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Research Article

Numerical Investigation of Fractional Kawahara Equation via Haar Scale Wavelet Method

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Abstract: The Kawahara equation is a fifth-order dispersive equation that plays a significant role in explaining the creation of non-linear water waves in the long-wavelength region. In this research, the Kawahara equation is solved numerically using the novel Haar scale-3 wavelet method in conjunction with the collocation method. The quasilinearisation approach and the Caputo derivative are used to characterise the non-linearity and fractional behaviour of the equation, respectively. To verify that the findings obtained are legitimate, residual and error estimates are generated. A thorough comparison is made between the present solutions and the numerical findings that have already been published in the literature, which demonstrates the advantages and effectiveness of the suggested technique. The Haar wavelet method reveals a dynamic system of alternative solutions for a wide variety of physical parameters.

***Keywords*:** Kawahara equation, dispersive equation, Haar scale-3 wavelet, fractional differential equations (FDEs)

MSC: 35Q35, 35Q53

1. Introduction

Over the past few years, various scientific and technological fields have become interested in fractional analysis. Due to its extraordinary capacity to carry out integration and differentiation operations on equations of any arbitrary order, it has found use in a variety of biological and physical science fields, including electromagnetic theory, fluid mechanics, electrical networks, and other physical sciences. Respective studies found that it is beneficial to explain these phenomena using the concept of fractional-order derivatives when dealing with practical problems. It was discovered during the investigation that the concept of derivatives having non-integer order has not only been widely used. But it has also been crucial in many scientific domains.

For solving differential equations (DEs) of non-integer order, various techniques have been employed. For simulating impulsive DEs of non-integer order, a generalised differential transform technique has been used [1]. Further, the authors suggested generalised versions of the three well-known fractional numerical methods, Euler, Runge-Kutta 2-step, and Runge-Kutta 4-step, to find the fractional Caputo derivatives [2]. In addition, a new numerical approach has been discussed for finding the solution of Caputo-Fabrizio derivative-containing fractional differential equations (FDEs) [3]. In their research, the authors proposed two arrays that contain the coefficients of the fractional Adams-Bashforth and Adams-Moulton as well as recursive relations to create the array elements method. By applying these arrays to suitable examples, the higher-order fractional linear multi-step methods in general form with expanded stability regions

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for solving FDEs could be constructed [4]. The authors recently developed a novel version of the L1-predictor-corrector (L1-PC) approach to solve multiple delay-type FDEs utilising various methods [5].

Numerical techniques based on wavelets are used to solve the system of equations quickly and efficiently. Wavelets have been utilised to solve partial differential equations (PDEs) for the past twenty years. The collocation method is the foundation of the wavelet algorithms used to solve PDEs. The Haar wavelets are the simplest orthonormal wavelets with compact support among them since they only include piecewise constant functions.

Dispersive wave equations are essential in practical physics and mathematics. The Kawahara equation, which plays a significant role in explaining the creation of non-linear water waves in the long-wavelength region, is the most exciting since it has weaker dispersion but large non-linearity. The Kawahara have been the focus of in-depth study for many years. In order to depict solitary-wave propagation in media, Kawahara first proposed the Kawahara equations in 1972. Both plasma magneto-acoustic waves and the theory of shallow water waves with surface tension contain them. The modified Kawahara equation is also widely applicable to plasma waves, capillary-gravity water waves, and other phenomena.

Numerous numerical and analytical techniques exist, such as the Mittag-Leffler law [6], the kernel method [7], the discontinuous Galerkin method [8], the Adomian decomposition method (ADM) [9, 10], the natural transform decomposition method [11], travelling wave solutions [12], the semi-analytical method [13], the B-spline method [14], the homotopy analysis method [15], and the cubic B-spline differential quadrature method [16] are used to discuss Kawahara and fractional Kawahara equations. But, it is not yet discussed by any wavelet method. In the previous study, Haar wavelet method (HWM) was found to be the simplest approach in solving non-linear higher-order FDEs. Apart from that, many other different equations have been solved by this method including fractional models [17], dispersive equations [18, 19], fractional advection dispersion equations [20], fractional integrodifferential equations [21, 22], integrodifferential equations [23], FDEs with singularity [24], fractional delay equation [25] and convection radiation equation [26]. The obtained solution to these equations is compatible as a comparison to other methods available in the literature. The application of the wavelet method for solving such non-linear time-fractional higher-order equations has motivated us to study the behaviour of this equation using HWM.

The motive of this article is to evaluate the numerical solution and success of applying the HWM to time fractional Kawahara, which are accordingly stated below as:

$$D_t^{\mu}\varphi(z,t)+m\varphi_z+\varphi_{zzz}-\varphi_{zzzzz}=f(z,t)$$
$$\varphi(z,0)=\varphi_0(z)$$

Here, the value of μ, which lies between 0 to 1 describes the fractional derivative of t.

The following sections of the manuscript contain: Section 2 offers the basic definitions of fractional calculus. In Section 3, it was briefly covered how to find the integrals of the explicit forms of Haar scale-3 wavelets and their families. The quasilinearisation technique to solve a non-linear term in a differential equation is described in Section 4. In Section 5, space and time approximations are done by the proposed technique. In Section 6, the performance and effectiveness of the current approach are assessed by solving two different models of FDEs. In the last section, the findings drawn from the data as well as suggestions for additional research have been discussed.

2. Basic definition of fractional calculus

In this section, we discussed the basic definitions of fractional differentiation and integration.

- Reimann-Liouville fractional differential operator of order α: for the positive real numbers, μ, t across the interval $[m, n]$, the fractional differential operator established by the mathematician Riemann-Liouville is given by [27]:

$$d^{\mu}f(t)=\frac{1}{\Gamma(p-\mu)}\left[\frac{d}{dt}\right]^{p}\int_{m}^{n}f(z)(t-z)^{p-\mu-1}dz$$

where μ denotes the order of derivative and $t \in [m, n]$.

- Caputo fractional differential operator of order α: for positive real numbers, μ, t, the fractional differential operator developed by the Italian mathematician Caputo is [27]:

$$d^{\mu} f(t) = \frac{1}{\Gamma(p-\mu)} \int_{m}^{n} \left[\frac{d}{dt}\right]^{p} f(z)(t-z)^{p-\mu-1} dz$$

where μ denotes the order of derivative and $t \in [m, n]$.

3. Basic structure of Haar wavelets

For the scale-2 wavelet family, detailed equations for scale-2 wavelet function, a father wavelet, a mother wavelet are provided below.

3.1 *Scale-2 Haar wavelet*

A group of square waves written in the form of set of Haar functions is defined as:

$$h_i(t) = \phi(t) = \begin{cases} 1, & \frac{\kappa}{m} \le t < \frac{\kappa + 0.5}{m} \\ -1, & \frac{\kappa + 0.5}{m} \le t < \frac{\kappa + 1}{m} \\ 0, & \text{otherwise.} \end{cases}$$

Here, the value of m represents level of wavelets in terms of 2^j, κ is a translation parameter and has j as its maximum number of revolution, here, i represents the wavelet number which can be expressed as $i - 1 = \kappa + m$. If we take $m = 1$, $\kappa = 0$, then the value of $i = 2$ and by further varying the value of k and m, different values of i can be obtained.

3.2 *Scale-3 Haar wavelet*

For the scale-3 wavelet family, detailed equations for scale-3 wavelet function, a father wavelet, two symmetric and antisymmetric mother wavelets are provided below.

$$h_i(t) = \phi(t) = \begin{cases} 1, & \propto_1 \le t < \propto_2 \\ 0, & \text{otherwise} \end{cases} \quad \text{for } i = 1.$$

$$h_i(t) = \varphi^1(3^m - k) = \frac{1}{\sqrt{2}} \begin{cases} -1, & \mu_1(i) \le t < \mu_2(i) \\ 2, & \mu_2(i) \le t < \mu_3(i) \\ -1, & \mu_3(i) \le t < \mu_4(i) \\ 0, & \text{otherwise} \end{cases} \quad \text{for } i = 2, 4, \ldots . 3p - 1.$$

$$h_i(t) = \varphi^2(3^m - k) = \sqrt{\frac{3}{2}} \begin{cases} 1, & \mu_1(i) \le t < \mu_2(i) \\ 0, & \mu_2(i) \le t < \mu_3(i) \\ -1, & \mu_3(i) \le t < \mu_4(i) \\ 0, & \text{otherwise} \end{cases} \quad \text{for } i = 3, 6, \ldots . 3p.$$

Here, $\mu_1(i) = \frac{\kappa}{p}$, $\mu_2(i) = \frac{3\kappa + 1}{3p}$, $\mu_3(i) = \frac{3k + 2}{3p}$, $\mu_4(i) = \frac{k + 1}{p}$. Here, the value of p represents level of wavelets in terms of 3^j, κ is a translation parameter and has j as its maximum number of revolution, here, i represents the wavelet number which can be expressed as $i - 1 = \kappa + m$. If we take $m = 1$, $\kappa = 0$, then the value of $i = 2$ and by further varying the value of k and m, different values of i can be obtained.

The main difference between Haar scale-2 and scale-3 wavelets is that in the former, the entire wavelet family is constructed by a single mother wavelet, whereas in the latter, for the construction of the entire wavelet family, two mother wavelets with different shapes are responsible. Because of this, Haar scale-3 wavelets improved the solution's convergence rate.

4. Quasilinearisation technique

Basically, quasilinearisation technique is generalised form of Newton-Raphson technique. It converges quadratically to an exact solution, if there is a convergence, then it must be a monotone convergence.

Let us consider the non-linear second-order differential equation:

$$\omega'' = k(\omega(z), z)$$

with boundary condition:

$$\omega(a_1) = \theta_1,\ \omega(b_1) = \theta_2; \quad a_1 \le z \le b_1$$

here, k is in terms of $\omega(z)$. Let us choose an approximation at the initial step of solution $\omega(z)$. Let us say $\omega_0(z)$. k can be expanded around $\omega_0(z)$ is written in the form:

$$k(\omega(z), z) = k(\omega_0(z), z) + (\omega(z) - \omega_0(z))k_{\omega_0(x)}(\omega_0(z), z)$$
$$\omega''(z) = k(\omega_0(z), z) + (\omega(z) - \omega_0(z))k_{\omega_0(z)}(\omega_0(z), z)$$
$$\omega''(z) = k(\omega_1(z), z) + (\omega(z) - \omega_1(z))k_{\omega_1(z)}(\omega_1(z), z).$$

The form of a recurrence relationship is

$$\omega''_{s+1}(z) = k(\omega_s(z), z) + (\omega_{s+1}(z) - \omega_s(z))k_{\omega_s(z)}(\omega_s(z), z).$$

$\omega_s(z)$ is known and can be used for obtaining $\omega_{s+1}(z)$. The above equation is always a non-linear DEs and the condition is $\omega_{s+1}(z) = \alpha$, $\omega_s(z) = \beta$.

Now, consider the non-linear second order differential equation of the form

$$\omega''(z) = k(\omega'(z), \omega(z), z).$$

Here, the first derivative $\omega'(\omega)$ can be considered as another function.

$$\omega''_{s+1}(z) = k(\omega'(z), \omega(z), z) + (\omega'_{s+1}(z) - \omega'_s(z))k_{\omega'_s(z)}(\omega'_s(z), \omega_s(z), z)$$
$$+(\omega_{s+1}(z) - \omega_s(z))\ \mathrm{k}\ (\omega'_s(z), \omega_s(z), z)$$

with boundary condition $\omega_{s+1}(z) = \alpha$, $\omega_s(z) = \beta$.

Follow the same technique to establish the recurrence relation for higher-order non-linear DEs.

$$L^j\omega_{s+1}(z) = k(\omega_s(z), \omega'_s(z) \ldots\ldots \omega_s^{j-1}(z), z) + \sum_{m=0}^{n-1}(\omega_{s+1}^m(z) - \omega_s^m(z))k_{\omega^m}(\omega'_s(z), \omega_s(z), \ldots., \omega_s^{j-1}(z), z).$$

The order of the DEs is j; the above equation is linear, and it can be solved recursively for $\omega_s(z)$ if it has a known value and can be used to get the value of $\omega_{s+1}(z)$.

5. Approximation of fractional Kawahara equation by scale-3 Haar wavelet

$$D_t^{\mu}\varphi(z,t)+m\varphi_z+\varphi_{zzz}-\varphi_{zzzzz}=F(z,t)$$
$$\varphi(z,0)=\varphi_0(z).$$

Approximate the higher order derivative:

$$\varphi_{tzzzzz}(z,t)=\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{iz}H_i(z)H_l(t).$$

Integrating the equation with respect to (w.r.t) t between 0 to t,

$$\varphi_{zzzzz}(z,t)-\varphi_{zzzzz}(z,0)=\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{iz}H_i(z)P_{1,l}(t)$$

$$\varphi_{zzzzz}(z,t)=\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{iz}H_i(z)P_{1,l}(t)+\varphi_{zzzzz}(z,0).$$

Integrating w.r.t z between 0 to z,

$$\varphi_{zzzz}(z,t)-\varphi_{zzzz}(0,t)=\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{iz}P_{1,i}(z)P_{1,l}(t)+\left[\varphi_{zzzz}(z,0)-\varphi_{zzzz}(0,0)\right]$$

$$\varphi_{zzzz}(z,t)=\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{iz}P_{1,i}(z)P_{1,l}(t)+\left[\varphi_{zzzz}(z,0)-\varphi_{zzzz}(0,0)\right]+\varphi_{zzzz}(0,t).$$

Again, integrating w.r.t z,

$$\varphi_{zzz}(z,t)-\varphi_{zzz}(0,t)=\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{iz}P_{2,i}(z)P_{1,l}(t)+\left[\varphi_{zzz}(z,0)-\varphi_{zzz}(0,0)\right]+z\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]$$

$$\varphi_{zzz}(z,t)=\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{iz}P_{2,i}(z)P_{1,l}(t)+\left[\varphi_{zzz}(z,0)-\varphi_{zzz}(0,0)\right]+z\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]+\varphi_{zzz}(0,t).$$

Again, integrating w.r.t z,

$$\begin{aligned}\varphi_{zz}(z,t)-\varphi_{zz}(0,t)=&\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{iz}P_{3,i}(z)P_{1,l}(t)+\left[\varphi_{zz}(z,0)-\varphi_{zz}(0,0)\right]\\&+\frac{z^2}{2}\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]+z[\varphi_{zzz}(0,t)-\varphi_{zzz}(0,0)]\end{aligned}$$

$$\begin{aligned}\varphi_{zz}(z,t)=&\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{iz}P_{3,i}(z)P_{1,l}(t)+\left[\varphi_{zz}(z,0)-\varphi_{zz}(0,0)\right]\\&+\frac{z^2}{2}\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]+z[\varphi_{zzz}(0,t)-\varphi_{zzz}(0,0)]+\varphi_{zz}(0,t).\end{aligned}$$

Again, integrating w.r.t z,

$$\varphi_z(z,t)-\varphi_z(0,t)=\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{iz}P_{4,i}(z)P_{1,l}(t)+\left[\varphi_z(z,0)-\varphi_z(0,0)\right]$$
$$+\frac{z^3}{6}\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]+\frac{z^2}{2}[\varphi_{zzz}(0,t)-\varphi_{zzz}(0,0)\left]+z[\varphi_{zz}(0,t)-\varphi_{zz}(0,0)\right]$$

$$\varphi_z(z,t)=\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{iz}P_{4,i}(z)P_{1,l}(t)+\left[\varphi_z(z,0)-\varphi_z(0,0)\right]$$
$$+\frac{z^3}{6}\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]+\frac{z^2}{2}[\varphi_{zzz}(0,t)-\varphi_{zzz}(0,0)\left]+z[\varphi_{zz}(0,t)-\varphi_{zz}(0,0)\right]+\varphi_z(0,t)$$

$$\varphi(z,t)-\varphi(0,t)=\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{iz}P_{5,i}(z)P_{1,l}(t)+\left[\varphi(z,0)-\varphi(0,0)\right]+\frac{z^4}{24}\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]$$
$$+\frac{z^3}{6}[\varphi_{zzz}(0,t)-\varphi_{zzz}(0,0)]+\frac{z^2}{2}[\varphi_{zz}(0,t)-\varphi_{zz}(0,0)\left]+z[\varphi_z(0,t)-\varphi_z(0,0)\right]$$

$$\varphi(z,t)=\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{iz}P_{5,i}(z)P_{1,l}(t)+\left[\varphi(z,0)-\varphi(0,0)\right]+\frac{z^4}{24}\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]$$
$$+\frac{z^3}{6}[\varphi_{zzz}(0,t)-\varphi_{zzz}(0,0)]+\frac{z^2}{2}[\varphi_{zz}(0,t)-\varphi_{zz}(0,0)\left]+z[\varphi_z(0,t)-\varphi_z(0,0)\right]+\varphi(0,t)$$

$$\varphi_t(z,t)=\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{iz}P_{5,i}(z)H_l(t)+\left[\varphi(z,0)-\varphi(0,0)\right]_t+\frac{z^4}{24}\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]_t$$
$$+\frac{z^3}{6}[\varphi_{zzz}(0,t)-\varphi_{zzz}(0,0)]_t+\frac{z^2}{2}[\varphi_{zz}(0,t)-\varphi_{zz}(0,0)]_t+z[\varphi_z(0,t)-\varphi_z(0,0)]_t+\left[\varphi(0,t)\right]_t$$

$$\varphi_t(z,t)+m\varphi_z+\varphi_{zzz}-\varphi_{zzzzz}=F(z,t)$$

$$\sum_{i=1}^{3p}\sum_{i=1}^{3p}a_{iz}P_{5,i}(z)H_l(t)+\left[\varphi(z,0)-\varphi(0,0)\right]_t+\frac{z^4}{24}\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]_t+\frac{z^3}{6}\left[\varphi_{zzz}(0,t)-\varphi_{zzz}(0,0)\right]_t$$
$$+\frac{z^2}{2}\left[\varphi_{zz}(0,t)-\varphi_{zz}(0,0)\right]_t+z\left[\varphi_z(0,t)-\varphi_z(0,0)\right]_t+\left[\varphi(0,t)\right]_t+m\left[\sum_{i=1}^{3p}\sum_{i=1}^{3p}a_{iz}P_{4,i}(z)P_{1,l}(t)\right.$$
$$\left.+\left[\varphi_z(z,0)-\varphi_z(0,0)\right]+\frac{z^3}{6}\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]+\frac{z^2}{2}\left[\varphi_{zzz}(0,t)-\varphi_{zzz}(0,0)\right]+z\left[\varphi_{zz}(0,t)-\varphi_{zz}(0,0)\right]+\varphi(0,t)\right]$$
$$+\sum_{i=1}^{3p}\sum_{i=1}^{3p}a_{iz}P_{2,i}(z)P_{1,l}(t)+\left[\varphi_{zzz}(z,0)-\varphi_{zzz}(0,0)\right]+z\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]+\varphi_{zzz}(0,t)$$
$$-\sum_{i=1}^{3p}\sum_{i=1}^{3p}a_{iz}H_i(z)P_{1,l}(t)+\varphi_{zzzzz}(z,0)=F(z,t)$$

$$\sum_{i=1}^{3p}\sum_{i=1}^{3p}a_{iz}P_{5,i}(z)H_l(t)+m\sum_{i=1}^{3p}\sum_{i=1}^{3p}a_{iz}P_{4,i}(z)P_{1,l}(t)+\sum_{i=1}^{3p}\sum_{i=1}^{3p}a_{iz}P_{2,i}(z)P_{1,l}(t)-\sum_{i=1}^{3p}\sum_{i=1}^{3p}a_{iz}H_i(z)P_{1,l}(t)$$
$$=F(z,t)-\left[\varphi(z,0)-\varphi(0,0)\right]_t-\frac{z^4}{24}\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]_t-\frac{z^3}{6}\left[\varphi_{zzz}(0,t)-\varphi_{zzz}(0,0)\right]_t-\frac{z^2}{24}\left[\varphi_{zz}(0,t)-\varphi_{zz}(0,0)\right]_t$$
$$-z\left[\varphi_z(0,t)-\varphi_z(0,0)\right]_t-\left[\varphi(0,t)\right]_t-m\left[\varphi_z(z,0)-\varphi_z(0,0)\right]-m\frac{z^3}{6}\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]-m\frac{z^2}{2}\left[\varphi_{zzz}(0,t)-\varphi_{zzz}(0,0)\right]$$
$$-mz\left[\varphi_{zz}(0,t)-\varphi_{zz}(0,0)\right]-m\varphi_z(0,t)-\left[\varphi_{zzz}(z,0)-\varphi_{zzz}(0,0)\right]-z\left[\varphi_{zzzz}(0,t)-\varphi_{zzzz}(0,0)\right]-\varphi_{zzz}(0,t)-\varphi_{zzzzz}(z,0).$$

Now, discretizing the variable as $z \to z_r$, $t \to t_s z_r$ where $z_r = \dfrac{2r-1}{6p}$ and $t_s = \dfrac{2s-1}{6p}$, r and s varies from 1, 2, …., $3p$ in the equation given above. We get the systems of equation in algebraic form as

$$\sum_{i=1}^{3p}\sum_{z=1}^{3p} a_{iz} R_{i,r,z,s} = F(r,s)$$

$$R_{i,r,z,s} = \sum_{i=1}^{3p}\sum_{l=1}^{3p} a_{iz}[P_{5,i}(z_r)H_l(t_s) + mP_{4,i}(z_r)P_{1,l}(t_s) + P_{2,i}(z_r)P_{1,l}(t_s) - H_i(z_r)P_{1,l}(t_s)]$$

$$\begin{aligned}
F(r,s) = {} & f(z_r,t_s) - \left[\varphi(z_r,0) - \varphi(0,0)\right]_{t_s} - \frac{z_r^{\,4}}{24}\left[\varphi_{z_r z_r z_r z_r}(0,t_s) - \varphi_{z_r z_r z_r z_r}(0,0)\right]_{t_s} \\
& - \frac{z_r^{\,3}}{6}\left[\varphi_{z_r z_r z_r}(0,t_s) - \varphi_{z_r z_r z_r}(0,0)\right]_{t_s} - \frac{z_r^{\,2}}{2}\left[\varphi_{z_r z_r}(0,t_s) - \varphi_{z_r z_r}(0,0)\right]_{t_s} \\
& - z_r\left[\varphi_{z_r}(0,t_s) - \varphi_{z_r}(0,0)\right]_{t_s} - \left[\varphi(0,t_s)\right]_{t_s} - m\left[\varphi_{z_r}(z_r,0) - \varphi_{z_r}(0,0)\right] \\
& - m\frac{z_r^{\,3}}{6}\left[\varphi_{z_r z_r z_r z_r}(0,t_s) - \varphi_{z_r z_r z_r z_r}(0,0)\right] - m\frac{z_r^{\,2}}{2}\left[\varphi_{z_r z_r z_r}(0,t_s) - \varphi_{z_r z_r z_r}(0,0)\right] \\
& - mz_r\left[\varphi_{z_r z_r}(0,t_s) - \varphi_{z_r z_r}(0,0)\right] - m\varphi_{z_r}(0,t_s) - \left[\varphi_{z_r z_r z_r}(z_r,0) - \varphi_{z_r z_r z_r}(0,0)\right] \\
& - z_r\left[\varphi_{z_r z_r z_r z_r}(0,t_s) - \varphi_{z_r z_r z_r z_r}(0,0)\right] - \varphi_{z_r z_r z_r}(0,t_s) - \varphi_{z_r z_r z_r z_r z_r}(z,0).
\end{aligned}$$

The above system is reduced to a system of algebraic equations, which is then reduced to the system of arrays in 4D mentioned below.

$$A_{3p\times 3p} R_{3p\times 3p\times 3p\times 3p} = F_{3p\times 3p}$$

By using transformations, the above array is reduced to a matrix of $a_{il} = b_\lambda$ and $F(r,s) = f_\mu$. Here, $\lambda = 3p(i-1)+l$ and $\mu = 3p(r-1)+s$. Value of b_λ can be calculated for the distinct values of n which varies from 1, 2, 3……… by solving the equations of system using MATLAB programmes.

6. Numerical observations

6.1 *Numerical experiment* 1

$$D_t^\mu \varphi(z,t) + \varphi\varphi_z + \varphi_{zzz} - \varphi_{zzzzz} = F(z,t)$$
$$\varphi(z,0) = 0.$$

With a forcing term,

$$F(z,t) = \frac{2t^{2-\mu}}{\Gamma(3-\mu)}\sin z + \frac{1}{2}t^4 \sin 2z - 2t^2 \cos z$$

the exact solution: $\varphi(z,t) = t^2 \sin z$; the solution computed by the proposed method for $z,t \in [0,2\pi]\times[0,1]$.

For $\mu = 1$,

$$\varphi_t(z,t) + \varphi\varphi_z + \varphi_{zzz} - \varphi_{zzzzz} = F(z,t)$$

$$F(z,t) = 2t\sin z + \frac{1}{2}t^4 \sin 2z - 2t^2 \cos z$$

$$\varphi_t(z,t)+\varphi\varphi_z+\varphi_{zzz}-\varphi_{zzzzz}=2t\sin z+\frac{1}{2}t^4\sin 2z-2t^2\cos z$$

by using quasilinearisation technique for non-linear term

$$\varphi_t(z,t)+\varphi\varphi_z+\varphi_{zzz}-\varphi_{zzzzz}=2t\sin z+\frac{1}{2}t^4\sin 2z-2t^2\cos z$$

$$\varphi_t(z,t)+\varphi_{r+1}\varphi_z-\varphi_r\varphi_z+(\varphi_z)_{r+1}\varphi+\varphi_{zzz}-\varphi_{zzzzz}=2t\sin z+\frac{1}{2}t^4\sin 2z-2t^2\cos z.$$

Table 1 shows the comparison of approximate and exact solutions with the value of absolute error for different values of z and t. Table 2 shows L_2 and L_∞ errors for different values of μ.

Table 1. Comparison of approximate and exact solutions for numerical experiment 1

z	t	Approximate solution	Exact solution	Value of absolute error
0.055555555555556	0.055555555555556	0.000057152654610	0.000057152654610	9.15e-18
0.166666666666667	0.166666666666667	0.000006350294957	0.000006350294957	1.06e-18
0.277777777777778	0.277777777777778	0.001428816365253	0.001428816365253	2.28e-16
0.388888888888889	0.388888888888889	0.002800480075895	0.002800480075896	4.48e-16
0.500000000000000	0.500000000000000	0.004629365023419	0.004629365023420	7.40e-16
0.611111111111111	0.611111111111111	0.006915471207824	0.006915471207825	1.10e-15
0.722222222222222	0.722222222222222	0.009658798629109	0.009658798629110	1.54e-15
0.833333333333333	0.833333333333333	0.012859347287275	0.012859347287277	9.76e-15
0.944444444444444	0.944444444444444	0.016517117182322	0.016517117182325	2.64e-15

Table 2. L_2 and L_∞ errors for numerical experiment 1

μ	0.25	0.50	0.75	1
L_2 error	2.32749967e-03	1.4542317e-03	6.74070884e-04	2.49115757e-06
L_∞ error	2.98853880e-03	1.8672269e-03	9.41931276e-04	2.97079307e-06

By discontinuous Galerkin method, for the different value of μ, norm representation attains third order level of accuracy [8]. But, the proposed method at similar values of μ, 3D graphical representation attains third as well as fourth level of accuracy.

Figure 1 shows the graphical representation of approximate, exact value and absolute error with contour view of the exact solution while Figure 2 depicts the graphical representation for μ = 0.25, μ = 0.50 and μ = 0.75 with the approximated solution at μ =1.

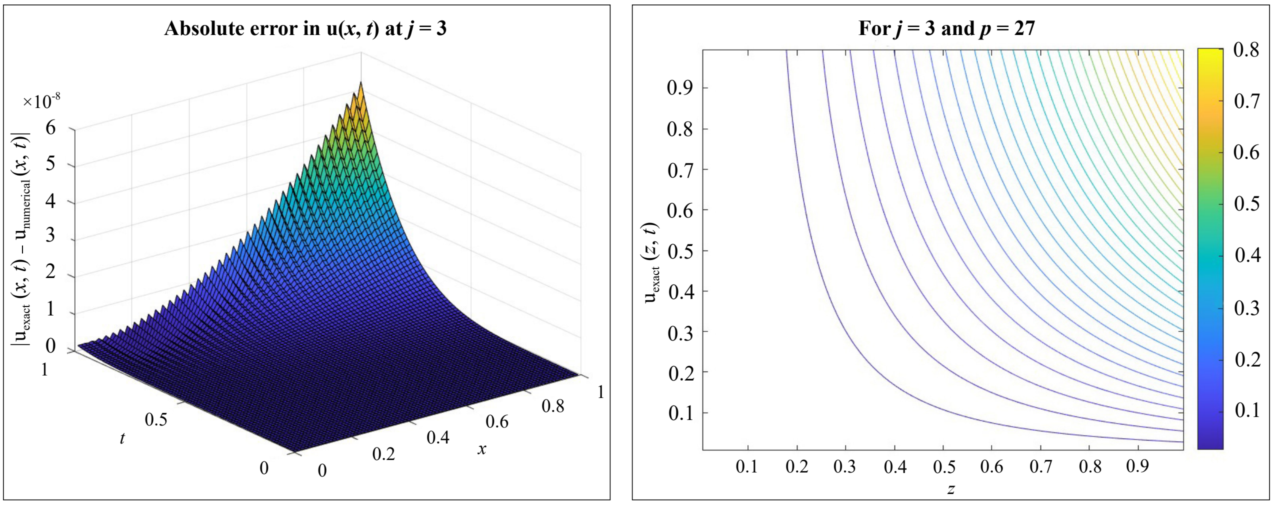


Figure 1. The graphical representation of approximate, exact value and absolute error with contour view of exact solution

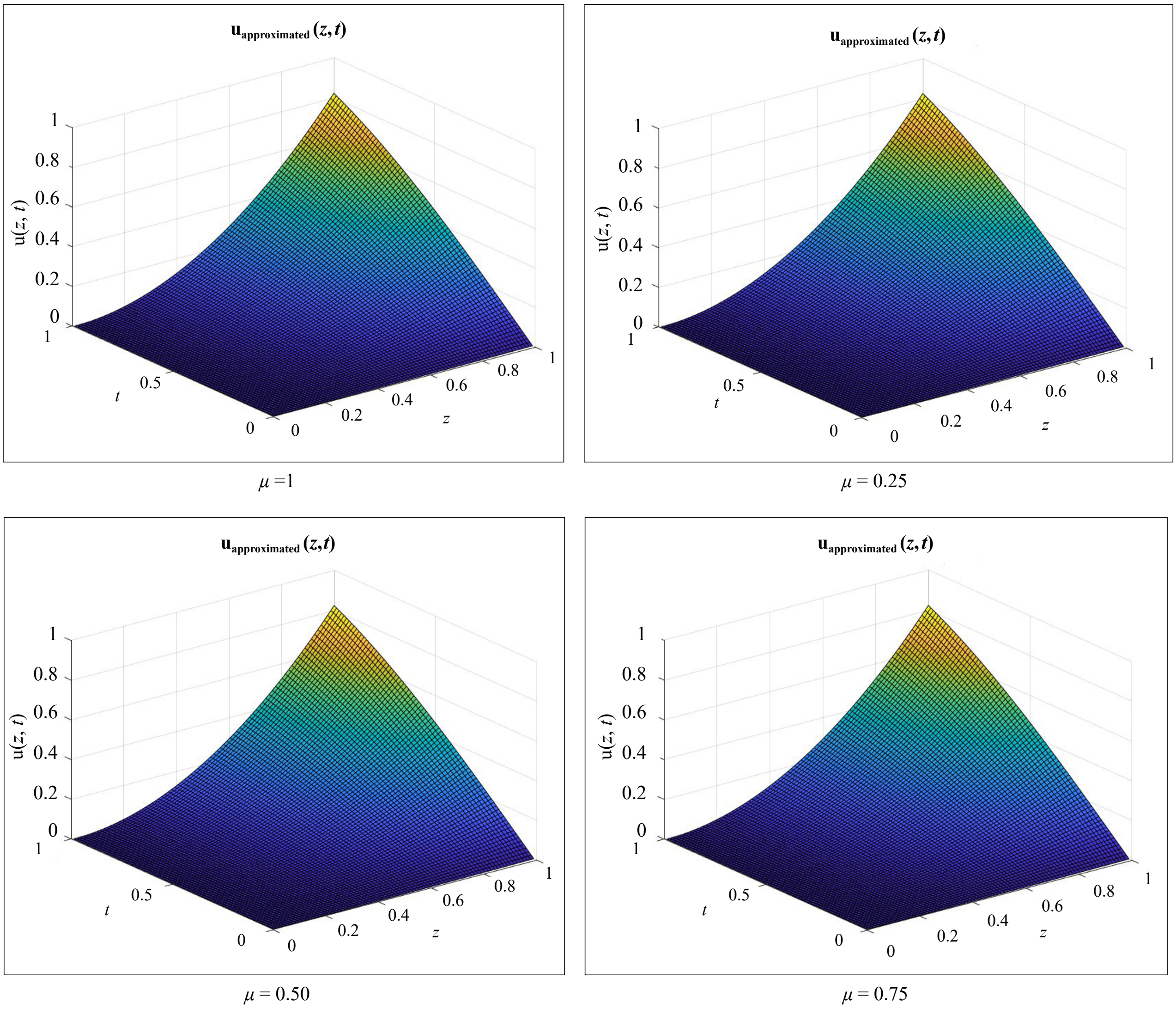


Figure 2. The graphical representation for μ = 0.25, μ = 0.50 and μ = 0.75 with approximated solution at μ =1

6.2 *Numerical experiment* 2

$$D_t^{\mu}\varphi(z,t)+\left(\frac{\varphi^2}{2}+\frac{\varphi^3}{3}\right)z+\varphi_{zzz}-\varphi_{zzzzz}=F(z,t)$$
$$\varphi(z,0)=0$$

With a source term,

$$F(z,t)=\frac{2t^{2-\mu}}{\Gamma(3-\mu)}\cos(2\pi z)-t^4\pi\sin(4\pi z)-t^6\pi\cos(2\pi z)\sin(4\pi z)+8\pi^3t^2\sin(2\pi z)+32\pi^5t^2\sin(2\pi z)$$

the exact solution: $\varphi(z,t)=t^2\cos(2\pi z)$, the calculation of accuracy of the proposed technique for $z,t\in[0,1]\times[0,1]$.

For $\mu = 1$,

$$\varphi_t(z,t)+\left(\frac{\varphi^2}{2}+\frac{\varphi^3}{3}\right)z+\varphi_{zzz}-\varphi_{zzzzz}=F(z,t).$$

After solving the above equation, it becomes

$$\begin{aligned}&\varphi_t+\varphi\varphi_z+\varphi^2\varphi_z+\varphi_{zzz}-\varphi_{zzzzz}\\&=2t\cos(2\pi z)-t^4\pi\sin(4\pi z)-t^6\pi\cos(2\pi z)\sin(4\pi z)+8\pi^3t^2\sin(2\pi z)+32\pi^5t^2\sin(2\pi z).\end{aligned}$$

After applying quasilinearisation technique on non-linear terms,

$$\begin{aligned}&\varphi_t+\varphi_{r+1}\varphi_z-\varphi_z\varphi+(\varphi_z)_{r+1}\varphi+2(\varphi_z)_{r+1}\varphi\varphi_z-2\varphi^2\varphi_z+\varphi^2\left(\varphi_z\right)_{r+1}+\varphi_{zzz}-\varphi_{zzzzz}\\&=2t\cos(2\pi z)-t^4\pi\sin(4\pi z)-t^6\pi\cos(2\pi z)\sin(4\pi z)+8\pi^3t^2\sin(2\pi z)+32\pi^5t^2\sin(2\pi z)\end{aligned}$$

Table 3 shows the comparison of approximate and exact solutions with the value of absolute error at $\mu = 1$ for different values of z and t, while Table 4 shows L_2 and L_∞ errors for numerical experiment 2 at $\mu = 0.2$, 0.4 and 0.6. Figure 3 is the graphical representation of approximated and exact value at $\mu = 1$ and Figure 4 shows the graphical representation for $\mu = 0.20$, 0.40 and 0.60 with the graph of collocation points.

Table 3. Comparison of approximate and exact solutions for numerical experiment 2

z	*t*	Approximate solution	Exact solution	Value of absolute error
0.055555555555556	0.055555555555556	0.004607110282356	0.004607110282465	1.09e-13
0.166666666666667	0.166666666666667	0.032021320226954	0.032021320227711	7.57e-13
0.277777777777778	0.277777777777778	0.084108319121690	0.084108319123679	2.61e-12
0.388888888888889	0.388888888888889	0.160868106966565	0.160868106970369	3.08e-12
0.500000000000000	0.500000000000000	0.262300683761578	0.262300683767781	6.20e-12
0.611111111111111	0.611111111111111	0.388406049506729	0.388406049515914	9.81e-11
0.722222222222222	0.722222222222222	0.539184204202020	0.539184204214769	1.27e-11
0.833333333333333	0.833333333333333	0.714635147847451	0.714635147864346	1.68e-11
0.944444444444444	0.944444444444444	0.914758880443024	0.914758880464644	2.16e-11

Table 4. L_2 and L_∞ errors for numerical experiment 2

μ	L_2 error (HSW3)	L_2 error [8]	L_∞ errors (HSW3)	L_∞ errors [8]
0.2	7.7e-003	1.013271e-002	9.9e-003	4.082594e-002
0.4	7.5e-003	1.013270e-002	9.7e-003	4.082593e-002
0.6	7.4e-003	1.013270e-002	9.3e-003	4.082592e-002

* Note: HSW3 = Haar scale-3 wavelet

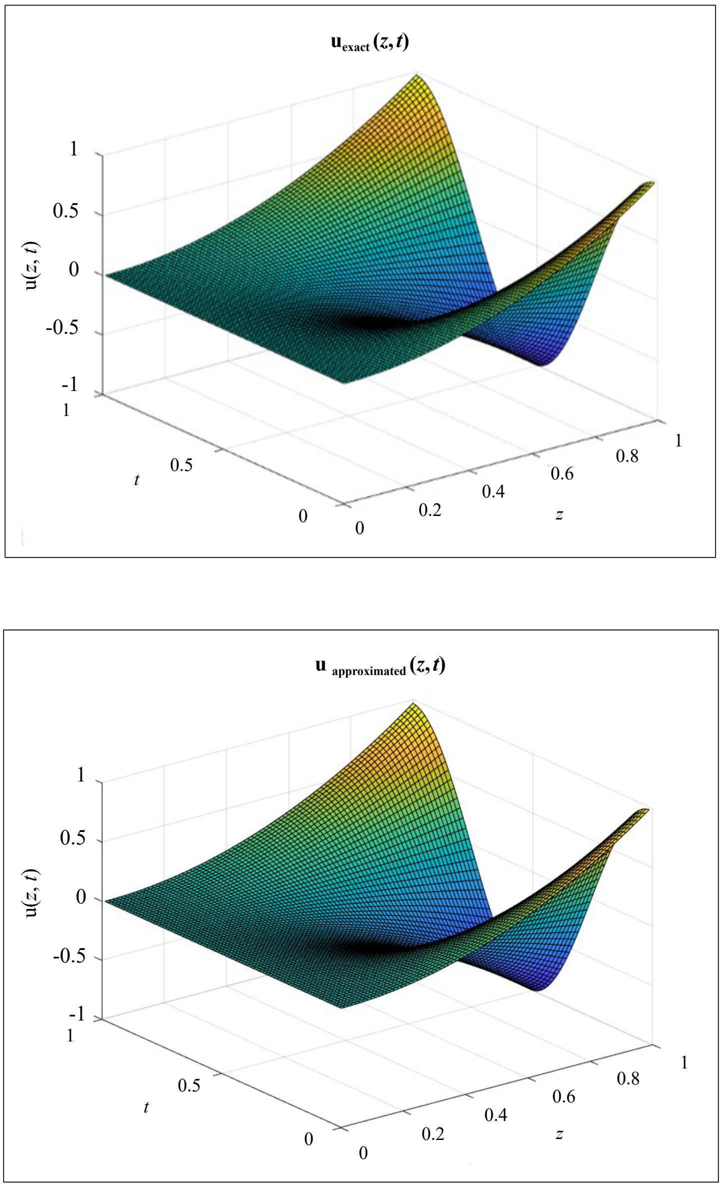
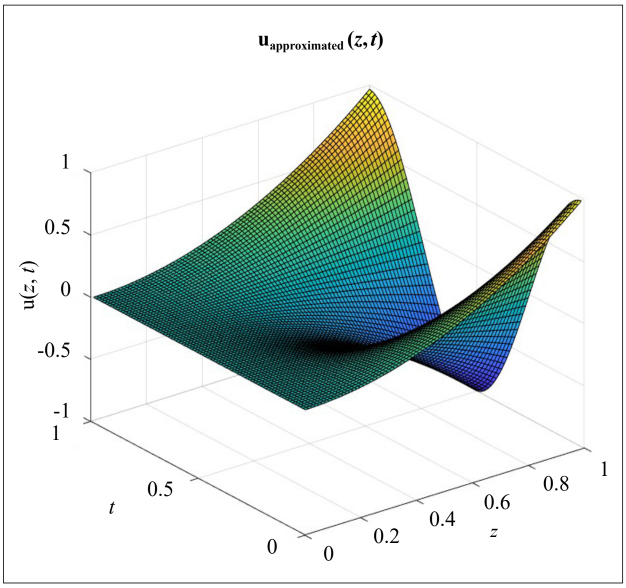
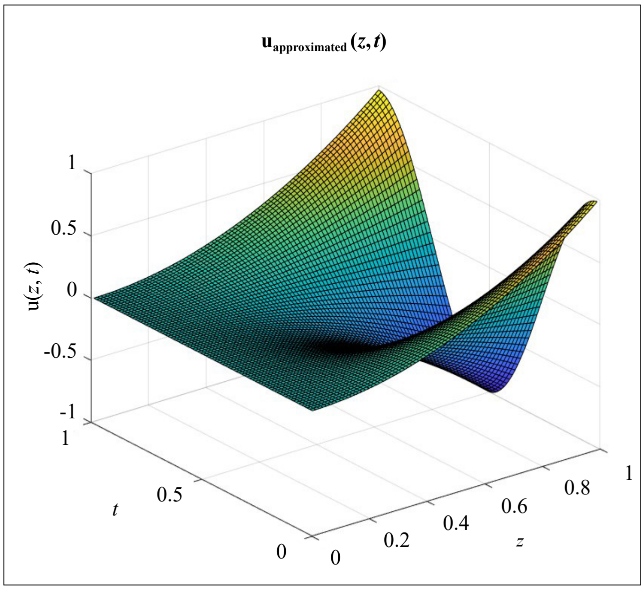


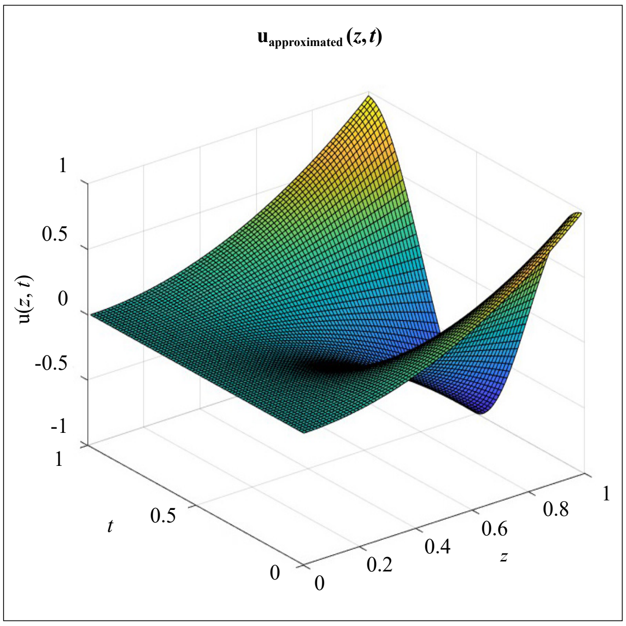
Figure 3. The graphical representation of approximated and exact value at $\mu = 1$



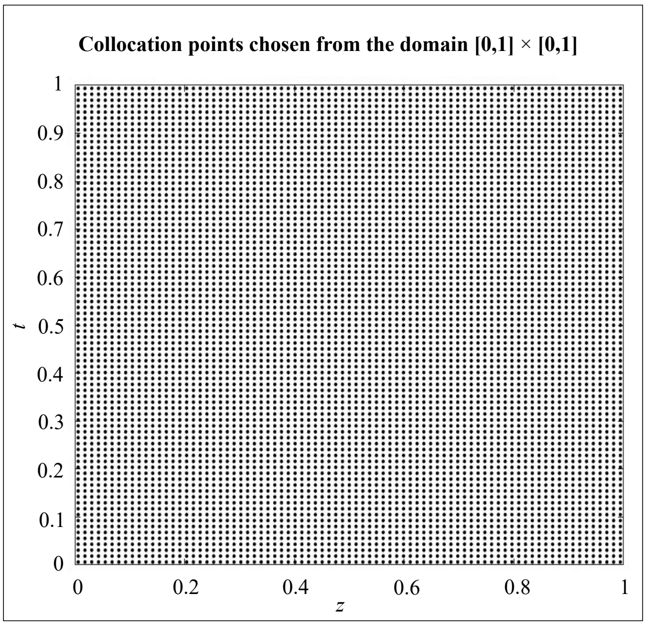
$\mu = 0.20$



$\mu = 0.40$



$\mu = 0.60$



Collocation points chosen from the domain [0,1] × [0,1]

Figure 4. The graphical representation for $\mu = 0.20$, 0.40 and 0.60 with the graph of collocation points

7. Conclusions

In this article, the time-fractional Kawahara equation is solved using the Haar scale wavelet method; the stability and error analysis for the fractional case are carried out. The numerical experiments are provided to show the method's accuracy and capacity. Higher-dimensional issues and other types of time-fractional equations can be easily solved using the same approach and analytical technique. With the support of graphics, the results are examined and discussed while taking various parameter values into account. Results show that the approximate solution converges to the exact solution as the value of the fractional-order derivative. Researchers may find our method attractive for solving higher-order time fractional problems that are emerging in the science of technology because it can be made more precise by assuming high approximations.

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Conflict of interest

The authors do not have any competing interests to declare that are relevant to the content of this article.

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