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Research Article

Existence and Stability Results of Nonlinear Random Impulsive Integro-Differential Evolution Equations with Time-Varying Delays

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Abstract: This study examines the existence, uniqueness, and stability of the nonlinear random impulsive integro-differential equations with time-varying delays under sufficient conditions. Our study is based on the Leray-Schauder alternative fixed point theorem, Pachpatte's inequality, and the Banach contraction principle. Besides, we generalize, extend, and develop some results in the existing literature. Our approach is generalizing the results mentioned above and also achieving better results with lesser hypotheses by using the Leray-Schauder alternative fixed point theorem, Pachpatte's inequality, and the Banach contraction principle.

Keywords: fixed point theorem, time-varying delays, integro-differential equations, random impulses, contraction principle

MSC: 34K20, 34K45, 45J05

1. Introduction

Mathematical modeling of systems in the scientific and technical sectors frequently leads to ordinary or partial differential equations (DEs), integral or integro-differential equations (IDEs), or stochastic equations in the real world. The integro-differential type equations will be used in fluid dynamics, chemical kinetics, and mechanics. For details, see [1-6]. Several evolution processes are defined by the fact that a change of state occurs at a predetermined time and is subject to short-term perturbations. In comparison to the lifespan of the process, the duration of this short-term disruption is insignificant. Instantaneous acts of disturbance are, of course, in the form of impulses. DEs with impulsive effects appear to be a natural explanation of real-world evolution occurrences. In the realms of science and technology, impulsive effects can be seen. One of the mathematical formulations of economic and biological events is randomness.

Impulses can occur at either predictable or random locations. The qualities of deterministic impulses have been studied by many researchers; for details, see [7-9], and the references therein. If impulses exist at random, on the other hand, the solution will act like a stochastic process. Different from deterministic impulsive DEs and stochastic DEs, it is a type of deterministic impulsive DE. The impulsive system has long been regarded as one of the more significant models in mathematical ecology, with numerous perfect existence and stability results for modified models.

In addition, the impulse perturbation coefficients must be selected based on the real condition, which may oscillate in certain ranges or alter irregularly. If we consider the disturbance of environmental factors at specific time moments,

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which cannot be ignored, there will be immediate and large changes in population density in the form of perturbations. As a result, we may naturally include impulsive effects in differential equations [10-12]. Many authors have studied the various kinds of random impulsive IDEs with time-varying delays [13-16]. However, there are not many papers considering the stability results of impulsive random IDEs with time-varying delays. Recently, Kumar et al. [17] developed the existence and uniqueness of mild solutions for nonlocal IDEs with time-varying delays. [18] studied the existence, uniqueness, and stability of random impulsive DEs.

Vinodkumar et al. [19] have investigated the existence, uniqueness, stability, and Ulam stability of nonlinear delay integro-differential equations (NDIDEs) with Random Impulse of the type

$$\begin{cases} z'(t) = \int_{t_0}^{t} \mathcal{H}(t,s,z(\beta(s)))ds, \ t \geq \tau, \ t \neq \zeta_q, \\ z(\zeta_q) = a_q(\tau_q)z(\zeta_q^-), \ q = 1,2,..., \\ z_{t_0} = \upsilon. \end{cases}$$

Kasinathan et al. [20] have investigated the existence, uniqueness, and stability of the NDIDEs system with Random Impulse of the type

$$\begin{cases} z'(t) = \mathcal{A}(t)z(t) + \int_{t_0}^{t} \mathcal{H}(t,s,z(\beta(s)))ds, \ t \geq \tau, \ t \neq \zeta_q, \\ z(\zeta_q) = a_q(\tau_q)z(\zeta_q^-), \ q = 1,2,..., \\ z_{t_0} = \upsilon. \end{cases}$$

Motivated by the above works, we consider nonlinear random impulsive integro-differential equations (NRIIDEs) with time-varying delays of the type

$$\begin{cases} z'(t) = \mathcal{A}(t)z(t) + \mathcal{H}\left(t, z(\sigma_1(t)),..., z(\sigma_n(t)), \int_{t_0}^{t} g(t,s,z(\sigma_{n+1}(s)))ds\right), \ t \geq \tau, \ t \neq \zeta_q, \\ z(\zeta_q) = a_q(\tau_q)z(\zeta_q^-), \ q = 1,2,..., \\ z_{t_0} = \upsilon, \end{cases} \tag{1}$$

where $\mathcal{A}$ is a family of linear operators, which generates an evolution operator $\mathfrak{U}(t,s), 0 \leq s \leq t \leq \mathcal{T}$, the function $\mathcal{H} : \mathbb{R}_\tau \times \mathcal{C} \times \mathbb{R}^n \to \mathbb{R}^n, g : \Delta \times \mathcal{C} \to \mathbb{R}^n$ and $\sigma_\iota : [t_0, \mathcal{T}] \to [t_0, \mathcal{T}], \iota = 1,2,\dots n+1$, are continuous functions with $\sigma_\iota(t) \leq t, \iota =$ $1,2,\dots,n+1$. The set of piecewise continuous functions is $\mathcal{C} = \mathcal{C}([-\rho, 0], \mathbb{R}^n)$, which maps $[-\rho, 0]$ in to $\mathbb{R}^n$ with some given $\rho > 0, \zeta_0 = t_0$ and $\zeta_q = \zeta_{q-1} + \tau_q$ for q = 1,2,... . Here, $t_0 \in \mathbb{R}_\tau$ is an arbitrary real number. Obviously, $t_0 = \zeta_0 < \zeta_1 < \zeta_2 < ... < \lim_{k \to \infty} \zeta_q = \infty, a_q : \mathcal{D}_q \to \mathbb{R}^{n \times n}$ is a matrix-valued function for all $q = 1,2,..., z(\zeta_q^-) = \lim_{t \uparrow \zeta_q} z(t)$. According to their paths with the norm $\|z\|_t = \sup_{t-\rho \leq s \leq t} |z(s)|$ for all t fulfilling $\tau \leq t \leq \mathcal{T}$ $\|.\|$ any given norm in $\mathbb{X}$, here the set $\{(t,s) : 0 \leq s \leq t < \infty\}$ denoted by Δ. Denote by $\{\mathbb{G}_t, t \geq 0\}$ the simple counting process generated by ζ_n, that is, $\{\mathbb{G}_t \geq n\} = \{\zeta_n \leq t\}$, and denote $\mathcal{H}_t$ the σ-algebra generated by $\{\mathbb{G}_t, t \geq 0\}$. Then, $(\Omega, \mathcal{P}, \{\mathcal{H}_t\})$ is a probability space.

Our approach is generalizing the results mentioned in [1, 20], and achieving better results with fewer hypotheses by using the Leray-Schauder alternative fixed-point theorem, Pachpatte's inequality, and the Banach contraction principle.

The structure of this article is as follows: In Section 2, we mention some basic concepts and preliminary. In Section 3, we investigate the existence of NRIIDEs with time-varying delays. Moreover, the existence and uniqueness of solutions for NRIIDEs with time-varying delays are examined by loosening growth assumptions. In Section 4, we investigate the stability of NRIIDEs with time-varying delays by looking at their continuous dependency on initial conditions. In Section 5, we examine the Ulam Hyers (UH) and Ulam Hyers-Rassias (UHR) for NRIIDEs with time-varying delays. Finally, Section 6 gives the conclusion with acknowledgements of the study.

2. Preliminaries

Consider an n-dimensional Euclidean space $\mathbb{R}^n$ and a non-empty set Ω. The random variable τ_q is defined from Ω to $\mathcal{D}_q \overset{def}{=} (0, d_q)$ for q = 1,2,..., where $0 < d_q < \infty$. Moreover, suppose that τ_q follow Erlang distribution, where q = 1,2,..., and let $\tau_\imath$ and $\tau_\jmath$ be independent of each other as $\imath \neq \jmath$ for $\imath, \jmath$ = 1,2... . For simplification, we denote $\mathbb{R}_\tau = [\tau, \infty)$. Let $\pounds_p = \pounds_p(\Omega, \mathcal{H}_t, \mathbb{R}^n)$ denote the Banach space of each $\mathcal{H}_t$-measurable square integrable random variables in $\mathbb{R}^n$. Suppose that $\mathcal{T} > t_0$ is any fixed time and G denotes the Banach space $G([t_0 - \rho, \mathcal{T}]\pounds_2)$, which is the family of all $\mathcal{H}_t$-measurable, $\mathcal{C}$-valued random variables with the norm

$$\|\phi\|_G = \left(\sup_{t_0 \le t \le \mathcal{T}} \mathbb{E}\|\phi\|_t^2 \right)^{1/2}.$$

The family of all $\mathcal{H}_0$- measurable, G-valued random variable υ is denoted by $\pounds_p^0(\Omega, G)$.

Definition 2.1. A map $\mathcal{H} : [\tau, \mathcal{T}] \times \mathcal{C} \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \Delta \times \mathcal{C} \to \mathbb{R}^n$ are given functions.

Moreover, $\sigma_\imath : [t_0, \mathcal{T}] \to [t_0, \mathcal{T}], \imath = 1, 2, \ldots, n+1$ are continuous functions with $\sigma_\imath(t) \le t, \imath = 1, 2, \ldots, n+1$. We assume the following:

(1) $t \to \mathcal{H}(t, z_1, ..., z_{n+1})$ is measurable for each $z_1, ..., z_n \in \mathcal{C}$.
(2) Each $z_1, ..., z_{n+1} \to \mathcal{H}(t, z_1, ..., z_{n+1})$ is continuous for almost all $t \in [\tau, \mathcal{T}]$.
(3) For all nonnegative integer $m > 0$, there exists $\alpha_m \in \pounds^1([\tau, \mathcal{T}], \mathbb{R}^+)$ such that

$$\sup_{\mathbb{E}\|z\|^p \le m} \mathbb{E}\left\| \mathcal{H}\left(t, z(\sigma_1(t)), ..., z(\sigma_n(t)), \int_{t_0}^t g(t, s, z(\sigma_{n+1}(s)))ds\right) \right\|^p \le \alpha_m(t), \, a.e.$$

We assume the following hypotheses hold for the linear operator family $\{\mathcal{A}(t) : 0 \le t \le \mathcal{T}\}$:

(H1) $\mathcal{A}(t)$ is a closed linear operator and the domain $\mathfrak{D}(\mathcal{A})$ of $\{\mathcal{A}(t) : 0 \le t \le \mathcal{T}\}$ is dense and independent of t in the Banach space $\mathbb{Y}$.
(H2) For every $t \in [0, \mathcal{T}]$, the resolvent $\Re(\eta, \mathcal{A}(t)) = (\eta I - \mathcal{A}(t))^{-1}$ of $\mathcal{A}(t)$ exists for each η with $\Re e\eta \le 0$ and $\|\Re(\eta, \mathcal{A}(t))\| \le \mathcal{C}(\eta + 1)^{-1}$.
(H3) For any $t, s, \tau \in [0, \mathcal{T}]$, there exists $0 < \delta < 1$ and $\ell > 0$ so that

$$\|\mathcal{A}(t) - \mathcal{A}(\tau)\mathcal{A}^{-1}(s)\| \le \ell \, | \, t - \tau \, |^\delta .$$

The assumptions (H1) and (H2) imply that there exists a family of evolution operator $\mathfrak{U}(t, s)$.

The family of two-parameter linear evolution systems $\mathfrak{U}(t, s) : 0 \le s \le t \le \mathcal{T}$ fulfills the following criteria:

(1) $\mathfrak{U}(t, s) \in \ell(\mathbb{Y})$ the space of bounded linear transformations define on $\mathbb{Y}$, whenever $0 \le s \le t \le \mathcal{T}$ and for each $y \in \mathbb{Y}$, the mapping $(t, s) \to \mathfrak{U}(t, s)y$ is continuous.
(2) $\mathfrak{U}(t, s)\mathfrak{U}(s, \tau) = (s, \tau)$ for $0 \le s \le t \le \mathcal{T}$
(3) $\mathfrak{U}(t, t) = I$

Definition 2.2. For a given $\mathcal{T} \in (t_0, +\infty)$, a stochastic process $\{z(t) \in G, t_0 - \rho \le t \le \mathcal{T}\}$ is said to be a solution to equation (1) in $(\Omega, \mathcal{P}, \{\mathcal{H}_t\})$, if

(1) $z(t) \in \mathbb{R}^n$ is $\mathcal{H}_t$-adapted for $t \ge t_0$,
(2) $z(t_0 + s) = \upsilon(s) \in \pounds_2^0(\Omega, \mathcal{H})$ when $s \in [-\rho, 0]$

$$\begin{aligned} z(t) = \sum_{q=0}^{\infty} \Big[\prod_{\imath=1}^{q} a_\imath(\tau_\imath) \mathfrak{U}(t, t_0)\upsilon(0) \\ + \sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} a_\jmath(\tau_\jmath) \int_{\zeta_{\imath-1}}^{\zeta_\imath} \mathfrak{U}(t, s)\mathcal{H}\left(s, z(\sigma_1(s)), ..., z(\sigma_n(s)), \int_{t_0}^s g(s, r, z(\sigma_{n+1}(r)))dr\right) ds \\ + \int_{\zeta_q}^t \mathfrak{U}(t, s)\mathcal{H}\left(s, z(\sigma_1(s)), ..., z(\sigma_n(s)), \int_{t_0}^s g(s, r, z(\sigma_{n+1}(r)))dr\right) ds \Big] I_{[\zeta_q, \zeta_{q+1})}(t), \, t \in [t_0, \mathcal{T}], \end{aligned}$$

where $\prod_{j=1}^{q} a_j(\tau_j) = a_q(\tau_q)a_{q-1}(\tau_{q-1}),...a_1(\tau_1)$ and $I_{\mathcal{A}}(.)$ is the index function.

The following Pachpatte's inequalities play a crucial role in our analysis.

Lemma 2.1. [21] (p.33), Let z, ϑ, and ψ be non-negative continuous functions defined on $\mathbb{R}_+$, for which the inequality

$$z(t) \leq z_0 + \int_0^t \vartheta(s)z(s)ds + \int_0^t \vartheta(s)\left(\int_0^s \psi(r)z(r)dr\right)ds, t \in \mathbb{R}_+$$

holds, where z_0 is non-negative constant. Then,

$$z(t) \leq z_0\left[1 + \int_0^t \vartheta(s)\exp\left(\int_0^s \vartheta(s) + \psi(r)dr\right)ds\right], t \in \mathbb{R}_+.$$

The following theorem, which is a variant of the topological transversality theorem, is known as the Leray-Schauder alternative.

Theorem 2.1. [22] Assume that the completely continuous operator is $\mathcal{H}$ maps from $\mathcal{S}$ into $\mathcal{S}$ and suppose that $0 \in \mathcal{S}$. Let $\mathcal{S}$ be a convex subset of a Banach space E, and let

$$\mathcal{U}(\mathcal{H}) = \{z \in \mathcal{S} : z = \eta\mathcal{H}z \text{ for some } 0 < \eta < 1\}.$$

Then, either $\mathcal{U}(\mathcal{H})$ is unbounded or $\mathcal{H}$ has a fixed point.

3. Existence results

Here, we list the following assumptions for our convenience:

(A1) There exists a continuous function $\varrho_1 : [t_0, \mathcal{T}] \to (0, +\infty)$ such that

$$\mathbb{E}\left\|\mathcal{H}\left(t, z_1, ..., z_{n+1}\right)\right\|^p \leq \varrho_1(t)W\mathbb{E}\left(\|z_1\|_s^p + ... + \|z_{n+1}\|_s^p\right), z_\iota \in \mathbb{R}^n, \iota = 1, 2, ..., n+1,$$

where $W : \mathbb{R}^+ \to (0, +\infty)$ is a continuous increasing function satisfying $W(\eta(t)z) \leq \eta(t)W(z)$.

(A2) There exists a continuous function $\varrho_2 : [t_0, \mathcal{T}] \to (0, +\infty)$ such that

$$\mathbb{E}\left\|\int_{t_0}^t g(t, s, z)ds\right\|^p \leq \varrho_2(t)\mathbb{E}\|z\|_s^p, t, s \geq 0, z \in \mathbb{R}^n.$$

(A3) $\mathcal{A}(t)$ generates a family of evolution operators $\mathfrak{U}(t, s) \in \mathbb{Y}$, and there is $K > 0$ such that

$$|\mathfrak{U}(t, s)| \leq K \text{ for } 0 \leq s \leq t \leq \mathcal{T}.$$

(A4) $\sigma_\iota : [t_0, \mathcal{T}] \to [t_0, \mathcal{T}], \iota = 1, 2, ..., n+1$ are continuous functions with $\sigma_\iota(t) \leq t$, $\iota = 1, 2, ..., n+1$.

(A5) $\mathbb{E}\left\{\max_{\iota, q} \prod_{j=\iota}^{q} \|a_\iota(\tau_j)\|\right\}$ is uniformly bounded if there is a constant $C > 0$ such that

$$\mathbb{E}\left\{\max_{\iota, q} \prod_{j=\iota}^{k} \|a_j(\tau_j)\|\right\} \leq C \text{ for each } \tau_j \in \mathcal{D}_j, j = 1, 2, ...$$

Theorem 3.1. Under the assumptions (A1)-(A5), the system (1) has a solution $z(t)$ defined on $[t_0, \mathcal{T}]$ provided that the following estimate is fulfilled

$$\int_{t_0}^{\mathcal{T}} M_*(s)ds < \int_{c_0}^{\infty} \frac{ds}{W(s)}, \tag{2}$$

where $M_*(s) = 2^{p-1}K^p \max\{1, C\}(\mathcal{T} - t_0)\varrho_1(t)(n + \varrho_2(t))$, $c_0 = 2^{p-1}K^pC^p\mathbb{E}\|\upsilon\|^p$ and $K^pC^p \geq \frac{1}{2^{p-1}}$.

Proof. Let $\mathcal{T}$ be an arbitrary number $t_0 < \mathcal{T} < \infty$ fulfilling (2). We transform the system (1) into a fixed-point problem. We consider the operator φ maps G into G defined as

$$\phi z(t) = \begin{cases} \upsilon(t - t_0), \quad t \in [t_0 - \rho, t_0] \\ \sum_{q=0}^{\infty} [\prod_{\iota=1}^{q} a_\iota(\tau_\iota)\mathfrak{U}(t, t_0)\upsilon(0) \\ + \sum_{\iota=1}^{q}\prod_{j=\iota}^{q} a_j(\tau_j)\int_{\zeta_{\iota-1}}^{\zeta_\iota} \mathfrak{U}(t,s)\mathcal{H}\Big(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r)))dr\Big)ds \\ + \int_{\zeta_q}^{t} \mathfrak{U}(t,s)\mathcal{H}\Big(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r)))dr\Big)ds\Big] I_{[\zeta_q, \zeta_{q+1})}(t), \ t \in [t_0, \mathcal{T}]. \end{cases}$$

To apply the transversality theorem, we must first establish a priori estimates for the solutions of the integral equation and $\eta \in (0,1)$

$$z(t) = \begin{cases} \eta\upsilon(t - t_0), \quad t \in [t_0 - \rho, t_0] \\ \eta\sum_{q=0}^{\infty} \Big[\prod_{\iota=1}^{q} a_\iota(\tau_\iota)\mathfrak{U}(t, t_0)\upsilon(0) \\ + \sum_{\iota=1}^{q}\prod_{\iota=\iota}^{q} a_\iota(\tau_\iota)\int_{\zeta_{\iota-1}}^{\zeta_\iota} \mathfrak{U}(t,s)\mathcal{H}\Big(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r)))dr\Big)ds \\ + \int_{\zeta_q}^{t} \mathfrak{U}(t,s)\mathcal{H}\Big(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r)))dr\Big)ds\Big] I_{[\zeta_q, \zeta_{q+1})}(t), \ t \in [t_0, \mathcal{T}]. \end{cases}$$

Then, by (A1)-(A5), we have

$$\begin{aligned} \|z(t)\|^p &\leq \eta^p \Bigg[\sum_{q=0}^{\infty} \Bigg[\Big\|\prod_{\iota=1}^{q} a_\iota(\tau_\iota)\Big\| \|\mathfrak{U}(t,t_0)\| \|\upsilon(0)\| \\ &+ \sum_{\iota=1}^{q}\Big\|\prod_{j=\iota}^{q} a_j(\tau_j)\Big\| \Big\{\int_{\zeta_{\iota-1}}^{\zeta_\iota} \Big\|\mathfrak{U}(t,s)\mathcal{H}\Big(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r)))dr\Big)\Big\| ds\Big\} \\ &+ \int_{\zeta_q}^{t} \Big\|\mathfrak{U}(t,s)\mathcal{H}\Big(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r)))dr\Big)\Big\| ds\Bigg] I_{[\zeta_q, \zeta_{q+1})}(t)\Bigg]^p \\ &\leq 2^{p-1}\Bigg[\sum_{q=0}^{\infty}\Bigg[\Big\|\prod_{\iota=1}^{q} a_\iota(\tau_\iota)\Big\|^p \|\mathfrak{U}(t,t_0)\|^p \|\upsilon(0)\|^p I_{[\zeta_q, \zeta_{q+1})}(t)\Bigg] \\ &+ \Bigg[\sum_{\iota=1}^{q}\Big\|\prod_{j=\iota}^{q} a_j(\tau_j)\Big\| \Big\{\int_{\zeta_{\iota-1}}^{\zeta_\iota} \|\mathfrak{U}(t,s)\| \Big\|\mathcal{H}\Big(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r))\Big)dr\Big\| ds\Big\} \\ &+ \int_{\zeta_q}^{t} \|\mathfrak{U}(t,s)\| \Big\|\mathcal{H}\Big(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r)))dr\Big\| ds\Bigg] I_{[\zeta_q, \zeta_{q+1})}(t)\Bigg]^p\Bigg] \\ &\leq 2^{p-1}\max_q \Big\{\prod_{j=\iota}^{q} \|a_\iota(\tau_\iota)\|^p\Big\} \|\mathfrak{U}(t,t_0)\|^p \|\upsilon(0)\|^p \end{aligned}$$

$$+ 2^{p-1}\left[\max_{\iota,q}\left\{1,\prod_{\jmath=\iota}^{q}\left\|a_\jmath(\tau_\jmath)\right\|\right\}\right]^p \left\|\mathfrak{U}(t,s)\right\|^p \left(\int_{t_0}^{t}\left\|\mathcal{H}\Big(s,z(\sigma_1(s)),\ldots,z(\sigma_n(s)),\int_{t_0}^{s} g(s,r,z(\sigma_{n+1}(r)))dr\right\|ds\right)^p.$$

We note that from the above inequality, the last term of the right side increases in t and choose $C^p \geq \frac{1}{2^{p-1}}$, we would get

$$\|z\|_t^p \leq 2^{p-1}\max_{q}\left\{\prod_{\iota=1}^{q}\left\|a_\iota(\tau_\iota)\right\|^p \left\|\mathfrak{U}(t,t_0)\right\|^p\right\}\left\|\upsilon(0)\right\|^p + 2^{p-1}\left[\max_{\iota,q}\left\{1,\prod_{\jmath=\iota}^{q}\left\|a_\jmath(\tau_\jmath)\right\|\right\}\right]^p$$
$$\times\left\|\mathfrak{U}(t,s)\right\|^p(\mathcal{T}-t_0)\left(\int_{t_0}^{t}\left\|\mathcal{H}\Big(s,z(\sigma_1(s)),\ldots,z(\sigma_n(s)),\int_{t_0}^{s} g(s,r,z(\sigma_{n+1}(r)))dr\right\|^p ds\right).$$

Then,

$$\begin{aligned}
\mathbb{E}\|z\|_t^p &\leq 2^{p-1}C^pK^p\mathbb{E}\left[\|\upsilon\|^p\right] + 2^{p-1}\max\left\{1,C^p\right\} \\
&\quad\times(\mathcal{T}-t_0)\int_{t_0}^{t}\mathbb{E}\left[\left\|\mathcal{H}\Big(s,z(\sigma_1(s)),\ldots,z(\sigma_n(s)),\int_{t_0}^{s} g(s,r,z(\sigma_{n+1}(r)))dr\right\|^p\right]ds \\
&\leq 2^{p-1}C^pK^p\mathbb{E}[\|\upsilon\|^p] + 2^{p-1}K^p\max\{1,C^p\} \\
&\quad\times(\mathcal{T}-t_0)\int_{t_0}^{t}\varrho_1(s)W\left[\mathbb{E}\left\|z(\sigma_1(s))\right\|^p+\ldots+\mathbb{E}\left\|z(\sigma_n(s))\right\|^p+\varrho_2(s)\mathbb{E}\left\|z(\sigma_{n+1}(r))\right\|^p dr\right]ds \\
&\leq 2^{p-1}C^pK^p\mathbb{E}[\|\upsilon\|^p] + 2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_0)\int_{t_0}^{t}\varrho_1(s)(n+\varrho_2(s))W\left[\mathbb{E}\left(\|z\|_s^p\right)\right]ds. \qquad (3)
\end{aligned}$$

From the above inequality, the last term of the right side increases in t, we would obtain

$$\begin{aligned}
\sup_{t_0\leq v\leq t}\mathbb{E}\|z\|_v^p &\leq 2^{p-1}K^pC^p\mathbb{E}[\|\upsilon\|^p] \\
&\quad+2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_0)\int_{t_0}^{t}\varrho_1(s)(n+\varrho_2(s))W\left[\mathbb{E}\left(\|z\|_s^p\right)\right]ds. \\
&\leq 2^{p-1}K^pC^p\mathbb{E}[\|\upsilon\|^p] \\
&\quad+2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_0)\int_{t_0}^{t}\varrho_1(s)(n+\varrho_2(s))\sup_{t_0\leq v\leq t}W\left[\mathbb{E}\left(\|z\|_v^p\right)\right]d
\end{aligned}$$

Let us define the function $\pounds(t)$ as follows:

$$\pounds(t)=\sup_{t_0\leq v\leq t}\mathbb{E}\left[\|z\|_v^p\right], t\in\left[t_0,\mathcal{T}\right].$$

Then, for any $t\in[t_0,\mathcal{T}]$, it follows that

$$\pounds(t)\leq 2^{p-1}K^pC^p\mathbb{E}[\|\upsilon\|^p]+2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_0)\int_{t_0}^{t}\varrho_1(s)(n+\varrho_2(s))W\left(\pounds(t)\right)ds. \qquad (4)$$

From the above inequality (4), the right side is denoted by $\rho(t)$, we get

$$£(t) \le \rho(t), t \in [t_0, \mathcal{T}], \rho(t_0) = 2^{p-1} K^p C^p \mathbb{E}\|\upsilon\|^p = c_0.$$

$$\rho'(t) = 2^{p-1} K^p \max\{1, C^p\}(\mathcal{T} - t_0)\varrho_1(s)(n + \varrho_2(s))W(£(t))$$

$$\le 2^{p-1} K^p \max\{1, C^p\}(\mathcal{T} - t_0)\varrho_1(s)(n + \varrho_2(s))W(\rho(t))$$

$$\frac{\rho'(t)}{W(\rho(t))} \le \varrho^*(t), t \in [t_0, \mathcal{T}].$$

This implies

$$\int_{\rho(t_0)}^{\rho(t)} \frac{ds}{W(s)} \le \int_{t_0}^{\mathcal{T}} M_*(t)ds < \int_{c_0}^{\infty} \frac{ds}{W(s)}, t \in [t_0, \mathcal{T}] \tag{5}$$

where the last inequality is given by (2). From (5) and by mean value theorem, there is a constant η_1 such that $\rho(t) \le \eta_1$, and here $£(t) \le \eta_1$. Since $£(t) = \sup_{t_0 \le v \le t} \mathbb{E}[\|z\|_v^p]$ holds for every $t \in [t_0, \mathcal{T}]$, we have $\eta_1 \ge \sup_{t_0 \le v \le \mathcal{T}} \mathbb{E}[\|z\|_v^p]$ where η_1 depends only on $\mathcal{T}$ and the functions ρ_2 and W

$$\eta_1 \ge \sup_{t_0 \le v \le \mathcal{T}} \mathbb{E}\|z\|_v^p = \mathbb{E}\|z\|_G^p.$$

Moreover, we will present the prove φ is continuous and completely continuous.

(I) To prove φ is continuous.

Let $\{z_i\}$ be a convergent sequence of elements in G. Then, for every $t \in [t_0, \mathcal{T}]$, we have

$$\begin{aligned}
\phi z_i(t) = \sum_{q=0}^{\infty} \Bigg[\prod_{\iota=1}^{q} a_\iota(\tau_\iota) \mathfrak{U}(t, t_0)\upsilon(0) \\
+ \sum_{\iota=1}^{q} \prod_{\jmath=\iota}^{q} a_\jmath(\tau_\jmath) \int_{\zeta_{\iota-1}}^{\zeta_\iota} \mathfrak{U}(t,s)\mathcal{H}\Big(s, z_i(\sigma_1(s)), ..., z_i(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z_i(\sigma_{n+1}(r)))dr\Big)ds \\
+ \int_{\zeta_q}^{t} \mathfrak{U}(t,s)\mathcal{H}\Big(s, z_i(\sigma_1(s)), ..., z_i(\sigma_n(s)) \int_{t_0}^{s} g(s, r, z_i(\sigma_{n+1}(r)))dr\Big)ds\Bigg] I_{[\zeta_q, \zeta_{q+1})}(t).
\end{aligned}$$

$$\begin{aligned}
\phi z_i(t) - \phi z(t) = \sum_{q=0}^{\infty} \Bigg[\sum_{\iota=1}^{q} \Bigg[\prod_{\jmath=\iota}^{q} a_\jmath(\tau_\jmath) \int_{\zeta_{\iota-1}}^{\zeta_\iota} \mathfrak{U}(t,s)\Big\{\mathcal{H}\Big(s, z_i(\sigma_1(s)), ..., z_n(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z_i(\sigma_{n+1}(r)))dr\Big) \\
- \mathfrak{U}(t,s)\mathcal{H}\Big(s, z_i(\sigma_1(s)), ..., z_i(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z_i(\sigma_{n+1}(r)))dr\Big)\Big\}ds + \int_{\zeta_q}^{t} \{\mathfrak{U}(t,s)\mathcal{H}\big(s, z_i(\sigma_1(s)), \\
\int_{t_0}^{s} g(s, r, z_i(\sigma_{n+1}(r)))dr\Big) - \mathfrak{U}(t,s)\mathcal{H}\Big(s, z_i(\sigma_1(s)), ..., z_i(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z_i(\sigma_{n+1}(r)))dr\Big)\Big\}ds] I_{[\zeta_q, \zeta_{q+1})}(t)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\|\phi z_n - \phi z\|_t^p \le K^p \max\{1, C^p\}(\mathcal{T} - t_\iota)\int_{t_0}^{t} \mathbb{E}\Bigg\|\Big[\mathcal{H}\Big(s, z_i(\sigma_1(s)), ..., z_i(\sigma_i(s)), \int_{t_0}^{s} g(s, r, z_i(\sigma_{n+1}(r)))dr\Big) \\
- \mathcal{H}\Big(s, z_i(\sigma_1(s)), ..., z_i(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z_i(\sigma_{n+1}(r)))dr\Big)\Big]\Bigg\|^p ds \\
\to 0 \text{ as } i \to \infty.
\end{aligned}$$

Therefore, ϕ is clearly continuous.

(II) To prove ϕ is completely continuous operator. Let us denote

$$\mathbb{G}_m = \left\{ z \in G / \|z\|_G^p \le m \right\}, m \ge 0.$$

(III) To prove ϕ maps $\mathbb{G}_m$ into an equicontinuous family. Let $x \in \mathbb{G}_m$ and $t_1, t_2 \in [t_0, \mathcal{T}]$. If $t_0 < t_1 < t_2 < \mathcal{T}$, then by assumptions (A1)-(A5) and inequality (2), we have

$$\begin{aligned}
\phi z(t_1) - \phi z(t_2) = \sum_{q=0}^{\infty} \Bigg[& \prod_{\iota=1}^{q} a_\iota(\tau_\iota) \mathfrak{U}(t, t_0) \upsilon(0) \\
& + \sum_{\iota=1}^{q} \prod_{\jmath=\iota}^{q} a_\jmath(\tau_\jmath) \int_{\zeta_{\iota-1}}^{\zeta_\iota} \mathfrak{U}(t,s) \mathcal{H}\left(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r))) dr \right) ds \\
& + \int_{\zeta_q}^{t_1} \mathfrak{U}(t,s) \mathcal{H}\left(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r))) dr \right) ds \Bigg] I_{[\zeta_q, \zeta_{q+1})}(t_1) \\
& - \sum_{q=0}^{\infty} \Bigg[\prod_{\iota=1}^{q} a_\iota(\tau_\iota) \mathfrak{U}(t, t_0) \upsilon(0) \\
& + \sum_{\jmath=1}^{q} \prod_{\jmath=\iota}^{q} a_\jmath(\tau_\jmath) \int_{\zeta_{\iota-1}}^{\zeta_\iota} \mathfrak{U}(t,s) \mathcal{H}\left(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r))) dr \right) ds \\
& + \int_{\zeta_q}^{t_2} \mathfrak{U}(t,s) \mathcal{H}\left(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r))) dr \right) ds \Bigg] I_{[\zeta_q, \zeta_{q+1})}(t_2).
\end{aligned}$$

Thus,

$$\begin{aligned}
\phi z(t_1) - \phi z(t_2) = \sum_{q=0}^{\infty} \Bigg[& \prod_{\iota=1}^{q} a_\iota(\tau_\iota) \mathfrak{U}(t, t_0) \upsilon(0) \\
& + \sum_{\iota=1}^{q} \prod_{\jmath=\iota}^{q} a_\jmath(\tau_\jmath) \int_{\zeta_{\iota-1}}^{\zeta_\iota} \mathfrak{U}(t,s) \mathcal{H}\left(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r))) dr \right) ds \\
& + \int_{\zeta_q}^{t_1} \mathfrak{U}(t,s) \mathcal{H}\left(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r))) dr \right) ds \Bigg] \left(I_{[\zeta_q, \zeta_{q+1})}(t_1) - I_{[\zeta_q, \zeta_{q+1})}(t_2) \right) \\
& + \sum_{q=0}^{\infty} \Bigg[\sum_{\iota=1}^{q} \prod_{\jmath=\iota}^{q} a_\jmath(\tau_\jmath) \int_{t_1}^{t_2} \mathfrak{U}(t,s) \mathcal{H}\left(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r))) dr \right) ds \Bigg] I_{[\zeta_q, \zeta_{q+1})}(t_2).
\end{aligned}$$

Then,

$$\mathbb{E}\left\| \phi z(t_1) - \phi z(t_2) \right\|^p \le 2^{p-1} \mathbb{E}\left\| I_1 \right\|^p + 2^{p-1} \mathbb{E}\left\| I_2 \right\|^p \tag{6}$$

where

$$\begin{aligned}
I_1 = \sum_{q=0}^{\infty} \Bigg[& \prod_{\iota=1}^{q} a_\iota(\tau_\iota) \mathfrak{U}(t - t_0) \upsilon(0) \\
& + \sum_{\iota=1}^{q} \prod_{\jmath=\iota}^{q} a_\jmath(\tau_\jmath) \int_{\zeta_{\iota-1}}^{\zeta_\iota} \mathfrak{U}(t,s) \mathcal{H}\left(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r))) dr \right) ds \\
& + \int_{\zeta_q}^{t_1} \mathfrak{U}(t,s) \mathcal{H}\left(s, z(\sigma_1(s)), \ldots, z(\sigma_n(s)), \int_{t_0}^{s} g(s, r, z(\sigma_{n+1}(r))) dr \right) ds \Bigg] \left(I_{[\zeta_q, \zeta_{q+1})}(t_1) - I_{[\zeta_q, \zeta_{q+1})}(t_2) \right),
\end{aligned}$$

and

$$I_2=\sum_{q=0}^{\infty}\left[\sum_{\iota=1}^{q}\prod_{\jmath=\iota}^{q}a_\jmath(\tau_\jmath)\int_{t_1}^{t_2}\mathfrak{U}(t,s)\mathcal{H}\left(s,z(\sigma_1(s),...,z(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1}(r)))dr\right)ds\right]I_{[\zeta_q,\zeta_{q+1})}(t_2).$$

$$\begin{aligned}\mathbb{E}\|I_1\|^p &\le 2^{p-1}C^pK^p\mathbb{E}\|\upsilon(0)\|^p\,\mathbb{E}\left(I_{[\zeta_q,\zeta_{q+1})}(t_1)-I_{[\zeta_q,\zeta_{q+1})}(t_2)\right)+2^{p-1}K^p\max\{1,C^p\}(t_1-t_0)\\
&\times\mathbb{E}\int_{t_0}^{t_1}\left\|\mathcal{H}\left(s,z(\sigma_1(s),...,z(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1}(r)))dr\right)\right\|^p ds\mathbb{E}\left(I_{[\zeta_q,\zeta_{q+1})}(t_1)-I_{[\zeta_q,\zeta_{q+1})}(t_2)\right)\\
&\le 2^{p-1}K^pC^p\mathbb{E}\|\upsilon(0)\|^p\,\mathbb{E}\left(I_{[\zeta_q,\zeta_{q+1})}(t_1)-I_{[\zeta_q,\zeta_{q+1})}(t_2)\right)+2^{p-1}K^p\max\{1,C^p\}(t_1-t_0)\\
&\times\int_{t_0}^{t_1}\varrho_1(s)W\left(n\mathbb{E}\|z\|_s^p+\int_{t_0}^{s}\varrho_2(s)\mathbb{E}\|z\|_s^p\right)ds\mathbb{E}\left(I_{[\zeta_q,\zeta_{q+1})}(t_1)-I_{[\zeta_q,\zeta_{q+1})}(t_2)\right)\\
&\le 2^{p-1}K^pC^p\mathbb{E}\|\upsilon(0)\|^p\,\mathbb{E}\left(I_{[\zeta_q,\zeta_{q+1})}(t_1)-I_{[\zeta_q,\zeta_{q+1})}(t_2)\right)+2^{p-1}K^p\max\{1,C^p\}(t_1-t_0)\\
&\times\int_{t_0}^{t_1}M_1W\left(\mathbb{E}\|z\|_s^p\right)ds\mathbb{E}\left(I_{[\zeta_q,\zeta_{q+1})}(t_1)-I_{[\zeta_q,\zeta_{q+1})}(t_2)\right)\\
&\to 0 \text{ as } t_2\to t_1, \qquad (7)\end{aligned}$$

where $M_1=\sup\{\varrho_1(t)(n+\varrho_2(t))\}$

$$\begin{aligned}\mathbb{E}\|I_2\|^p &\le K^pC^p(t_2-t_1)(\mathbb{E}\int_{t_1}^{t_2}\left\|\mathcal{H}\left(s,z(\sigma_1(s),\ldots,z)(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1})(r)dr)\right)\right\|^p ds\\
&\le K^pC^p(t_2-t_1)\int_{t_1}^{t_2}M_1\varrho_1(s)W\left(n\mathbb{E}\|z\|_s^p+\int_{t_0}^{s}\varrho_2(s)\mathbb{E}\|z\|_s^p\right)ds\\
&\to 0 \text{ as } t_2\to t_1. \qquad (8)\end{aligned}$$

The right side of equations (7) and (8) is independent of $z\in\mathbb{G}_m$. It shows that the right side of (5) tends to zero as $t_2\to t_1$. Thus, ϕ maps $\mathbb{G}_m$ into an equicontinuous family of functions.

(IV) To prove $\phi\mathbb{G}_m$ is uniformly bounded. From (2), $\|z\|_G^p\le m$ and by (A1)-(A5) it yields that

$$\|\phi z(t)\|^p\le 2^{p-1}\max_q\left\{\prod_{\iota=1}^{q}\|a_\iota(\tau_\iota)\|^p\|a_\iota(\tau_\iota)\|^p\right\}\|\mathfrak{U}(t,t_0)\|^p\|\upsilon(0)\|^p+2^{p-1}\|\mathfrak{U}(t,s)\|^p\left[\max_{\iota,q}\left\{1,\prod_{\iota=1}^{q}\|a_\iota(\tau_\iota)\|\right\}\right]^p$$
$$\left[\sum_{q=0}^{\infty}\int_{t_0}^{t}\left\|\mathcal{H}(s,z(\sigma_1(s),...,z(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1}(r)))dr)\right\|dsI_{[\zeta_q,\zeta_{q+1})}(t)\right]^p.$$

Thus,

$$\begin{aligned}\mathbb{E}\|\phi z\|_t^p &\le 2^{p-1}K^pC^p\mathbb{E}\|\upsilon(0)\|^p+2^{p-1}K^p\max\{1,C^p\}\\
&\times(\mathcal{T}-t_0)\left(\int_{t_0}^{t}\mathbb{E}\left\|\mathcal{H}\left(s,z(\sigma_1(s),...,z(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1}(r)))dr\right)\right\|^p ds\right)\\
&\le 2^{p-1}K^pC^p\mathbb{E}\|\upsilon(0)\|^p+2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_0)^2\|\alpha_m\|_{L^1}.\end{aligned}$$

This gives that the set $\{\phi z(t),\|w\|_G^p\le m\}$ is uniformly bounded, so $\phi\mathbb{G}_m$ is uniformly bounded. It is to show that $\phi\mathbb{G}_m$ is an equicontinuous collection. Now, by Arzela-Ascoli theorem, we may show ϕ maps $\mathbb{G}_m$ into a precompact set in $\mathbb{R}^n$.

(V) To prove $\phi\mathbb{G}_m$ is compact. Let $t_0 < t \le \mathcal{T}$ be fixed and ε a real number fulfilling $\varepsilon \in (0, t-t_0)$. For $z \in \mathbb{G}_m$, we define

$$(\phi_\varepsilon z)(t) = \sum_{q=0}^{\infty}\left[\prod_{\iota=1}^{q} a_\iota(\tau_\iota)\mathfrak{U}(t,t_0)\upsilon(0)\right.$$
$$+\sum_{\iota=1}^{q}\prod_{j=\iota}^{q} a_j(\tau_j)\int_{\zeta_{\iota-1}}^{\zeta_\iota}\mathfrak{U}(t,s)\left[\mathcal{H}\left(s, z(\sigma_1(s)),...,z(\sigma_n(s)), \int_{t_0}^{s} g(s,r,z(\sigma_{n+1}(r)))dr\right)\right]ds$$
$$\left.+\int_{\zeta_q}^{t-\varepsilon}\mathfrak{U}(t,s)\left[\mathcal{H}\left(s, z(\sigma_1(s)),...,z(\sigma_n(s)), \int_{t_0}^{s} g(s,r,z(\sigma_{n+1}(r)))dr\right)\right]ds\right] I_{[\zeta_q,\zeta_{q+1})}(t), t \in (t_0, t-\varepsilon).$$

The set

$$W_\varepsilon(t) = \left\{\phi_\varepsilon z(t) : z \in \mathbb{G}_m\right\}$$

is precompact in $\mathbb{R}^n$ for all $\varepsilon \in (0, t_0 - t)$. Using (A1)-(A5), (2), and $\mathbb{E}\|z\|_{\mathbb{G}}^p \le m$, we obtain

$$\mathbb{E}\left\|(\phi z) - (\phi_\varepsilon z)\right\|_t^p \le K^p \max\left\{1, C^p\right\}\left(\mathcal{T} - t_0\right)\int_{t-\varepsilon}^{t} M_1 W(m) ds.$$

Thus, there are precompact sets arbitrarily close to the set $\left\{\phi z(t) : z \in \mathbb{G}_m\right\}$. Hence, the set $\left\{\phi z(t) : z \in \mathbb{G}_m\right\}$ is precompact in $\mathbb{R}^n$. Therefore, ϕ is a completely continuous operator.

Furthermore, the set $U(\phi) = \left\{z \in G : z = \eta\phi z \text{ for some } 0 < \eta < 1\right\}$ is bounded.

Subsequently, by Theorem 2.1, the operator φ has a fixed point in G. Therefore, the system (1) has a solution, which completes the proof.

Theorem 3.2. Assume that there exist the functions $\varrho_1, \varrho_2 : [t_0, \mathcal{T}] \to (0, +\infty)$ are continuous such that

$$\mathbb{E}\left\|\mathcal{H}(t, z_1, \ldots z_{n+1})\right\|^p \le \varrho_1(t)\mathbb{E}\left(\left\|z_1\right\|_s^p + \ldots + \left\|z_{n+1}\right\|_s^p\right), z_\iota \in \mathbb{R}^n, \iota = 1, 2, \ldots, n+1,$$
$$\mathbb{E}\left\|g(t,s,z)ds\right\|^p \le \varrho_2(t)\mathbb{E}\|z\|_s^p, \; t, s \ge 0, \; z \in \mathbb{R}^n.$$

Under the assumptions (A3)-(A5), the system (1) has a solution $z(t)$ defined on $[t_0, \mathcal{T}]$.

Proof. Let us define operator φ as in Theorem 3.1. Then, from equation (3), we get,

$$\mathbb{E}\|z\|_t^p \le 2^{p-1}C^p K^p \mathbb{E}\left[\|\upsilon\|^p\right]$$
$$+ 2^{p-1}\max\{1, C^p\}(\mathcal{T} - t_0)\int_{t_0}^{t}\mathbb{E}\left[\left\|\mathcal{H}\left(s, z(\sigma_1(s)),...,z(\sigma_n(s)), \int_{t_0}^{s} g(s,r,z(\sigma_{n+1}(r)))dr\right)\right\|^p\right]ds$$
$$\le 2^{p-1}C^p K^p \mathbb{E}\left[\|\upsilon\|^p\right] + 2^{p-1}K^p \max\{1, C^p\}$$
$$\times(\mathcal{T} - t_0)\int_{t_0}^{t}\varrho_1(s)\left[\mathbb{E}\left\|z(\sigma_1(s))\right\|^p + \ldots + \mathbb{E}\left\|z(\sigma_n(s))\right\|^p + \int_{t_0}^{s}\varrho_2(r)\mathbb{E}\left\|z(\sigma_{n+1}(r))\right\|^p dr\right]ds$$
$$\le 2^{p-1}C^p K^p \mathbb{E}\left[\|\upsilon\|^p\right] + 2^{p-1}K^p \max\{1, C^p\}$$
$$\times(\mathcal{T} - t_0)\int_{t_0}^{t}\varrho_1(s)\left[\mathbb{E}\left\|z(s)\right\|^p + \ldots + \mathbb{E}\left\|z(s)\right\|^p + \int_{t_0}^{s}\varrho_2(r)\mathbb{E}\left\|z(r)\right\|^p dr\right]ds$$
$$\le 2^{p-1}C^p K^p \mathbb{E}\left[\|\upsilon\|^p\right] + 2^{p-1}K^p \max\{1, C^p\}(\mathcal{T} - t_0)\int_{t_0}^{t}\varrho_1(s)\left(n\mathbb{E}\|z\|_s^p + \int_{t_0}^{s}\varrho_2(r)dr\mathbb{E}\|z\|_s^p\right)ds.$$

We note that from the above inequality, the last term of the right side increases in t, we would obtain

$$\sup_{t_0\le v\le t}\mathbb{E}\|z\|_v^p \le 2^{p-1}K^pC^p\mathbb{E}[\|\upsilon\|^p]+2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_0)\int_{t_0}^t \varrho_1(s)\Big(n\mathbb{E}\|z\|_s^p+\int_{t_0}^s\varrho_2(r)\mathbb{E}\|z\|_s^p\,dr\Big)ds.$$

$$\le 2^{p-1}K^pC^p\mathbb{E}[\|\upsilon\|^p]$$

$$+2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_0)\int_{t_0}^t\varrho_1(s)\Big(n\sup_{t_0\le v\le t}\left[\mathbb{E}\|z\|_v^p\right]+\int_{t_0}^s\varrho_2(r)dr\sup_{t_0\le v\le t}\left[\mathbb{E}\|z\|_v^p\right]\Big)ds.$$

Let us define the function £(t) as follows:

$$£(t)=\sup_{t_0\le v\le t}\mathbb{E}[\|z\|_v^p],t\in[t_0,\mathcal{T}].$$

Then, for any $t\in[t_0,\mathcal{T}]$, it follows that

$$£(t)\le 2^{p-1}K^pC^p\mathbb{E}[\|\upsilon\|^p]+2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_0)\int_{t_0}^t\varrho_1(s)\left(n£(s)+\int_{t_0}^s\varrho_2(r)£(r)dr\right)ds.$$

$$£(t)\le 2^{p-1}K^pC^p\mathbb{E}[\|\upsilon\|^p]+2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_0)\left(\int_{t_0}^t n\varrho_1(s)£(s)ds+\int_{t_0}^t\varrho_1(s)\int_{t_0}^s\varrho_2(r)£(r)drds\right).$$

$$£(t)\le 2^{p-1}K^pC^p\mathbb{E}[\|\upsilon\|^p]+2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_0)\left(\int_{t_0}^t n\varrho_1(s)£(s)ds+\int_{t_0}^t\varrho_1(s)\int_{t_0}^s\varrho_2(r)£(r)drds\right). \tag{9}$$

where $c_0=2^{p-1}K^pC^p\mathbb{E}[\|\upsilon\|^p]$. Let $N(t)=\sup\{M_*\}$ and $N_*(t)=\sup\{N(t):t\in[t_0,\mathcal{T}]\}$, then from equation (9), we have

$$£(t)\le c_0+\int_{t_0}^t N(s)£(s)ds+\int_{t_0}^t N(s)\int_{t_0}^s N(r)£(r)drds. \tag{10}$$

Applying Lemma 2.1. to inequality (10), we get

$$£(t)\le c_0\left[1+\int_{t_0}^t\vartheta(s)\exp\left(\int_{t_0}^s N(r)+N(r)dr\right)ds\right]$$

$$\le c_0\left[1+N_*\exp\big(2N_*(\mathcal{T}-t_0)\big)(\mathcal{T}-t_0)\right]=\mathfrak{A}.$$

Remaining proof is similar as Theorem 3.1.

To prove the existence and uniqueness results by Banach contraction principle, we need the following hypotheses.

(A6) The function $\mathcal{H}:[t_0,\mathcal{T}]\times\mathcal{C}\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous and there exist $L_{\mathcal{H}}>0$ such that

$$\mathbb{E}\|\mathcal{H}(t,z_1,...,z_{n+1})-\mathcal{H}(t,w_1,...,w_{n+1})\|^p\le L_{\mathcal{H}}\mathbb{E}\Big(\|z_1-w_1\|_s^p+...+\|z_{n+1}-w_{n+1}\|_s^p\Big),$$
$$z_\iota,w_\iota\in\mathbb{R}^n,\iota=1,2,...,n+1.$$

(A7) Let $g:[t_0,\mathcal{T}]\times[t_0,\mathcal{T}]\times\mathcal{C}\to\mathbb{R}^n$. There exist constants $L_g>0$ such that

$$A(\mathcal{T})=2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_0)^2L_{\mathcal{H}}(n+L_g)<1.$$

Theorem 3.3. If the hypotheses (A3)-(A7) holds, then the system (1) with initial value $\upsilon(0)$ has a unique solution on $[t_0,\mathcal{T}]$ provided

$$A(\mathcal{T})=2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_0)^2L_{\mathcal{H}}(n+L_g)<1. \tag{11}$$

Proof. As shown in Theorem 3.1, the nonlinear operator ϕ maps from G into G

$$\mathbb{E}\|\phi z-\phi w\|_t^p \le 2^{p-1}K^p \max\{1,C^p\}(\mathcal{T}-t_0)\int_{t_0}^{t}\left\{\mathbb{E}\left\|\mathcal{H}\left(s,z(\sigma_1(s),...,z(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1}(r)))dr\right)\right.\right.$$

$$\left.\left.-\mathcal{H}\left(s,z(\sigma_1(s),...,z(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1}(r)))dr\right)ds\right\|^p\right\}ds$$

$$\le 2^{p-1}\max K^p\{1,C^p\}(\mathcal{T}-t_0)\int_{t_0}^{t}\Big[\mathbb{E}\|z(\sigma_1(s))-w(\sigma_1(s))\|^p+...$$

$$+\int_{t_0}^{s}\mathbb{E}\|g(s,r,z(\sigma_{n+1}(r)))-g(s,r,w(\sigma_{n+1}(r)))\|^p\,dr\Big]ds$$

$$\le 2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_0)L_{\mathcal{H}}\int_{t_0}^{t}\Big[n\mathbb{E}\|z-w\|_s^p+L_g\mathbb{E}\|z-w\|_s^p\Big]ds$$

$$\le 2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_0)L_{\mathcal{H}}\int_{t_0}^{t}(n+L_g)\mathbb{E}\|z-w\|_s^p\,ds.$$

Then, taking supremum over t, we obtain

$$\|\phi z-\phi w\|_G^p \le 2^{p-1}K^p\max\{1,C^p\}(\mathcal{T}-t_{t_0})^2L_{\mathcal{H}}(n+L_g)\|z-w\|_G^p\,ds$$

$$\le A(\mathcal{T})\|z-w\|_{\mathbb{G}}^p.$$

From (11), ϕ is a contraction on G. By the Banach contraction principle, there is a unique fixed point for ϕ in space G and this fixed point is a solution of (1).

4. Stability

Theorem 4.1. Let $z(t)$ and $\overline{z}(t)$ be solutions of the system (1) with initial values $\upsilon(0)$ and $\overline{\upsilon(0)}\in G$, respectively. If the hypotheses of Theorem 3.3 is fulfilled, then the solution of the system (1) is stable in the pth mean.

Proof: Using the hypotheses, $z(t)$ and $\overline{z}(t)$ are the two solutions of the system (1) for $t\in[t_0,\mathcal{T}]$. Then,

$$z(t)-\overline{z}(t)=\sum_{q=0}^{\infty}[\prod_{\iota=1}^{q}a_\iota(\tau_\iota)\mathfrak{U}(t,t_0)[\upsilon(0)-\overline{\upsilon(0)}]$$

$$+\sum_{\iota=1}^{q}\prod_{j=\iota}^{q}a_j(\tau_j)\int_{\zeta_{\iota-1}}^{\zeta_\iota}\mathfrak{U}(t,s)\Big[\mathcal{H}(s,z(\sigma_1(s)),...,z(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1}(r)))dr$$

$$-\mathcal{H}(s,\overline{z}(\sigma_1(s)),...,\overline{z}(\sigma_n(s)),\int_{t_0}^{s}g(s,r,\overline{z}(\sigma_{n+1}(r)))dr\Big]ds$$

$$+\int_{\zeta_q}^{t}\mathfrak{U}(t,s)\Big[\mathcal{H}(s,z(\sigma_1(s)),...,z(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1}(r)))dr$$

$$-\mathcal{H}(s,\overline{z}(\sigma_1(s)),...,\overline{z}(\sigma_n(s)),\int_{t_0}^{s}g(s,r,\overline{z}(\sigma_{n+1}(r)))drds\Big]I_{[\zeta_q,\zeta_{q+1})}(t).$$

Using the assumptions (A3)-(A7), we obtain

$$\mathbb{E}\|z-\overline{z}\|_t^p \leq 2^{p-1}K^p\sum_{q=0}^{\infty}\left[\prod_{\imath=1}^{q}\|a_\imath(\tau_\imath)\|^p\mathbb{E}\left\|\upsilon(0)-\overline{\upsilon(0)}\right\|^p I_{[\zeta_q,\zeta_{q+1})}(t)\right]+2^{p-1}K^p\mathbb{E}\left[\sum_{q=0}^{\infty}\left[\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}\|a_\jmath(\tau_\jmath)\|\right.\right.$$

$$\times\int_{\zeta_{i-1}}^{\zeta_i}\left\|\mathcal{H}(s,z(\sigma_1(s)),\ldots,z(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1}(r)))dr\right.$$

$$\left.-\mathcal{H}(s,\overline{z}(\sigma_1(s)),\ldots,\overline{z}(\sigma_n(s)),\int_{t_0}^{s}g(s,r,\overline{z}(\sigma_{n+1}(r)))dr\right\|ds$$

$$+\int_{\zeta_k}^{t}\left\|\mathcal{H}(s,z(\sigma_1(s)),\ldots,z(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1}(r)))dr\right.$$

$$\left.\left.\left.-\mathcal{H}(s,\overline{z}(\sigma_1(s)),\ldots,\overline{z}(\sigma_n(s)),\int_{t_0}^{s}g(s,r,\overline{z}(\sigma_{n+1}(r)))dr\right\|ds\right]I_{[\zeta_q,\zeta_{q+1})}(t)\right]^p$$

$$\leq 2^{p-1}K^p\mathbb{E}\left\{\max_q\left\{\prod_{\imath=1}^{q}\|a_\imath(\tau_\imath)\|\right\}\right\}\mathbb{E}\left\|\upsilon(0)-\overline{\upsilon(0)}\right\|^p+2^{p-1}K^p\left[\max_{\imath,q}\left(1,\prod_{\jmath=\imath}^{q}\|a_\jmath(\tau_\jmath)\|\right)\right]^p$$

$$\times\mathbb{E}\left(\left\|\mathcal{H}(s,z(\sigma_1(s)),\ldots,z(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1}(r)))dr\right.\right.$$

$$\left.\left.-\mathcal{H}(s,\overline{z}(\sigma_1(s)),\ldots,\overline{z}(\sigma_n(s)),\int_{t_0}^{s}g(s,r,\overline{z}(\sigma_{n+1}(r)))dr\right\|dsI_{[\zeta_q,\zeta_{q+1})}(t)\right)^p$$

$$\leq 2^{p-1}K^pC^p\mathbb{E}\left\|\upsilon(0)-\overline{\upsilon(0)}\right\|^p+2^{p-1}K^p\max\{1,C^p\}(t-t_0)$$

$$\times\int_{t_0}^{t}\mathbb{E}\left\|\mathcal{H}(s,z(\sigma_1(s)),\ldots,z(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1}(r)))dr\right.$$

$$\left.-\mathcal{H}(s,\overline{z}(\sigma_1(s)),\ldots,\overline{z}(\sigma_n(s)),\int_{t_0}^{s}g(s,r,\overline{z}(\sigma_{n+1}(r)))dr\right\|^p ds$$

and

$$\sup_{t\in[t_0,\mathcal{T}]}\mathbb{E}\|z-\overline{z}\|_t^p \leq 2^{p-1}K^pC^p\mathbb{E}\left\|\upsilon(0)-\overline{\upsilon(0)}\right\|^p$$
$$+2^{p-1}\max K^p\{1,C^p\}(\mathcal{T}-t_0)L_{\mathcal{H}}(n+L_g)\int_{t_0}^{t}\sup_{s\in[t_0,\mathcal{T}]}\mathbb{E}\|z-\overline{z}\|_s^p\,ds$$

Using Grownwall's inequality,

$$\sup_{t\in[t_0,\mathcal{T}]}\mathbb{E}\|z-\overline{z}\|_t^p \leq 2^{p-1}K^pC^p\mathbb{E}\left\|\upsilon(0)-\overline{\upsilon(0)}\right\|^p$$
$$\times\exp\left[2^{p-1}\max K^p\left\{1,C^p\right\}(\mathcal{T}-t_0)^2L_{\mathcal{H}}(n+L_g)\right]$$

Here, given $\epsilon>0$, choose $\delta=\dfrac{\epsilon}{\mathfrak{T}}$ such that $\mathbb{E}\left\|\upsilon(0)-\overline{\upsilon(0)}\right\|^p<\delta$. Then, $\sup_{t\in[t_0,\mathcal{T}]}\mathbb{E}\|z-\overline{z}\|_t^p<\epsilon$.

This complete the proof.

5. UHR type stability

This section looks at the UH stability for NRIIDE with time-varying delay. Let $\varepsilon>0,\rho>0$ and $\Phi:[t_0,\mathcal{T}]\to\mathbb{R}_+$ be a piecewise continuous function. We consider the following estimates

$$\begin{cases} \mathbb{E}\left\| z'(t) - \mathcal{A}(t)z(t) + \mathcal{H}\left(t, z(\sigma_1(t)), \ldots, z(\sigma_n(t)), \int_{t_0}^{t} g(t,s,z(\sigma_{n+1}(s)))ds\right)\right\|^p < \varepsilon, t \neq \zeta_q, t \geq t_0. \\ \mathbb{E}\left\| z(\zeta_q) - a_q(\tau_k)z(\zeta_q^-)\right\|^p < \varepsilon, q = 1,2,\ldots. \end{cases} \tag{12}$$

$$\begin{cases} \mathbb{E}\left\| z'(t) - \mathcal{A}(t)z(t) + \mathcal{H}\left(t, z(\sigma_1(t)), \ldots, z(\sigma_n(t)), \int_{t_0}^{t} g(t,s,z(\sigma_{n+1}(s)))ds\right)\right\|^p < \Phi(t), t \neq \zeta_q, t \geq t_0. \\ \mathbb{E}\left\| z(\zeta_q) - a_q(\tau_k)z(\zeta_q^-)\right\|^p < \rho, q = 1,2,\ldots. \end{cases} \tag{13}$$

$$\begin{cases} \mathbb{E}\left\| z'(t) - \mathcal{A}(t)z(t) + \mathcal{H}\left(t, z(\sigma_1(t)), \ldots, z(\sigma_n(t)), \int_{t_0}^{t} g(t,s,z(\sigma_{n+1}(s)))ds\right)\right\|^p < \varepsilon\Phi(t), t \neq \zeta_q, t \geq t_0. \\ \mathbb{E}\left\| z(\zeta_q) - a_q(\tau_k)z(\zeta_q^-)\right\|^p < \varepsilon\rho, q = 1,2,\ldots. \end{cases} \tag{14}$$

Definition 5.1. The problem (1) is called UH stable in the pth mean if there is a constant $K > 0$ such that for every $\varepsilon > 0$ and for every solution $z \in G$ of the estimate (12), there is a solution $\mathfrak{D} \in G$ of (1) satisfying

$$\mathbb{E}\|z(t) - \mathfrak{D}(t)\|^p \leq K\varepsilon, t \in [t_0, \mathcal{T}].$$

Definition 5.2. The problem (1) is called generalized UH stable in the pth mean if there is a constant $\lambda \in G$, $\lambda(0) = 0$ such that for every $\varepsilon > 0$ and for every solution $z \in G$ of the estimate (12), there is a solution $\mathfrak{D} \in G$ of (1) satisfying

$$\mathbb{E}\|z(t) - \mathfrak{D}(t)\|^p \leq \lambda(\varepsilon), t \in [t_0, \mathcal{T}].$$

Definition 5.3. The problem (1) is called UHR stable in the pth mean with respect to (Φ, ρ) if there is a constant $\gamma > 0$ such that for every $\varepsilon > 0$ and for every solution $z \in G$ of the estimate (14), there is a solution $\mathfrak{D} \in G$ of (1) satisfying

$$\mathbb{E}\|z(t) - \mathfrak{D}(t)\|^p \leq \gamma\varepsilon(\Phi + \rho), t \in [t_0, \mathcal{T}].$$

Definition 5.4. The problem (1) is called generalized UHR stable in the pth mean with respect to (Φ, ρ) if there is a constant $\gamma > 0$ such that for every solution $z \in G$ of the estimate (13), there is a solution $\mathfrak{D} \in G$ of (1) satisfying

$$\mathbb{E}\|z(t) - \mathfrak{D}(t)\|^p \leq \gamma(\Phi + \rho), t \in [t_0, \mathcal{T}].$$

Remark 5.1. It is clear that

1. Inequality (12) ⇒ Inequality (13).
2. Inequality (14) ⇒ Inequality (15).
3. Inequality (14) for $\Phi(t) = \rho = 0$ ⇒ Inequality (12).

Remark 5.2. A function $z \in G$ is a solution of the estimate (10) if and only if there is a function $P \in G$ and the sequence P_q, $q = 1,2,\ldots$ (which depend on x) such that

(I) $\mathbb{E}\|P(t)\|^p \leq \varepsilon\Phi(t), t \in [t_0, \mathcal{T}]$ and $\mathbb{E}\|P_q(t)\|^p \leq \varepsilon\rho, q = 1,2,\ldots,$

(II) $z'(t) = \mathcal{A}(t)z(t) + \mathcal{H}(t, z(\sigma_1(t)), \ldots, z(\sigma_n(t)), \int_0^t g(t,s,z(\sigma_{n+1}(s))ds + P(t), t \neq \zeta_q, t \geq t_0.$

(III) $z(\zeta_q) = a_q(\tau_q)z(\zeta_q^-) + P_q, q = 1,2,\ldots$

Similar arguments can be made about inequality (12) and (13).

Remark 5.3. If $z \in G$ is a solution of the estimate (14), then z is a solution of the following integral estimate

$$\mathbb{E}\Bigg\| z(t)-\sum_{q=0}^{\infty}\Bigg[\prod_{\imath=1}^{q} a_{\jmath}(\tau_{\jmath})\mathfrak{U}(t,t_0)\upsilon(0)$$

$$+\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} a_{\jmath}(\tau_{\jmath})\int_{\zeta_{\imath-1}}^{\zeta_{\imath}}\mathfrak{U}(t,s)\Big[\mathcal{H}\Big(s,z(\sigma_1(s)),...,z(\sigma_n(s)),\int_{t_0}^{s} g(s,r,z(\sigma_{n+1}(r))\Big)dr\Big]ds$$

$$+\int_{\zeta_k}^{t}\mathfrak{U}(t,s)\Big[\mathcal{H}(s,z(\sigma_1(s)),...,z(\sigma_n(s)),\int_{t_0}^{s} g(s,r,z(\sigma_{n+1}(r)))dr\Big]ds\Bigg] I_{[\zeta_q,\zeta_{q+1})}(t)\Bigg\|^p$$

$$\le 2^{p-1}K^p\varepsilon\Big\{e^p\rho+\max\{1,e^p\}(\mathcal{T}-t_0)\int_{t_0}^{t}\Phi(t)ds\Big\},\ t\in[t_0,\mathcal{T}].$$

From above remark, we get

$$\begin{cases} z'(t)=\mathcal{A}(t)z(t)+\mathcal{H}(t,z(\sigma_1(t)),...,z(\sigma_n(t)),\int_{t_0}^{t} g(t,s,z(\sigma_{n+1}(s)))ds+P(t),t\ne\zeta_q,t\ge t_0.\\ z(\zeta_q)=a_q(\tau_q)z(\zeta_q^-)+P_q,\ q=1,2,... \end{cases} \quad (15)$$

Then,

$$z(t_0+s)=\upsilon(s)\ \text{when}\ t\in[-\rho,0]$$

$$z(t)=\sum_{q=0}^{\infty}\Bigg[\prod_{\imath=1}^{q} a_{\imath}(\tau_{\imath})\mathfrak{U}(t,t_0)\upsilon(0)+\prod_{\imath=1}^{q} a_{\imath}(\tau_{\imath})\mathfrak{U}(t,t_0)g_{\imath}$$

$$+\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} a(\tau_{\jmath})\int_{\zeta_{\imath-1}}^{\zeta_{\imath}}\mathfrak{U}(t,s)\Big[\mathcal{H}(s,z(\sigma_1(s)),\ldots z(\sigma_1(s)),\int_{t_0}^{s} g(s,r,z(\sigma_{n+1}(r)))dr)ds$$

$$+\int_{\zeta_q}^{t}\mathfrak{U}(t,s)\mathcal{H}(s,z(\sigma_1)(s)),\ldots,z\Big(\sigma_n(s),\int_{t_0}^{s} g(s,r,z(\sigma_{n+1}(r)))dr\Big)ds$$

$$+\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} a(\tau_{\jmath})\mathfrak{U}(t,s)\int_{\zeta_{\imath-1}}^{\zeta_{\imath}} P(s)ds+\int_{\zeta_q}^{t}\mathfrak{U}(t,s)P(s)ds\Bigg] I_{[\zeta_q,\zeta_{q+1})}(t),t\in[t_0,\mathcal{T}].$$

Therefore,

$$\mathbb{E}\Bigg\| z(t)-\sum_{q=0}^{\infty}\Bigg[\prod_{\imath=1}^{q} a_{\jmath}(\tau_{\jmath})\mathfrak{U}(t,t_0)\upsilon(0)$$

$$+\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} a_{\jmath}(\tau_{\jmath})\int_{\zeta_{i-1}}^{\zeta_{i}}\mathfrak{U}(t,s)\Big[\mathcal{H}\Big(s,z(\sigma_1(s)),...,z(\sigma_n(s)),\int_{t_0}^{s} g(s,r,z(\sigma_{n+1}(r))\Big)dr\Big]ds$$

$$+\int_{\zeta_k}^{t}\mathfrak{U}(t,s)\Big[\mathcal{H}\Big(s,z(\sigma_1(s)),...,z(\sigma_n(s)),\int_{t_0}^{s} g(s,r,z(\sigma_{n+1}(r))\Big)dr\Big]ds\Bigg] I_{[\zeta_q,\zeta_{q+1})}(t)\Bigg\|^p$$

$$=\mathbb{E}\Bigg\|\sum_{q=0}^{\infty}\Big[\prod_{\imath=1}^{q} a_{\jmath}(\tau_{\jmath})\mathfrak{U}(t,t_0)P_i+\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} a_{\jmath}(\tau_{\jmath})\int_{\zeta_{i-1}}^{\zeta_{i}}\mathfrak{U}(t,s)P(s)ds+\int_{\zeta_q}^{t}\mathfrak{U}(t,s)P(s)ds\Big] I_{[\zeta_q,\zeta_{q+1})}(t)\Bigg\|^p$$

$$\le 2^{p-1}K^p\mathbb{E}\Big\{\max_{q}\Big\{\prod_{\imath=1}^{q}\|a_{\imath}(\tau_{\imath})\|^p\Big\}\Big\}\mathbb{E}\|P_i\|^p+2^{p-1}K^p\mathbb{E}\Bigg[\max_{q}\Big\{\prod_{\imath=1}^{q}\|a_{\imath}(\tau_{\imath})\|^p\Big\}\Bigg]^p(\mathcal{T}-t_0)\int_{t_0}^{t}\mathbb{E}\|P(t)\|^p ds$$

$$\le 2^{p-1}K^p\varepsilon\Big\{e^p\rho+\max\{1,e^p\}(\mathcal{T}-t_0)\int_{t_0}^{t}\Phi(t)ds\Big\},\ t\in[t_0,\mathcal{T}].$$

For the solutions of the inequalities (12) and (13), we have similar remarks. In this section, we present the primary results, UHR results.

Theorem 5.1. Assumption (A3)-(A7) hold. Let $\Phi \in ([t_0, \mathcal{T}], \mathbb{R}_+)$ is continuous increasing, and there is a constant $\eta > 0$ such that

$$\int_0^t \Phi(s)ds \leq \eta\Phi(t), \text{ for } t \in [t_0, \mathcal{T}].$$

Then, the problem (1) is UHR stable in the pth mean.

Proof. Let $z \in G$ be a solution of the estimate (14). By Theorem 3.3, there is a unique solution w of the system NRIIDE with time-varying delays

$$\begin{cases} \mathfrak{w}'(t) = \mathcal{A}(t)\mathfrak{w}(t) + \mathcal{H}\left(t, \mathfrak{w}(\sigma_1(t)), ..., \mathfrak{w}(\sigma_n(t)), \int_{t_0}^t g(t, s, \mathfrak{w}(\sigma_{n+1}(s)))ds\right), t \geq \tau, t \neq \zeta_q, \\ \mathfrak{w}(\zeta_q) = a_q(\tau_q)\mathfrak{w}(\zeta_q^-), q = 1, 2, ..., \\ \mathfrak{w}_{t_0} = \upsilon. \end{cases}$$

Then, we have $\mathfrak{w}(t_0 + s) = \upsilon(s)$ when $s \in [-\rho, 0]$,

$$\begin{aligned} \mathfrak{w}(t) = \sum_{q=0}^{\infty}\Bigg[&\prod_{\imath=1}^{q} a_\imath(\tau_\imath)\mathfrak{U}(t, t_0)\upsilon(0) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} a_\jmath(\tau_\jmath)\int_{\zeta_{\imath-1}}^{\zeta_\imath} \mathfrak{U}(t, s)\mathcal{H}\left(s, \mathfrak{w}(\sigma_1(s)), ..., \mathfrak{w}(\sigma_n(s)), \int_{t_0}^s g(s, r, \mathfrak{w}(\sigma_{n+1}(r)))dr\right)ds \\ &+ \int_{\zeta_q}^t \mathfrak{U}(t, s)\mathcal{H}\left(s, \mathfrak{w}(\sigma_1(s)), ..., \mathfrak{w}(\sigma_n(s)), \int_{t_0}^s g(s, r, \mathfrak{w}(\sigma_{n+1}(r)))dr\right)ds\Bigg] I_{[\zeta_q, \zeta_{q+1})}(t), t \in [t_0, \mathcal{T}]. \end{aligned}$$

By differential inequality (14), we have

$$\begin{aligned} \mathbb{E}\Bigg\| z(t) - \sum_{q=0}^{\infty}\Bigg[&\prod_{\imath=1}^{q} a_\jmath(\tau_\jmath)\mathfrak{U}(t, t_0)\upsilon(0) \\ &+ \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} a_\jmath(\tau_\jmath)\int_{\zeta_{i-1}}^{\zeta_i} \mathfrak{U}(t, s)\left[\mathcal{H}\left(s, z(\sigma_1(s)), ..., z(\sigma_n(s)), \int_{t_0}^s g(s, r, z(\sigma_{n+1}(r)))dr\right]ds \\ &+ \int_{\zeta_k}^t \mathfrak{U}(t, s)\left[\mathcal{H}(s, z(\sigma_1(s)), ..., z(\sigma_n(s)), \int_{t_0}^s g(s, r, z(\sigma_{n+1}(r)))dr\right]ds\Bigg] I_{[\zeta_q, \zeta_{q+1})}(t)\Bigg\|^p \\ = \mathbb{E}\Bigg\| &\sum_{q=0}^{\infty}\left[\prod_{\imath=1}^{q} a_\jmath(\tau_\jmath)\mathfrak{U}(t, t_0)P_i + \sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} a_\jmath(\tau_\jmath)\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(t, s)P(s)ds + \int_{\zeta_q}^t \mathfrak{U}(t, s)P(s)ds\right]ds\Bigg] I_{[\zeta_q, \zeta_{q+1})}(t)\Bigg\|^p \\ \leq 2^{p-1}&K^p\mathbb{E}\left\{\max_q\left\{\prod_{\imath=1}^{q}\|a_\imath(\tau_\imath)\|^p\right\}\right\}\mathbb{E}\|P_i\|^p + 2^{p-1}K^p\mathbb{E}\left[\max_q\left\{\prod_{\imath=1}^{q}\|a_\imath(\tau_\imath)\|^p\right\}\right]^p \varepsilon\int_{t_0}^t \Phi(s)ds \\ \leq 2^{p-1}&K^p\varepsilon\left\{e^p\rho + \max\{1, e^p\}(\mathcal{T} - t_0)\eta\Phi(t)ds\right\}, t \in [t_0, \mathcal{T}]. \end{aligned}$$

Hence, for all $t \in [t_0, \mathcal{T}]$, we get

$$\mathbb{E}\left\|z(t)-\mathfrak{w}(t)\right\|^{p}=\mathbb{E}\left\|z(t)-\sum_{q=0}^{\infty}\left[\prod_{\imath=1}^{q}a_{\jmath}(\tau_{\jmath})\mathfrak{U}(t,t_0)\upsilon(0)\right.\right.$$

$$+\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}a_{\jmath}(\tau_{\jmath})\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(t,s)\left[\mathcal{H}(s,\mathfrak{w}(\sigma_1(s)),...,\mathfrak{w}(\sigma_n(s)),\int_{t_0}^{s}g(s,r,\mathfrak{w}(\sigma_{n+1}(r)))dr\right]ds$$

$$\left.\left.+\int_{\zeta_q}^{t}\mathfrak{U}(t,s)\left[\mathcal{H}(s,\mathfrak{w}(\sigma_1(s)),...,\mathfrak{w}(\sigma_n(s)),\int_{t_0}^{s}g(s,r,\mathfrak{w}(\sigma_{n+1}(r)))dr\right]ds\right]I_{[\zeta_q,\zeta_{q+1})}(t)\right\|^{p}$$

$$\|z-\mathfrak{w}\|_t^p\le 2^{p-1}\mathbb{E}\left\|z(t)-\sum_{q=0}^{\infty}\left[\prod_{\imath=1}^{q}a_{\jmath}(\tau_{\jmath})\mathfrak{U}(t,t_0)\upsilon(0)\right.\right.$$

$$+\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}a_{\jmath}(\tau_{\jmath})\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(t,s)\left[\mathcal{H}\left(s,\mathfrak{w}(\sigma_1(s)),...,w(\sigma_n(s)),\int_{t_0}^{s}g(s,r,\mathfrak{w}(\sigma_{n+1}(r))\right)dr\right]ds$$

$$\left.\left.+\int_{\zeta_q}^{t}\mathfrak{U}(t,s)\left[\mathcal{H}(s,\mathfrak{w}(\sigma_1(s)),...,\mathfrak{w}(\sigma_n(s)),\int_{t_0}^{s}g(s,r,\mathfrak{w}(\sigma_{n+1}(r)))dr\right]ds\right]I_{[\zeta_q,\zeta_{q+1})}(t)\right\|^{p}$$

$$+2^{p-1}\mathbb{E}\left\|\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q}a_{\jmath}(\tau_{\jmath})\int_{\zeta_{i-1}}^{\zeta_i}\mathfrak{U}(t,s)\left(\mathcal{H}(s,z(\sigma_1(s)),...,z(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1}(r)))dr\right.\right.$$

$$\left.-\mathcal{H}\left(s,\mathfrak{w}(\sigma_1(s)),...,\mathfrak{w}(\sigma_n(s)),\int_{t_0}^{s}g(s,r,\mathfrak{w}(\sigma_{n+1}(r)))dr\right)ds\right.$$

$$+\int_{\zeta_q}^{t}\mathfrak{U}(t,s)\left(\mathcal{H}(s,z(\sigma_1(s)),...,z(\sigma_n(s)),\int_{t_0}^{s}g(s,r,z(\sigma_{n+1}(r)))dr\right.$$

$$\left.\left.-\mathcal{H}(s,\mathfrak{w}(\sigma_1(s)),...,\mathfrak{w}(\sigma_n(s)),\int_{t_0}^{s}g(s,r,\mathfrak{w}(\sigma_{n+1}(r)))dr\right)dsI_{[\zeta_q,\zeta_{q+1})}(t)\right\|^{2}$$

$$\le 4^{p-1}K^p\varepsilon\left\{e^p\rho+\max\{1,e^p\}(\mathcal{T}-t_0)\eta\Phi(t)\right\}$$

$$+2^{p-1}K^p\max\{1,e^p\}(\mathcal{T}-t_0)L_{\mathcal{H}}\int_{t_0}^{t}(n+L_g)\mathbb{E}\|z-\mathfrak{w}\|_t^2\,ds.$$

$$\sup_{t\in[t_0,\mathcal{T}]}\mathbb{E}\|z-\mathfrak{w}\|_t^p\le 4^{p-1}K^p\varepsilon\left\{e^p\rho+\max\{1,e^p\}(\mathcal{T}-t_0)\eta\Phi(t)\right\}$$

$$+2^{p-1}K^p\max\{1,e^p\}(\mathcal{T}-t_0)L_{\mathcal{H}}\int_{t_0}^{t}(n+L_g)\sup_{t\in[t_0,\mathcal{T}]}\mathbb{E}\|z-\mathfrak{w}\|_s^2\,ds.$$

$$\le 4^{p-1}K^p\varepsilon\left\{e^p\rho+\max\{1,e^p\}(\mathcal{T}-t_0)\eta\Phi(t)\right\}$$

$$+2^{p-1}K^p\max\{1,e^p\}(\mathcal{T}-t_0)L_{\mathcal{H}}\int_{t_0}^{\mathcal{T}}(n+L_g)ds\sup_{t\in[t_0,\mathcal{T}]}\mathbb{E}\|z-\mathfrak{w}\|_t^2.$$

There exists a real number $\mathfrak{P}=\dfrac{1}{1-2^{p-1}K^p\max\{1,e^p\}(\mathcal{T}-t_0)L_{\mathcal{H}}\int_{t_0}^{\mathcal{T}}(n+L_g)ds}>0$ independent of $\eta\Phi(t)$ such that

$$\sup_{t\in[t_0,\mathcal{T}]}\mathbb{E}\|z-\mathfrak{w}\|_t^p\le\mathfrak{P}4^{p-1}K^p\varepsilon\left\{e^p\rho+\max\{1,e^p\}(\mathcal{T}-t_0)\eta\Phi(t)\right\},\ t\in[t_0,\mathcal{T}].$$

Then, the problem (1) is UHR stable in the pth mean. Thus, the proof.

6. Conclusion

In this paper, we have obtained the existence and various types of stability results for the NRIIDEs with time-varying delays using the Leray-Schauder alternative fixed-point theorem, Pachpatte's inequality, and the Banach contraction principle. A delightful existence of our results would be to discuss finite-time stability and UH-Mittag-Leffler stability for a class of NRIIDEs with time-varying delays. This will be the focus of future research.

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Conflict of interest

There is no conflict of interest in this study.

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