

To begin, for a positive integer n, let us recall the general definition of an absolutely continuous copula in n dimensions.

Definition 1. Let $n \geq 2$ be an integer. The function $C(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in [0,1]^n$, is said to be an absolutely continuous n-dimensional copula if and only if the properties described into the following items are satisfied:

(I) $C(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0$ for any $(x_1, \ldots, x_n) \in [0,1]^n$ and $i = 1, ..., n$,

(II) $C(1, \ldots, 1, x, 1, \ldots, 1) = x$ for any $x \in [0,1]$, and this, in each of the n vector components,

(III) $\partial_{x_1,\ldots,x_n} C(x_1, \ldots, x_n) \geq 0$ for any $(x_1, \ldots, x_n) \in [0,1]^n$, where $\partial_{x_1,\ldots,x_n} = \partial^n / (\partial x_1 \ldots \partial x_n)$ denotes the mixed n-th order partial derivatives according to $x_1, \ldots, x_n$.

We refer to [3] for more detail about the notion of an absolutely continuous n-dimensional copula. Hereafter, to lighten the text, we omit the expression "absolutely continuous". From a mathematical viewpoint, the more challenging item remains (III). Naturally, a 3D copula satisfies Definition 1 with $n = 3$. This definition will be at the center of our main results, beginning with the one below, which presents a novel 3D polynomial-power function copula with a distinctive form and adaptable properties.

Proposition 2.1. Let $(\lambda, \theta) \in \mathbb{R}^2$, and m be a positive integer. Let us consider the 3D function

$$C(x,y,z) = xyz\left[1+\lambda(x-y)^m z^\theta(1-z)\right],\ (x,y,z)\in[0,1]^3. \tag{2}$$

Then, for $\theta \geq 0$ and

$$|\lambda| \leq \frac{1}{(\theta+1)\max[m(m-1), m+1]}, \tag{3}$$

$C(x, y, z)$ is a valid 3D copula.

Proof. The proof consists of demonstrating that $C(x, y, z)$ satisfies the items (I), (II), and (III) of Definition 1 with $n = 3$.

(I) For any $(x,y)\in[0,1]^2$, it is clear that

$$C(x,y,0) = xy\times 0\times\left[1+\lambda(x-y)^m\times 0^\theta\times(1-0)\right] = 0,$$

with the usual convention $0^0 = 1$ for $\theta = 0$. Using a similar development, for any $(y,z)\in[0,1]^2$, we obtain $C(0, y, z) = 0$, and for any $(x,z)\in[0,1]^2$, we have $C(x, 0, z) = 0$. The item (I) is proved.

(II) For any $x\in[0,1]$, we have

$$C(x,1,1) = x\times 1\times 1\times\left[1+\lambda(x-1)^m\times 1^\theta\times(1-1)\right] = x.$$

Similarly, for any $y\in[0,1]$, we obtain

$$C(1,y,1) = 1\times y\times 1\times\left[1+\lambda(1-y)^m\times 1^\theta\times(1-1)\right] = y$$

and for any $z\in[0,1]$, since $m \geq 1$, we get

$$C(1,1,z) = 1\times 1\times z\times\left[1+\lambda(1-1)^m z^\theta(1-z)\right] = z.$$

The item (II) is proved.

(III) By the differentiation of $C(x,y,z)$ with respect to x, y, and z, we obtain

$$\partial_{x,y,z}C(x,y,z) = 1-\lambda z^\theta[(\theta+2)z-(\theta+1)](x-y)^{m-2}\left[(m+1)x^2+(m+1)y^2-(m^2+m+2)xy\right].$$

Let us now distinguish the case $m = 1$ and the case $m \geq 2$.

- For $m = 1$, after simplifications, we can write

$$\partial_{x,y,z}C(x,y,z) = 1-2\lambda z^\theta[(\theta+2)z-(\theta+1)](x-y).$$

It is clear that, for any $(x,y,z)\in[0,1]^3, z^\theta\in[0,1], x-y\in[-1,1]$, and since $\theta \geq 0$, we have $-(\theta+1)\leq(\theta+2)z-(\theta+1)\leq 1$, which implies that

$$|(\theta+2)z-(\theta+1)| \leq \max(1,\theta+1) = \theta+1. \tag{4}$$

Therefore, with the use of absolute values, we have

$$\partial_{x,y,z}C(x,y,z) \geq 1-2|\lambda|z^\theta|(\theta+2)z-(\theta+1)||x-y| \geq 1-2(\theta+1)|\lambda|.$$

Furthermore, since $\max[m(m-1), m+1] = \max(0,2) = 2$ and $|\lambda| \leq 1/[2(\theta+1)]$, we have

$$\partial_{x,y,z}C(x,y,z) \geq 0.$$

- For $m \geq 2$, we can write

$$\partial_{x,y,z}C(x,y,z) = 1 - \lambda z^{\theta}[(\theta+2)z - \theta - 1](x-y)^{m-2}F(x,y),$$

where

$$F(x,y) = (m+1)x^2 + (m+1)y^2 - (m^2+m+2)xy$$
$$= (m+1)(x-y)^2 - m(m-1)xy.$$

For any $(x,y,z) \in [0,1]^3$, we have $z^{\theta} \in [0,1]$, and since m is an integer such that $m \geq 2$, we have $(x-y)^{m-2} \in [-1,1]$. These results, combined with the use of absolute values and equation (4), give

$$\partial_{x,y,z}C(x,y,z) \geq 1 - |\lambda| z^{\theta} |(\theta+2)z - (\theta+1)| |x-y|^{m-2} |F(x,y)|$$
$$\geq 1 - |\lambda|(\theta+1)|F(x,y)|.$$

Let us now find a sharp upper bound for $|F(x,y)|$. Since $m(m-1) \geq 0$ and $(x-y)^2 \in [0,1]$, we have

$$F(x,y) \leq (m+1)(x-y)^2 \leq m+1$$

and

$$-m(m-1) \leq -m(m-1)xy \leq F(x,y),$$

implying that

$$|F(x,y)| \leq \max[m(m-1), m+1].$$

Therefore, since $|\lambda| \leq 1/\{(\theta+1)\max[m(m-1), m+1]\}$, the above inequalities give

$$\partial_{x,y,z}C(x,y,z) \geq 1 - |\lambda|(\theta+1)\max[m(m-1), m+1] \geq 0.$$

The item (III) is proved.

To conclude, the items (I), (II), and (III) of Definition 1 with $n = 3$ are demonstrated; $C(x, y, z)$ is a valid 3D copula.

In a similar manner to the copula in equation (1), the copula in equation (2) is based on a perturbation of the 3D independence copula $\Pi(x,y,z)$, with the perturbed function defined as $Q(x,y,z) = \lambda xy(x-y)^m z^{\theta+1}(1-z)$. This last function has the features of being non-separable with respect to x, y, and z due to the term $(x - y)^m$, and of depending on three tuning parameters. To our knowledge, such perturbed functionalities are new in the literature. In particular, it seems to have no resemblance between the copula in equation (1) and the proposed copula. Some of its properties are described below.

Clearly, for $\lambda \neq 0$, the proposed copula is not exchangeable: there exist infinite examples of triplet values (x_*, y_*, z_*) such that $C(x_*, y_*, z_*) \neq C(x_*, z_*, y_*)$, for instance. In addition, one can remark that, for any permutation (x_0, y_0, z_0) of (x, y, z), $C(x_0, y_0, z_0)$ is a copula; six different copulas can be derived from $C(x, y, z)$ (listed in the three first lines of Table 1, with the use of the "plus or minus sign ±", each sign corresponding to a different copula). The corresponding copula density is given by

$$
\begin{aligned}
c(x,y,z) &= \partial_{x,y,z} C(x,y,z) \\
&= 1-\lambda z^{\theta}[(\theta+2)z-(\theta+1)](x-y)^{m-2}\left[(m+1)x^2+(m+1)y^2-(m^2+m+2)xy\right], \\
&\quad (x,y,z)\in[0,1]^3.
\end{aligned} \tag{5}
$$

The following 2D copulas can be derived from $C(x, y, z)$:

$$
C_{\dagger}(x,y)=C(x,y,1)=xy,\ C_{\dagger\dagger\dagger}(x,y)=C(1,x,y)=xy\left[1+\lambda(1-x)^m y^{\theta}(1-y)\right]
$$

and

$$
\begin{aligned}
C_{\dagger\dagger\dagger}(x,y) &= C(x,1,y)=xy\left[1+\lambda(x-1)^m y^{\theta}(1-y)\right] \\
&= xy\left[1+(-1)^m\lambda(1-x)^m y^{\theta}(1-y)\right].
\end{aligned}
$$

These 2D copulas are not new: the first one is the 2D independence copula, and the others are special cases of the extended FGM copulas as described in [19]. The corresponding survival copula is expressed as

$$
\begin{aligned}
\tilde{C}(x,y,z) &= x+y+z-2+C(1-x,1-y,1) \\
&\quad +C(1-x,1,1-z)+C(1,1-y,1-z)-C(1-x,1-y,1-z) \\
&= x+y+z-2+(1-x)(1-y)+(1-x)(1-z)\left[1+\lambda(-1)^m x^m(1-z)^{\theta}z\right] \\
&\quad +(1-y)(1-z)\left[1+\lambda y^m(1-z)^{\theta}z\right]-(1-x)(1-y)(1-z)\left[1+\lambda(y-x)^m(1-z)^{\theta}z\right] \\
&= xyz+\lambda z(1-z)^{\theta+1}\left[(-1)^m(1-x)x^m+(1-y)y^m-(1-x)(1-y)(y-x)^m\right].
\end{aligned}
$$

To the best of our knowledge, this survival copula also constitutes a new three-parameter, 3D, mainly non-exchangeable copula in the literature. It has a higher level of functional complexity than its parent copula, with the presence of several power functions.

When m is even and $\lambda \geq 0$, the following copula ordering holds: $C(x,y,z)\geq xyz=\Pi(x,y,z)$ for any $(x,y,z)\in[0,1]^3$. The reversed inequality holds for $\lambda \leq 0$.

As a standard copula property, the Fréchet-Hoeffding bounds hold for the proposed copula; for any $(x,y,z)\in[0,1]^3$, we have

$$
\max\left(x+y+z-2,0\right)\leq C(x,y,z)\leq \min(x,y,z),
$$

that is,

$$
\max\left(x+y+z-2,0\right)\leq xyz\left[1+\lambda(x-y)^m z^{\theta}(1-z)\right]\leq \min(x,y,z)
$$

(it is understood in the context of the condition in equation (3)). These 3D inequalities can be of independent interest.

According to [3], the 3D medial correlation is given by

$$
\mathcal{M}=\frac{1}{3}\left[8C\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)-1\right]=0.
$$

Hence, the proposed copula has no medial correlation, exactly as the 3D independence copula $\Pi(x,y,z)$.

According to [20], based on $C(x, y, z)$ the 3D Spearman's rho is given by

$$\rho = 8\int_0^1\int_0^1\int_0^1 C(x, y, z)dxdydz - 1.$$

After several integrations, we obtain

$$\rho = 8\lambda\int_0^1\int_0^1\int_0^1 xy(x-y)^m z^{\theta+1}(1-z)dxdydz$$

$$= 8\lambda\frac{1+(-1)^m}{(6+5\theta+\theta^2)(m+1)(m+2)(m+4)}.$$

Thus, the 3D Spearman's rho has a closed form. Surprisingly, it is equal to 0 if m is an odd integer, which is an intriguing feature already observed in some 3D trigonometric copulas (see [21]).

New 3D distributions can be generated for the proposed copula. Indeed, for three unidimensional distribution functions, say $F(x)$, $G(x)$, and $H(x)$, the following 3D function defines a valid 3D distribution function:

$$K(x, y, z) = C[F(x), G(y), H(z)], \quad (x, y, z) \in \mathbb{R}^3.$$

Owing to the originality of $C(x, y, z)$, a plethora of 3D distributions can be introduced. For motivated choices of uni-dimensional lifetime distribution functions $F(x)$, $G(x)$, and $H(x)$, we may refer to [22].

Table 1 presents some examples of simple copulas derived from the proposed copula.

Table 1. Some examples of copulas derived from the copula in equation (2) (with $\theta \geq 0$ and m an integer)

New general copulas	λ values
$xyz\left[1 \pm \lambda(x-y)^m z^\theta(1-z)\right]$	Equation (3)
$xyz\left[1 \pm \lambda(x-z)^m y^\theta(1-y)\right]$	Equation (3)
$xyz\left[1 \pm \lambda(y-z)^m x^\theta(1-x)\right]$	Equation (3)
Examples of new simple copulas	λ values
$xyz\left[1 + \lambda(x-y)(1-z)\right]$	$\lvert\lambda\rvert \leq \frac{1}{2}$
$xyz\left[1 + \lambda(x-y)z(1-z)\right]$	$\lvert\lambda\rvert \leq \frac{1}{4}$
$xyz\left[1 + \lambda(x-y)^2(1-z)\right]$	$\lvert\lambda\rvert \leq \frac{1}{3}$
$xyz\left[1 + \lambda(x-y)^2 z(1-z)\right]$	$\lvert\lambda\rvert \leq \frac{1}{6}$

To the best of our knowledge, all the copulas presented in Table 1 are new in the literature and can be used in a theoretical way, or for fresh 3D analysis in various statistical scenarios.

Remark 2.2. A generalized form for the proposed copula may be as follows:

$$C(x, y, z) = xyz\left(1 + \lambda\phi(x-y)\psi(z)\right), \quad (x, y, z) \in [0,1]^3,$$

or

$$C(x,y,z)=xyz\left(1+\lambda\phi(z(x-y))\psi(z)\right),\quad (x,y,z)\in[0,1]^3,$$

where φ and ψ denote continuous functions satisfying $\varphi(0)=0$ and $\psi(1)=0$, in order to satisfy the items (I) and (II) of Definition 1 with $n=3$. However, additional conditions on φ and ψ are necessary to satisfy (III), which remains a mathematical challenge that we leave for future work.

Remark 2.3. For the estimation of the parameters λ, θ, and m in a data analysis setting, one can think of the so-called omnibus estimation method (see [23, 24]). To present this method, let us consider observations drawn from a continuous random vector, such as (X, Y, Z). Thus, let n be the number of observations of this vector, where these observations are triplets of values, say $(x_1,y_1,z_1),\ldots,(x_n,y_n,z_n)$. These triplets represent the data. Then, the omnibus estimates of λ, θ, and m are obtained as the following argmax-values:

$$(\hat{\lambda},\theta,\hat{m})=\operatorname{argmax}_{(\lambda,\theta,m)\in T}\sum_{i=1}^{n}\log\left\{c\left[\hat{F}(x_i),G(y_i),H(z_i);\lambda,\theta,m\right]\right\},$$

where $c(x,y,z;\lambda,\theta,m)=c(x,y,z)$, as described in equation (5), T represents, the restrictions imposed on the parameters, i.e., $T=\{m$ positive integer, $\theta\geq 0$ and $|\lambda|\leq 1/\{(\theta+1)\max[m(m-1), m+1]$,

$$\hat{F}(x)=\frac{1}{n}\sum_{j=1}^{n}1\{x_j\leq x\},\quad \hat{G}(y)=\frac{1}{n}\sum_{j=1}^{n}1\{y_j\leq y\},\quad H(z)=\frac{1}{n}\sum_{j=1}^{n}1\{z_j\leq z\},$$

and $1S$ denotes the indicator function over a given set S. In other words, when the transformed data $(\hat{F}(x_1),\hat{G}(y_1),\hat{H}(z_1)),\ldots,(\hat{F}(x_n),\hat{G}(y_n),\hat{H}(z_n))$ are considered, the omnibus estimates $\hat{\lambda},\theta$, and $\hat{m}$ correspond to their respective maximum likelihood estimates. The global effectiveness of this method is guaranteed. We refer to [23, 24] for more information. The contributions of the paper have a theoretical focus; the statistical application with real-world data is actually a perspective.

This ends this section; another 3D copula approach is described in the next section.

3. Second 3D copula

A new 3D polynomial copula with an original form and attractive functionalities is demonstrated in the result below as a complement to the copula described in the section above.

Proposition 3.1. Let $(\lambda,\alpha,\beta)\in\mathbb{R}^3$. Let us consider the 3D function

$$C(x,y,z)=xyz\left\{1+\lambda\left[1-(\alpha x+\beta y+\gamma xy)\right](1-z)\right\},\quad (x,y,z)\in[0,1]^3, \tag{6}$$

with $\gamma=1-(\alpha+\beta)$ (so that γ is not a new additional parameter). Let us set

$$\Omega=\begin{cases}\max(|1-2\alpha-2\beta-4\gamma|,1) & \text{if} \quad \gamma\geq 0,\alpha\geq 0,\beta\geq 0\\ \max(|1-2\beta-4\gamma|,1-2\alpha) & \text{if} \quad \gamma\geq 0,\alpha\leq 0,\beta\geq 0\\ \max(|1-2\alpha-4\gamma|,1-2\beta) & \text{if} \quad \gamma\geq 0,\alpha\geq 0,\beta\leq 0\\ \max(|1-4\gamma|,1-2\alpha-2\beta) & \text{if} \quad \gamma\geq 0,\alpha\leq 0,\beta\leq 0\\ \max(|1-2\alpha-2\beta|,1-4\gamma) & \text{if} \quad \gamma\leq 0,\alpha\geq 0,\beta\geq 0\\ \max(|1-2\beta|,1-2\alpha-4\gamma) & \text{if} \quad \gamma\leq 0,\alpha\leq 0,\beta\geq 0\\ \max(|1-2\alpha|,1-2\beta-4\gamma) & \text{if} \quad \gamma\leq 0,\alpha\geq 0,\beta\leq 0\end{cases} \tag{7}$$

and

$$\Xi=\max(|1-2\alpha|,1)\max(|1-2\beta|,1)+4|1-\alpha||1-\beta|. \tag{8}$$

Then, for

$$| \lambda | \leq \frac{1}{\min(\Omega, \Xi)}, \tag{9}$$

$C(x, y, z)$ is a valid 3D copula.

Proof. The proof consists in demonstrating that $C(x, y, z)$ satisfies the items (I), (II), and (III) of Definition 1 with $n = 3$.

(I) For any $(x, y) \in [0,1]^2$, it is clear that

$$C(x, y, 0) = xy \times 0 \times \left\{1 + \lambda\left[1 - (\alpha x + \beta y + \gamma xy)\right](1 - 0)\right\} = 0.$$

Using a similar development, for any $(y, z) \in [0,1]^2$, we obtain $C(0, y, z) = 0$, and for any $(x, z) \in [0,1]^2$, we have $C(x, 0, z) = 0$. The item (I)) is proved.

(II) For any $x \in [0,1]$, we have

$$C(x,1,1) = x \times 1 \times 1 \times \left\{1 + \lambda\left[1 - (\alpha x + \beta \times 1 + \gamma x \times 1)\right](1 - 1)\right\} = x.$$

Similarly, for any $y \in [0,1]$, we obtain

$$C(1, y,1) = 1 \times y \times 1 \times \left\{1 + \lambda\left[1 - (\alpha \times 1 + \beta y + \gamma \times 1 \times y)\right](1 - 1)\right\} = y$$

and for any $z \in [0,1]$, since $\alpha + \beta + \gamma = 1$, we obtain

$$\begin{aligned} C(1,1,z) &= 1 \times 1 \times z\left\{1 + \lambda\left[1 - (\alpha \times 1 + \beta \times 1 + \gamma \times 1 \times 1)\right](1 - z)\right\} \\ &= z\left\{1 + \lambda\left[1 - (\alpha + \beta + \gamma)\right](1 - z)\right\} = z. \end{aligned}$$

The item (II) is proved.

(III) By a differentiation with respect to x, y, and z, we obtain

$$\partial_{x,y,z} C(x, y, z) = 1 - \lambda(2z - 1)F(x, y),$$

where

$$F(x, y) = 1 - 2\alpha x - 2\beta y - 4\gamma xy. \tag{10}$$

It is clear that, for any $z \in [0,1]$, we have $2z - 1 \in [-1,1]$, so, with the use of absolute values, we have

$$\partial_{x,y,z} C(x, y, z) \geq 1 - | \lambda || 2z - 1 || F(x, y) | \geq 1 - | \lambda || F(x, y) |.$$

Let us now find some sharp and tractable upper bounds for

As a first approach, let us remark that, for any $(x, y) \in [0,1]^2$ and $\gamma \geq 0$,

- $\alpha \geq 0$ and $\beta \geq 0$, we have $1 - 2\alpha - 2\beta - 4\gamma \leq F(x, y) \leq 1$, so

$$| F(x, y) | \leq \max(| 1 - 2\alpha - 2\beta - 4\gamma |, 1).$$

- $\alpha \leq 0$ and $\beta \geq 0$, we have $1 - 2\beta - 4\gamma \leq F(x, y) \leq 1 - 2\alpha$, so

$$| F(x, y) | \leq \max(| 1 - 2\beta - 4\gamma |, 1 - 2\alpha).$$

- $\alpha \geq 0$ and $\beta \leq 0$, we have $1-2\alpha-4\gamma \leq F(x,y) \leq 1-2\beta$, so

$$|F(x,y)| \leq \max(|1-2\alpha-4\gamma|, 1-2\beta).$$

- $\alpha \leq 0$ and $\beta \leq 0$, we have $1-4\gamma \leq F(x,y) \leq 1-2\alpha-2\beta$, so

$$|F(x,y)| \leq \max(|1-4\gamma|, 1-2\alpha-2\beta).$$

On the other hand, for any $(x,y) \in [0,1]^2$ and $\gamma \leq 0$,

- $\alpha \geq 0$ and $\beta \geq 0$, we have $1-2\alpha-2\beta \leq F(x,y) \leq 1-4\gamma$, so

$$|F(x,y)| \leq \max(|1-2\alpha-2\beta|, 1-4\gamma).$$

- $\alpha \leq 0$ and $\beta \geq 0$, we have $1-2\beta \leq F(x,y) \leq 1-2\alpha-4\gamma$, so

$$|F(x,y)| \leq \max(|1-2\beta|, 1-2\alpha-4\gamma).$$

- $\alpha \geq 0$ and $\beta \leq 0$, we have $1-2\alpha \leq F(x,y) \leq 1-2\beta-4\gamma$, so

$$|F(x,y)| \leq \max(|1-2\alpha|, 1-2\beta-4\gamma).$$

Hence, we get

$$|F(x,y)| \leq \Omega,$$

where Ω is defined in equation (7).

Now, as an alternative approach, let us remark that we can write

$$F(x,y) = (1-2\alpha x)(1-2\beta y) - 4xy(1-\alpha)(1-\beta).$$

It is clear that, for any $(x,y) \in [0,1]^2$, we have $xy \in [0,1], 1-2\alpha x \leq \max(|1-2\alpha|, 1)$, and $1-2\beta y \leq \max(|1-2\beta|, 1)$. With the use of absolute values, this implies that

$$\begin{aligned} |F(x,y)| &\leq |1-2\alpha x||1-2\beta y| + 4xy|1-\alpha||1-\beta| \\ &\leq \max(|1-2\alpha|, 1)\max(|1-2\beta|, 1) + 4|1-\alpha||1-\beta|. \end{aligned}$$

Hence, we get

$$|F(x,y)| \leq \Xi,$$

where Ξ is defined in equation (8).

The obtained upper bounds for $|F(x,y)|$ imply that $|F(x,y)| \leq \min(\Omega, \Xi)$. Therefore, since $|\lambda| \leq 1/\min(\Omega, \Xi)$, we have

$$\partial_{x,y,z} C(x,y,z) \geq 1 - |\lambda| \min(\Omega, \Xi) \geq 0.$$

The item (III) is proved.

The items (I), (II), and (III) of Definition 1 with $n = 3$ are established; $C(x, y, z)$ constitutes a valid 3D copula.

In Proposition 3.1, it is important to note that α, β, or γ can be negative (but not all three simultaneously), which can make the situation more complex than it seems at a first glance.

Similarly to the copulas in equations (1) and (2), the copula in equation (6) is based on a perturbation of the 3D independence copula $\Pi(x, y, z)$, with the perturbed function defined as $R(x, y, z) = \lambda xyz[1-(\alpha x+\beta y+\gamma xy)](1-z)$. This last function has the features of being non-separable with respect to x, y, and z and of depending on three tuning parameters. In the functional sense, it seems to have no relation between the copulas in equations (1), (2), and (6). Some basic properties of the proposed copula are described below.

When $\alpha = 1$, $\beta = 1$, and $\gamma = -1$, after some algebra, we remark that the copula in equation (6) can be expressed as

$$C(x, y, z)) = xyz\left\{1+\lambda(1-x)(1-y)(1-z)\right\}, \quad (x, y, z) \in [0,1]^3,$$

with the condition $|\lambda| \leq 1$, which corresponds to the standard 3D FGM copula. It is thus exchangeable. For all the other admissible values of α, β, and γ, and for $\lambda \neq 0$, it is obviously not exchangeable: there exist infinite examples of triplet values (x_*, y_*, z_*) such that $C(x_*, y_*, z_*) \neq C(x_*, z_*, y_*)$, for instance. Furthermore, for any permutation (x_0, y_0, z_0) of $(x, y, z), C(x_0, y_0, z_0)$ is a copula. With this method, three different copulas can be derived from $C(x, y, z)$ (see Table 2).

The corresponding copula density is given by

$$\begin{aligned} c(x, y, z) &= \partial_{x,y,z} C(x, y, z) \\ &= 1-\lambda(2z-1)\left[(1-2\alpha x)(1-2\beta y)-4xy(1-\alpha)(1-\beta)\right], \quad (x, y, z) \in [0,1]^3. \end{aligned}$$

The following 2D copulas can be derived from $C(x, y, z)$:

$$C_{\dagger}(x, y) = C(x, y, 1) = xy,$$

$$C_{\dagger\dagger}(x, y) = C(1, x, y) = xy\left\{1+\lambda\left[1-(\alpha+(1-\alpha)x)\right](1-y)\right\}$$

and

$$C_{\dagger\dagger\dagger}(x, y) = C(x, 1, y) = xy\left\{1+\lambda\left[1-(\beta+(1-\beta)x)\right](1-y)\right\}.$$

For the first one, we recognize the 2D independence copula. The others can be apparent as modified (non-diagonally symmetric) versions of the 2D FGM copula. The copula $C_{\dagger\dagger}(x, y)$ is defined with $|\lambda| \leq 1/|1-\alpha|$ for $\alpha \neq 1$, and the copula $C_{\dagger\dagger\dagger}(x, y)$ is defined with $|\lambda| \leq 1/|1-\beta|$ for $\beta \neq 1$.

The corresponding survival copula is expressed as

$$\begin{aligned} \tilde{C}(x, y, z) &= x+y+z-2+C(1-x, 1-y, 1)+C(1-x, 1, 1-z) \\ &\quad + C(1, 1-y, 1-z)-C(1-x, 1-y, 1-z) \\ &= x+y+z-2+(1-x)(1-y)+(1-x)(1-z)\left\{1+\lambda\left[1-(\beta+(1-\beta)(1-x))\right]z\right\} \\ &\quad +(1-y)(1-z)\left\{1+\lambda\left[1-(\alpha+(1-\alpha)(1-y))\right]z\right\} \\ &\quad -(1-x)(1-y)(1-z)\left\{1+\lambda\left[1-(\alpha(1-x)+\beta(1-y)+\gamma(1-x)(1-y))\right]z\right\} \\ &= xyz+\lambda z(1-z)\left\{(1-x)\left[1-(\beta+(1-\beta)(1-x))\right]+(1-y)\left[1-(\alpha+(1-\alpha)(1-y))\right]\right. \\ &\quad \left. -(1-x)(1-y)\left[1-(\alpha(1-x)+\beta(1-y)+\gamma(1-x)(1-y))\right]\right\}. \end{aligned}$$

To the best of our knowledge, this survival copula is a new three-parameter, 3D, mainly non-exchangeable copula in the literature. From a functional standpoint, it remains more complicated than its parent copula.

The Fréchet-Hoeffding bounds hold, i.e., or any $(x, y, z) \in [0,1]^3$, we have

$$\max\left(x+y+z-2,0\right) \le C(x,y,z) \le \min(x,y,z),$$

that is,

$$\max\left(x+y+z-2,0\right) \le xyz\left\{1+\lambda\left[1-(\alpha x+\beta y+\gamma xy)\right](1-z)\right\} \le \min(x,y,z),$$

(it is understood in the context of the condition in equation (9)). These 3D inequalities can be used as novel multivariate analytic tools and are potentially of independent interest. The 3D medial correlation follows immediately; it is given by

$$\mathcal{M} = \frac{1}{3}\left[8C\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)-1\right] = \frac{\lambda}{24}(3-\alpha-\beta).$$

This expression is quite manageable, and M can be negative or positive. After several integrations, the 3D Spearman's rho is given by

$$\begin{aligned}\rho &= 8\int_0^1\int_0^1\int_0^1 C(x,y,z)dxdydz-1\\ &= 8\lambda\int_0^1\int_0^1\int_0^1 xy\left[1-(\alpha x+\beta y+\gamma xy)\right]z(1-z)dxdydz\\ &= \frac{\lambda}{27}(5-2\alpha-2\beta).\end{aligned}$$

Thus, the 3D Spearman's rho has a very simple form, which is ideal for further correlation analysis. Furthermore, it can be positive or negative.

Like for the first introduced copula, by considering three unidimensional distribution functions, say $F(x)$, $G(x)$, and $H(x)$, the following 3D function defines a valid 3D distribution function:

$$K(x,y,z) = C[F(x),G(y),H(z)], \quad (x,y,z)\in\mathbb{R}^3.$$

Owing to the originality of $C(x, y, z)$ a plethora of 3D distributions can be introduced. Table 2 presents some examples of simple copulas derived from the copula in equation (6).

Table 2. Some examples of copulas derived to the copula in equation (6) (with $(\alpha, \beta) \in \mathbb{R}^2$ and $\gamma = 1 - (\alpha + \beta)$)

New general copulas	λ values
$xyz\{1+\lambda[1-(\alpha x+\beta y+\gamma xy)](1-z)\}$	Equation (9)
$xyz\{1+\lambda[1-(\alpha x+\beta z+\gamma xz)](1-y)\}$	Equation (9)
$xyz\{1+\lambda[1-(\alpha z+\beta y+\gamma zy)](1-x)\}$	Equation (9)
Examples of new simple copulas	**λ values**
$xyz\{1+\lambda(1-x)(1-y)(1-z)\}$ (FGM)	$\lvert\lambda\rvert \leq 1$
$xyz\left\{1+\lambda\left[1-\left(\frac{x}{2}+\frac{y}{2}\right)\right](1-z)\right\}$	$\lvert\lambda\rvert \leq 1$
$xyz\left\{1+\lambda\left[1-\left(2xy-\frac{x}{2}-\frac{y}{2}\right)\right](1-z)\right\}$	$\lvert\lambda\rvert \leq \frac{1}{7}$
$xyz[1+\lambda(1-xy)(1-z)]$	$\lvert\lambda\rvert \leq \frac{1}{3}$
$xyz\left\{1+\lambda\left[1-\frac{y}{3}(1+2x)\right](1-z)\right\}$	$\lvert\lambda\rvert \leq \frac{3}{7}$
$xyz\{1+\lambda[1-(5x+2y-6xy)](1-z)\}$	$\lvert\lambda\rvert \leq \frac{1}{25}$

To the best of our knowledge, except for the 3D FGM copula, all the copulas presented in Table 2 are new in the literature and can be used in a theoretical way or for fresh 3D analysis in various statistical scenarios.

Remark 3.2. The condition in equation (9) has the advantage of being manageable and taking into account all the possible values for α and β (positive or negative), but it is not optimal in the mathematical sense. In some cases, it can be refined, but only with a more complex expression involving the precise study of $F(x, y)$ in equation (10).

Remark 3.3. A generalized form for the copula in equation (6) may be as follows:

$$C(x,y,z) = xyz\{1+\lambda\phi[1-(\alpha x+\beta y+\gamma xy)]\psi(z)\}, \quad (x,y,z)\in[0,1]^3,$$

where φ and ψ denote continuous functions satisfying $\varphi(0) = 0$ and $\psi(1) = 0$, in order to satisfy the items (I) and (II) of Definition 1 with $n = 3$. However, additional conditions on φ and ψ are necessary to satisfy (III), which remains a mathematical challenge that we leave for future work.

4. Discussion and perspectives

Original functional constructions of copulas in high dimensions are still relatively rare. This paper contributed to the topic by proposing two new and simple 3D copulas that have the following characteristics:

(a) They are only defined with polynomial-power functions and depend on three tuning parameters, making them manageable and adaptable.

(b) They are based on the perturbation of the 3D independence copula with original non-separable perturbation functions.

(c) Wide ranges for the admissible values of the parameters are identified.

(d) They are mainly not exchangeable, which, in addition to their simplicity, remains a demanded property.

(e) One of them generalizes the 3D FGM copula and produces various generalizations of the 2D FGM copula, which has been involved in many important applications.
(f) The medial correlation and Spearman's rho have closed-form expressions.

We provided the main theoretical ingredients to make them operational for further purposes. In particular, they can be used as 3D dependence models for concrete data analyses, as in those performed in [9-13]. This applied aspect necessitates a high level of expertise, which we leave to future research. On the theoretical side, based on the suggested constructions, the development of comparable 3D copulas with richer perturbation functions, such as trigonometric or ratio functions, continues to be an interesting area of research.

Acknowledgments

The author expresses gratitude to the three reviewers, the associate editor, and the editor for their constructive feedback on the article.

Conflict of interest

There are no conflicts of interest in this study.

References

[1] Sklar A. Fonctions de r\'epartition à n dimensions et leurs marges. *Publications de l'Institut Statistique de l'Université de Paris*. 1959; 8: 229-231.
[2] Sklar A. Random variables, joint distribution functions, and copulas. *Kybernetika*. 1973; 9: 449-460. Available from: https://www.kybernetika.cz/content/1973/6/449/paper.pdf.
[3] Nelsen RB. *An introduction to copulas*. 2nd ed. New York: Springer; 2006. Available from: https://doi.org/10.1007/0-387-28678-0.
[4] Joe H. *Dependence modeling with copulas*. Boca Raton: CRC Press; 2015.
[5] Durante F, Sempi C. *Principles of copula theory*. Boca Raton: CRC Press; 2016.
[6] Nadarajah S, Afuecheta E, Chan S. A compendium of copulas. *Statistica*. 2017; 77(4): 279-328. Available from: https://doi.org/10.6092/issn.1973-2201/7202.
[7] Bedford T, Cooke RM. Probability density decomposition for conditionally dependent random variables modeled by Vines. *Annals of Mathematics and Artificial Intelligence*. 2001; 32: 245-268. Available from: https://doi.org/10.1023/A:1016725902970.
[8] Bedford T, Cooke RM. Vines - A new graphical model for dependent random variables. *Annals of Statistics*. 2002; 30(4): 1031-1068. Available from: https://doi.org/10.1214/aos/1031689016.
[9] Jadhav S, Daruwala R. Development and analysis of 3D-copula model for statistical dependencies in wireless sensor networks. In: *2016 IEEE Region 10 Conference (TENCON)*. Singapore: IEEE; 2016. p.1356-1360. Available from: https://doi.org/10.1109/TENCON.2016.7848235.
[10] Bevacqua E, Maraun D, Haff IH, Widmann M, Vrac M. Multivariate statistical modelling of compound events via pair-copula constructions: Analysis of floods. *Hydrology and Earth System Sciences*. 2017: 21(6): 2701-2723. Available from: https://doi.org/10.5194/hess-21-2701-2017.
[11] Hou W, Yan P, Feng G, Zuo D. A 3D copula method for the impact and risk assessment of drought disaster and an example application. *Frontiers in Physics*. 2021; 9: 656253. Available from: https://doi.org/10.3389/fphy.2021.656253.
[12] Liu J, Sriboonchitta S, Wiboonpongse A, Denoeux T. A trivariate Gaussian copula stochastic frontier model with sample selection. *International Journal of Approximate Reasoning*. 2021; 137: 181-198. Available from: https://doi.org/10.1016/j.ijar.2021.06.016.

[13] Orcel O, Sergent P, Ropert F. Trivariate copula to design coastal structures. *Natural Hazards and Earth System Sciences*. 2020; 21(1): 239-260. Available from: https://doi.org/10.5194/nhess-21-239-2021.
[14] Liebscher E. Construction of asymmetric multivariate copulas. *Journal of Multivariate Analysis.* 2008; 99(10): 2234-2250. Available from: https://doi.org/10.1016/j.jmva.2008.02.025.
[15] Zimmer DM, Trivedi PK. Using trivariate copulas to model sample selection and treatment effects: Application to family health care demand. *Journal of Business & Economic Statistics*. 2006; 24(1): 63-76.
[16] Úbeda-Flores M. A method for constructing trivariate distributions with given bivariate margins. *Far East Journal of Theoretical Statistics*. 2005; 15(1): 115-120.
[17] Cerqueti R, Lupi C. Non-exchangeable copulas and multivariate total positivity. *Information Sciences*. 2016: 360: 163-169. Available from: https://doi.org/10.1016/j.ins.2016.04.026.
[18] Ignazzi C, Durante F. On a new class of trivariate copulas. *Joint Proceedings of the 19th World Congress of the International Fuzzy Systems Association (IFSA), the 12th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT), and the 11th International Summer School on Aggregation Operators (AGOP).* Netherlands: Atlantis Press; 2021. p.654-660. Available from: https://doi.org/10.2991/asum.k.210827.088.
[19] Huang JS, Kotz S. Modifications of the Farlie-Gumbel-Morgenstern distributions. A tough hill to climb. *Metrika*. 1999; 49(2): 135-145. Available from: https://doi.org/10.1007/s001840050030.
[20] Schmid F, Schmidt R. Multivariate conditional versions of Spearman's rho and related measures of tail dependence. *Journal of Multivariate Analysis*. 2007; 98(6): 1123-1140. Available from: https://doi.org/10.1016/j.jmva.2006.05.005.
[21] Chesneau C. On new types of multivariate trigonometric copulas. *AppliedMath*. 2021; 1(1): 3-17. Available from: https://doi.org/10.3390/appliedmath1010002.
[22] Taketomi N, Yamamoto K, Chesneau C, Emura T. Parametric distributions for survival and reliability analyses, a review and historical sketch. *Mathematics*. 2022; 10(20): 3907. Available from: https://doi.org/10.3390/math10203907.
[23] Genest C, Ghoudi K, Rivest L-P. A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*. 1995; 82(3): 543-552. Available from: https://doi.org/10.1093/biomet/82.3.543.
[24] Silvapulle P, Kim G, Silvapulle MJ. Robustness of a semiparametric estimator of a copula. *Econometric Society 2004 Australasian Meeting*. 2004; 317.