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Research Article

Stability and Data Dependence Results for Jungck-Type Iteration Scheme

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Abstract: We propose and study a new Jungck-type iteration scheme to approximate coincidence points of contractive mappings. The strong convergence, stability, and data dependency results have been discussed. Numerical experiments demonstrate that the newly introduced Jungck-type iteration scheme yields a higher convergence rate in comparison with other Jungck-type iteration schemes available in the literature.

***Keywords*:** strong convergence, non-linear equation, Banach space, contraction-type mappings

MSC: 47H09, 47H10, 54H25

1. Introduction

Most of the equations, whether linear algebraic equations, non-linear algebraic and transcendental equations, differential and integral equations [1, 2], non-linear optimization problems, variational inequality, equilibrium problems [3-5], etc., arising in the various physical formulations may be transformed into fixed point problems. The fixed points, common fixed points, coincidence points, and fixed points theorem in general deal with the study and solutions of the above-mentioned problems. Further, many graphical and geometrical shapes, such as fractal patterns and chaotic systems [6, 7], are discovered as fixed points of certain mappings.

Jungck [8] introduced an iteration scheme that involves the use of two coupled mappings. This scheme is very useful to approximate the common and coincidence points of the mappings. For more details on non-self mappings, the reader is referred to [9-20].

Let $(E, \|.\|)$ be a Banach space, X an arbitrary set, S, $T: X \to E$ be arbitrary non-self mapping with $T(X) \subseteq S(X)$, S is injective, $S(X)$ is a complete subspace of E, and $x_0 \in X$.

The Jungck-Mann (J-Mann) iteration scheme was introduced by Singh et al. [9] as

$$Sx_{k+1} = (1-\alpha_k)Sx_k + \alpha_k\, Tx_k, \tag{1}$$

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where $\{\alpha_k\}_{k=0}^{\infty} \subseteq [0,1]$.

Olatinwo [10] developed the Jungck-Ishikawa (J-Ishikawa) iteration scheme as

$$\begin{cases} Sy_k = (1-\beta_k)Sx_k + \beta_k Tx_k, \\ Sx_{k+1} = (1-\alpha_k)Sx_k + \alpha_k Ty_k, \end{cases} \tag{2}$$

where $\{\beta_k\}_{k=0}^{\infty}$ and $\{\alpha_k\}_{k=0}^{\infty} \subseteq [0,1]$. For $\beta_k = 0$, (2) reduces to J-Mann iteration (1).

Olatinwo [11] defined the Jungck-Noor (J-Noor) iteration scheme as

$$\begin{cases} Sz_k = (1-\gamma_k)Sx_k + \gamma_n\ Tx_k, \\ Sy_k = (1-\beta_k)Sx_k + \beta_k\ Tz_k, \\ Sx_{k+1} = (1-\alpha_k)Sx_k + \alpha_k\ Ty_k, \end{cases} \tag{3}$$

where $\{\gamma_k\}_{k=0}^{\infty}$, $\{\beta_k\}_{k=0}^{\infty}$, and $\{\alpha_k\}_{k=0}^{\infty} \subseteq [0,1]$. For $\gamma_k = 0$, (3) becomes the J-Ishikawa iteration scheme (2), and for $\gamma_k = \beta_k = 0$, (3) becomes the J-Mann iteration scheme (1).

The Jungck-Sahu [12] (J-Sahu) iteration scheme is the Jungck version of Sahu and Petruşel [21] defined as

$$\begin{cases} Sy_k = (1-\alpha_k)Sx_k + \alpha_k Tx_k, \\ Sx_{k+1} = Ty_k, \end{cases} \tag{4}$$

where $\{\alpha_k\}_{k=0}^{\infty} \subseteq [0,1]$.

Khan et al. [13] introduced a new three-step the Jungck-Khan (J-Khan) iteration scheme defined as

$$\begin{cases} Sz_k = (1-\alpha_k)Sx_k + \alpha_k Tx_k, \\ Sy_k = (1-\beta_k-\lambda_k)Sx_k + \beta_k Tz_k + \lambda_k Tx_k, \\ Sx_{k+1} = (1-\gamma_k-\zeta_k)Sx_k + \gamma_k Ty_k + \zeta_k Tx_k, \end{cases} \tag{5}$$

where $\{\gamma_k\}_{k=0}^{\infty}, \{\beta_k\}_{k=0}^{\infty}, \{\alpha_k\}_{k=0}^{\infty}, \{\lambda_k\}_{k=0}^{\infty}$, and $\{\zeta_k\}_{k=0}^{\infty} \subseteq [0,1]$, satisfying $\gamma_k + \zeta_k, \beta_k + \lambda_k \in [0,1]$.

Khan et al. [14] defined a special case of the Jungck-Khan (J-Khan Special) iteration scheme [17] as

$$\begin{cases} Sz_k = (1-\alpha_k)Sx_k + \alpha_k Tx_k, \\ Sy_k = (1-\beta_k-\lambda_k)Sx_k + \beta_k Tz_k + \lambda_k Tx_k, \\ Sx_{k+1} = Ty_k, \end{cases} \tag{6}$$

where $\{\beta_k\}_{k=0}^{\infty}, \{\alpha_k\}_{k=0}^{\infty}$, and $\{\lambda_k\}_{k=0}^{\infty} \subseteq [0,1]$, satisfying $\beta_k + \lambda_k \in [0,1]$.

Kanwar et al. [22] recently defined the geometrically constructed iteration scheme given as

$$x_{k+1} = \frac{mx_k + Tx_k}{1+m}, \tag{7}$$

where m is a non-negative real number.

Our main aim in this work is to propose a new Jungck-type iteration scheme to approximate the coincidence points of the contractive type mappings. The proposed scheme has a higher convergence rate and stability in comparison with other Jungck-type iteration schemes. Further, we have also proved the data dependence results for our proposed scheme.

2. Preliminaries

Olatinwo and Imoru [15] introduced the following contractive definition to prove the strong convergence of the

J-Mann and J-Ishikawa iterations.

$$\|Tu - Tv\| \le 2\delta \|Su - Tu\| + \delta \|Su - Sv\|, \forall u, v \in X, \tag{8}$$

where $0 \le \delta < 1$.

Olatinwo [10] proved the stability of the J-Ishikawa iteration scheme by using the following definition: there exists a real number $\delta \in [0,1)$ and continuous and monotone increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ and for all $u, v \in X$, one gets

$$\|Tu - Tv\| \le \phi(\|Su - Tu\|) + \delta \|Su - Sv\|. \tag{9}$$

Lemma 2.1. [23] If $\delta \in [0,1)$ and $\{\varepsilon_k\}_{k=0}^{\infty}$ is a sequence of positive numbers with $\lim_{k\to\infty} \varepsilon_k = 0$, then for every sequence of positive numbers $\{u_k\}_{k=0}^{\infty}$, which satisfies

$$u_{k+1} \le \delta u_k + \varepsilon_k, k = 0, 1, 2, \dots$$

one gets $\lim_{k\to\infty} u_k = 0$.

Lemma 2.2. [24] Let $\{\psi_k\}_{k=0}^{\infty}$ be a sequence of non-negative real numbers. Suppose that there exists $k_0 \in N$, such that for all $k \ge k_0$, one gets the inequality

$$\psi_{k+1} \le (1 - \theta_k)\psi_k + \theta_k \sigma_k,$$

where $\theta_k \in (0,1), \Sigma_{k=0}^{\infty} \theta_k = \infty$, and $\sigma_k \ge 0$, for all $k \in N$. Then, one gets

$$0 \le \limsup_{k\to\infty} \psi_k \le \limsup_{k\to\infty} \sigma_k.$$

Definition 2.1. [25] Assume that $\{a_k\}$ and $\{b_k\}$ are real convergent sequences with limits a and b, respectively. Then, $\{a_k\}$ converges faster than $\{b_k\}$ if $\lim_{k\to\infty} \left| \frac{a_k - a}{b_k - b} \right| = 0$.

Definition 2.2. [8, 19] Let Y be a non-empty set and f, g: $Y \to Y$ be mappings. If $p = fu = gu$ for some $u \in Y$, then u is called a coincidence point and p is called a point of coincidence of f and g. If $u = fu = gu$ for some u in Y, then u is called as common fixed point of f and g. A pair (f, g) is called weakly compatible if they commute at their coincidence point.

Definition 2.3. [9] Let $(E, \|.\|)$ be a Banach space, X an arbitrary set, S, T: $X \to E$ be non-self mapping with $T(X) \subseteq S(X)$, and $Tx = Sx = p$. For any $x_0 \in X$, let $\{Sx_k\}_{k=0}^{\infty}$ be generated by the iteration process $Sx_{k+1} = f(T, x_k)$ converges to p. Let $\{Su_k\}_{k=0}^{\infty} \subset E$ be an arbitrary sequence and set $\varepsilon_k = \|Su_{k+1} - f(T, u_k)\|, k = 0, 1, 2, \dots$.

Then, the iteration process $f(T, x_k)$ will be called (S, T)-stable if and only if $\lim_{k\to\infty} \varepsilon_k = 0$ implies that $\lim_{k\to\infty} Su_k = p$.

Definition 2.4. [13] Let (S, T), (S_1, T_1): $X \to E$ be non-self mapping pairs on an arbitrary set X with $T(X) \subseteq S(X)$ and $T_1(X) \subseteq S_1(X)$. Then, (S_1, T_1) is an approximate mapping pair of (S, T) if for all $x \in X$, and for fixed $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, one gets

$$\|Tx - T_1x\| \le \varepsilon_1,$$

$$\|Sx - S_1x\| \le \varepsilon_2.$$

3. Data dependence, stability, and strong convergence of new iteration scheme

We introduce the new Jungck-type iteration scheme based on iteration scheme (7) defined as:

For x_0 in X, the sequence $\{Sx_k\}_{k=0}^{\infty}$ in E is given by

$$\begin{cases} Sz_k = Tx_k, \\ Sy_k = Tz_k, \\ Sx_{k+1} = \dfrac{mSy_k + Ty_k}{1+m}, \end{cases} \tag{10}$$

where m is a positive real number.

Theorem 3.1. Let $(E, \|.\|)$ be a Banach space and X be an arbitrary set. Suppose S, $T: X \to E$ be non-self mappings with $T(X) \subseteq S(X)$, S is injective and $S(X)$ is a complete subspace of E. Suppose S and T have a coincidence point z, (i.e., $Sz = p = Tz$). Suppose also that T and S satisfies the contractive condition (9). Let $\{Sx_k\}_{k=0}^{\infty}$ be the new iteration process defined by (10) and $x_0 \in X$, where m is a positive real number. Then, $\{Sx_k\}_{k=0}^{\infty}$ is strongly convergent to p. Moreover, (S, T) has a unique common fixed-point p provided that $X = E$, and T and S are weakly compatible.

Proof. We will show that $\lim_{k\to\infty} Sx_k = p$. Using (10), one has

$$\begin{aligned} \|Sz_k - p\| &= \|Tx_k - p\| \\ &= \|Tx_k - Tz\| \\ &\le \phi(\|Sz - Tz\|) + \delta\|Sz - Sx_k\| \\ &= \delta\|Sx_k - p\|, \end{aligned} \tag{11}$$

Also,

$$\begin{aligned} \|Sy_k - p\| &= \|Tz_k - Tz\| \\ &\le \phi(\|Sz - Tz\|) + \delta\|Sz - Sz_k\| \\ &= \delta\|Sz_k - p\| \end{aligned} \tag{12}$$

and

$$\begin{aligned} \|Sx_{k+1} - p\| &= \left\|\frac{mSy_k + Ty_k}{1+m} - p\right\| \\ &\le \frac{m}{1+m}\|Sy_k - p\| + \frac{1}{1+m}\|Ty_k - p\| \\ &\le \frac{m}{1+m}\|Sy_k - p\| + \frac{1}{1+m}\left(\phi(\|Sz - Tz\|) + \delta\|Sz - Sy_k\|\right) \\ &= \frac{m}{1+m}\|Sy_k - p\| + \frac{\delta}{1+m}\|Sy_k - p\| \\ &= \frac{\delta + m}{1+m}\|Sy_k - p\|. \end{aligned} \tag{13}$$

Substituting (12) in (13), one gets

$$\|Sx_{k+1} - p\| \le \delta\left(\frac{\delta + m}{1+m}\right)\|Sz_k - p\|. \tag{14}$$

Using (11) in (14), we get

$$\|Sx_{k+1} - p\| \le \delta^2\left(\frac{\delta + m}{1+m}\right)\|Sx_k - p\|. \tag{15}$$

By repeating the above procedure, one gets

$$\|Sx_k - p\| \le \delta^2 \left(\frac{\delta + m}{1+m}\right)\|Sx_{k-1} - p\|$$
$$\|Sx_{k-1} - p\| \le \delta^2 \left(\frac{\delta + m}{1+m}\right)\|Sx_{k-2} - p\|$$
$$\vdots$$
$$\|Sx_1 - p\| \le \delta^2 \left(\frac{\delta + m}{1+m}\right)\|Sx_0 - p\|.$$

Using the above inequalities, one gets

$$\|Sx_{k+1} - p\| \le \delta^{2(k+1)} \left(\frac{\delta + m}{1+m}\right)^{k+1}\|Sx_0 - p\|. \tag{16}$$

As $\delta < 1$, one can have

$$0 < \frac{\delta + m}{1+m} < 1,$$

which implies that

$$\delta^{2(k+1)} \left(\frac{\delta + m}{1+m}\right)^{k+1} \to 0 \text{ as } k \to \infty.$$

Hence, from (16), we get $\lim_{n\to\infty}\|Sx_{k+1} - p\| = 0$. Therefore, $\{Sx_k\}$ is strongly convergent to p. Now, we will prove the uniqueness of coincidence point. Let z and z_1 be the coincidence points of T and S, such that $Tz = p = Sz$ and $Tz_1 = p_1 = Sz_1$. Suppose that $p_1 \neq p$.

$$\begin{aligned}\|p - p_1\| &= \|Tz - Tz_1\| \\ &\le \phi(\|Tz - Sz\|) + \delta\|Sz - Sz_1\| \\ &= \delta\|p - p_1\| \\ &< \|p - p_1\|,\end{aligned}$$

which is a contradiction. Thus, $p_1 = p$. Further, S is injective leads to $z_1 = z$. Hence, the pair (S, T) have a unique coincidence point z. As T and S are weakly compatible, and $Sz = p = Tp$ so $Sp = SSz = STz = TSz$ gives $Sp = Tp$. Thus, $Sp = Tp = p$ implies that S and T have a unique common fixed-point p.

Theorem 3.2. Let $(E, \|.\|)$ be a Banach space and X be an arbitrary set. Suppose $S, T: X \to E$ be non-self mappings with $T(X) \subseteq S(X)$, S is injective and $S(X)$ is a complete subspace of E. Suppose S and T has a coincidence point z, (i.e., $Sz = p = Tz$). Suppose also that T and S satisfies the contractive condition (9). Let the iterative scheme $\{Sx_k\}_{k=0}^{\infty}$ be defined by (10) converges to p. Then, $\{Sx_k\}_{k=0}^{\infty}$ is (S, T)-stable.

Proof. Suppose $\{Su_k\}_{k=0}^{\infty} \subset E$ be an arbitrary sequence, such that $\varepsilon_k = \left\|Su_{k+1} - \frac{mSv_k + Tv_k}{1+m}\right\|$, where $Sv_k = Tw_k$, $Sw_k = Tu_k$.

Let $\lim_{k\to\infty}\varepsilon_k = 0$. We will show that $\lim_{k\to\infty}Su_k = p$.

$$
\begin{aligned}
\|Su_{k+1} - p\| &\le \left\| Su_{k+1} - \frac{mSv_k + Tv_k}{1+m} \right\| + \left\| \frac{mSv_k + Tv_k}{1+m} - p \right\| \\
&= \varepsilon_k + \left\| \frac{mSv_k + Tv_k}{1+m} - p \right\| \\
&\le \varepsilon_k + \frac{m}{1+m}\|Sv_k - p\| + \frac{1}{1+m}\|Tv_k - p\| \\
&\le \varepsilon_k + \frac{m}{1+m}\|Sv_k - p\| + \frac{1}{1+m}\left(\phi\left(\|Sz - Tz\|\right) + \delta\|Sv_k - Sz\|\right) \\
&= \varepsilon_k + \frac{m}{1+m}\|Sv_k - p\| + \frac{\delta}{1+m}\|Sv_k - p\| \\
&= \varepsilon_k + \frac{\delta + m}{1+m}\|Sv_k - p\|.
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\|Sv_k - p\| &= \|Tw_k - p\| \\
&\le \phi\left(\|Sz - Tz\|\right) + \delta\|Sw_k - Sz\| \\
&= \delta\|Sw_k - Sz\| \\
&= \delta\|Sw_k - p\|.
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\|Sw_k - p\| &= \|Tu_k - p\| \\
&= \|Tu_k - Tz\| \\
&\le \phi\left(\|Sz - Tz\|\right) + \delta\|Su_k - Sz\| \\
&= \delta\|Su_k - Sz\| \\
&= \delta\|Su_k - p\|.
\end{aligned} \tag{19}
$$

Using (19) in (18), we get

$$
\|Sv_k - p\| \le \delta^2 \|Su_k - p\|. \tag{20}
$$

Putting (20) in (17), one gets

$$
\|Su_{k+1} - p\| \le \varepsilon_k + \delta^2\left(\frac{\delta + m}{1+m}\right)\|Su_k - p\|. \tag{21}
$$

As $0 < \delta < 1$, one can have $\frac{\delta + m}{1+m} < 1.$ By taking limits as $k \to \infty$ of both sides of (21) and using Lemma 2.1, we get $\lim_{k\to\infty} Su_k = p.$

Conversely, assume that $\lim_{k\to\infty} Su_k = p.$ Now, we will prove that $\lim_{k\to\infty} \varepsilon_k = 0.$

$$
\begin{aligned}
\varepsilon_k &= \left\| Su_{k+1} - \frac{mSv_k + Tv_k}{1+m} \right\| \\
&\le \left\| Su_{k+1} - p \right\| + \frac{m}{1+m}\left\| Sv_k - p \right\| + \frac{1}{1+m}\left\| Tv_k - p \right\| \\
&\le \left\| Su_{k+1} - p \right\| + \frac{m}{1+m}\left\| Sv_k - p \right\| + \frac{\delta}{1+m}\left\| Sv_k - p \right\| \\
&= \left\| Su_{k+1} - p \right\| + \frac{\delta + m}{1+m}\left\| Sv_k - p \right\| \\
&= \left\| Su_{k+1} - p \right\| + \frac{\delta + m}{1+m}\left\| Tw_k - p \right\| \\
&\le \left\| Su_{k+1} - p \right\| + \delta\left(\frac{\delta + m}{1+m}\right)\left\| Sw_k - p \right\| \\
&= \left\| Su_{k+1} - p \right\| + \delta\left(\frac{\delta + m}{1+m}\right)\left\| Tu_k - p \right\| \\
&\le \left\| Su_{k+1} - p \right\| + \delta^2\left(\frac{\delta + m}{1+m}\right)\left\| Su_k - p \right\|. \qquad (22)
\end{aligned}
$$

By taking limits as $k \to \infty$ of both sides of (22), we get $\lim_{k\to\infty} \varepsilon_k = 0$.

Theorem 3.3. Let $(E, \|.\|)$ be a Banach space, (S, T), $(S_1, T_1) : X \to E$ be non-self map, $T(Y) \subseteq S(Y), T_1(Y) \subseteq S_1(Y)$, $S_1(Y)$ and $S(Y)$ are complete subspaces of E. Suppose S and T have a coincidence point z, and S_1 and T_1 have a coincidence point z_1, i.e., $Sz = p = Tz$ and $Sz_1 = q = Tz_1$. Suppose also that T and S satisfy the contractive condition (9). Let $\{Sx_k\}_{k=0}^{\infty}$ be the iteration scheme generated by (10) converging to p, where m is a positive real number and $\{S_l a_k\}_{k=0}^{\infty}$ be the sequence defined by

$$
\begin{cases}
S_1 a_k = \dfrac{mS_1 b_k + T_1 b_k}{1+m}, \\
S_1 b_k = T_1 c_k, \\
S_1 c_k = T_1 a_k.
\end{cases}
$$

Assume that $\{S_1 a_k\}_{k=0}^{\infty}$ converges to q. Then, for $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, we have $\|p - q\| \le \dfrac{6\varepsilon}{1-\delta}$.

Proof.

$$
\begin{aligned}
\left\| S_1 a_{k+1} - Sx_{k+1} \right\| &= \left\| \frac{mS_1 b_k + T_1 b_k}{1+m} - \frac{mSy_k + Ty_k}{1+m} \right\| \\
&\le \frac{m}{1+m}\left\| S_1 b_k - Sy_k \right\| + \frac{1}{1+m}\left\| T_1 b_k - Ty_k \right\| \\
&\le \frac{m}{1+m}\left\| S_1 b_k - Sy_k \right\| + \frac{1}{1+m}\left\| T_1 b_k - Tb_k \right\| + \frac{1}{1+m}\left\| Tb_k - Ty_k \right\| \\
&\le \frac{m}{1+m}\left\| S_1 b_k - Sy_k \right\| + \frac{1}{1+m}\left(\phi\left\| Sy_k - Ty_k \right\| + \delta\left\| Sb_k - Sy_k \right\|\right) + \frac{\varepsilon_1}{1+m} \\
&\le \frac{m}{1+m}\left\| S_1 b_k - Sy_k \right\| + \frac{1}{1+m}\phi\left(Sy_k - Ty_k\right) + \frac{\delta}{1+m}\left(\left\| Sb_k - S_1 b_k \right\| + \left\| S_1 b_k - Sy_k \right\|\right) + \frac{\varepsilon_1}{1+m} \\
&\le \frac{m}{1+m}\left\| S_1 b_k - Sy_k \right\| + \frac{\delta}{1+m}\left\| S_1 b_k - Sy_k \right\| + \frac{1}{1+m}\phi\left(\left\| Sy_k - Ty_k \right\|\right) + \frac{\varepsilon_1}{1+m} + \frac{\delta\varepsilon_2}{1+m} \\
&= \frac{\delta + m}{1+m}\left\| S_1 b_k - Sy_k \right\| + \frac{1}{1+m}\phi\left(\left\| Sy_k - Ty_k \right\|\right) + \frac{\varepsilon_1}{1+m} + \frac{\delta\varepsilon_2}{1+m}. \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
\left\|S_1 b_k - S y_k\right\| &= \left\|T_1 c_k - T z_k\right\| \\
&\leq \left\|T_1 c_k - T c_k\right\| + \left\|T c_n - T z_k\right\| \\
&\leq \varepsilon_1 + \phi\left(\left\|S z_k - T z_k\right\|\right) + \delta\left\|S c_k - S z_k\right\| \\
&\leq \varepsilon_1 + \phi\left(\left\|S z_k - T z_k\right\|\right) + \delta\left\|S c_k - S_1 c_k\right\| + \delta\left\|S_1 c_k - S z_k\right\| \\
&\leq \delta\left\|S_1 c_k - S z_k\right\| + \phi\left(\left\|S z_k - T z_k\right\|\right) + \varepsilon_1 + \delta\varepsilon_2.
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\left\|S_1 c_k - S z_k\right\| &= \left\|T_1 a_k - T x_k\right\| \\
&\leq \left\|T_1 a_k - T a_k\right\| + \left\|T a_k - T x_k\right\| \\
&\leq \varepsilon_1 + \phi\left(\left\|S x_k - T x_k\right\|\right) + \delta\left\|S a_k - S x_k\right\| \\
&\leq \varepsilon_1 + \phi\left(\left\|S x_k - T x_k\right\|\right) + \delta\left\|S a_k - S_1 a_k\right\| + \delta\left\|S_1 a_k - S x_k\right\| \\
&\leq \delta\left\|S_1 a_k - S x_k\right\| + \phi\left(\left\|S x_k - T x_k\right\|\right) + \varepsilon_1 + \delta\varepsilon_2.
\end{aligned} \tag{25}
$$

Combining (23), (24), and (25), we get

$$
\begin{aligned}
\left\|S_1 a_{k+1} - S x_{k+1}\right\| \leq{}& \frac{\delta^2(\delta+m)}{1+m}\left\|S_1 a_k - S x_k\right\| + \frac{\delta(\delta+m)}{1+m}\phi\left(\left\|S x_k - T x\right\|\right) + \frac{1}{1+m}\phi\left(\left\|S y_k - T y_k\right\|\right) \\
&+ \frac{\delta+m}{1+m}\phi\left(\left\|S z_k - T z_k\right\|\right) + \frac{\varepsilon_1}{1+m} + \frac{\delta\varepsilon_2}{1+m} + \frac{\delta(\delta+m)}{1+m}\varepsilon_1 + \frac{\delta^2(\delta+m)}{1+m}\varepsilon_2 \\
&+ \frac{\delta+m}{1+m}\varepsilon_1 + \frac{\delta(\delta+m)}{1+m}\varepsilon_2 \\
\leq{}& \delta\left\|S_1 a_k - S x\right\| + 3\varepsilon_1 + \phi\left(\left\|S x_k - T x_k\right\|\right) + 3\varepsilon_2 + \phi\left(\left\|S z_k - T z_k\right\|\right) + \phi\left(\left\|S y_k - T y_k\right\|\right).
\end{aligned} \tag{26}
$$

Define

$$
\begin{aligned}
\psi_k &= \left\|S_1 a_k - S x_k\right\|, \\
\theta_k &= 1-\delta \in (0,1), \\
\sigma_k &= \frac{1}{1-\delta}\left(3\varepsilon_1 + \phi\left(\left\|S x_k - T x_k\right\|\right) + 3\varepsilon_2 + \phi\left(\left\|S x_k - T x_k\right\|\right) + \phi\left(\left\|S y_k - T y_k\right\|\right)\right).
\end{aligned}
$$

Thus, (26) can be written as

$$
\psi_{k+1} \leq (1-\theta_k)\psi_k + \theta_k \sigma_k. \tag{27}
$$

Now,

$$
\begin{aligned}
\left\|S x_k - T x_k\right\| &\leq \left\|S x_k - p\right\| + \left\|p - T x_k\right\| \\
&\leq \left\|S x_k - p\right\| + \phi\left(\left\|S z - T z\right\|\right) + \delta\left\|S x_k - S z\right\| \\
&= (1+\delta)\left\|S x_k - p\right\|.
\end{aligned} \tag{28}
$$

From (28), we get

$$\lim_{k\to\infty}\|Sx_k - Tx_k\| \le (1+\delta)\lim_{k\to\infty}\|Sx_k - p\|.$$

So, $\lim_{k\to\infty}\|Sx_k - Tx_k\| = 0,$ which further implies $\lim_{k\to\infty}\phi\left(\|Sx_k - Tx_k\|\right) = 0.$

$$\begin{aligned}
\|Sz_k - Tz_k\| &\le \|Sz_k - p\| + \|p - Tz_k\| \\
&\le \|Sz_k - p\| + \phi(\|Sz - Tz\|) + \delta\|Sz_k - Sz\| \\
&= (1+\delta)\|Sz_k - p\| \\
&= (1+\delta)\|Tx_k - Tz\| \\
&\le (1+\delta)\left(\phi(\|Sz - Tz\|) + \delta\|Sx_k - p\|\right) \\
&= \delta(1+\delta)\|Sx_k - p\|
\end{aligned} \tag{29}$$

From (29), one gets

$$\lim_{k\to\infty}\|Sz_k - Tz_k\| = 0.$$

So,

$$\lim_{k\to\infty}\phi\left(\|Sz_k - Tz_k\|\right) = 0.$$

$$\begin{aligned}
\|Sy_k - Ty_k\| &\le \|Sy_k - p\| + \|p - Ty_k\| \\
&\le \|Sy_k - p\| + \phi(\|Sz - Tz\|) + \delta\|Sy_k - Sz\| \\
&= (1+\delta)\|Sy_k - p\| \\
&= (1+\delta)\|Tz_k - p\| \\
&\le \delta(1+\delta)\|Sz_k - p\| \\
&\le \delta^2(1+\delta)\|Sx_k - p\|.
\end{aligned} \tag{30}$$

From (30), one can have

$$\lim_{k\to\infty}\|Sy_k - Ty_k\| = 0.$$

So, $\lim_{k\to\infty}\phi\left(\|Sy_k - Ty_k\|\right) = 0.$

As (26) satisfies all the requirements of Lemma 2.2, one gets

$$0 \le \limsup_{k\to\infty}\psi_k \le \limsup_{k\to\infty}\sigma_k,$$

which implies

$$\limsup_{k\to\infty}\|Sx_k - S_1 a_k\| \le \limsup_{k\to\infty}\sigma_k. \tag{31}$$

Also, we have shown that

$$\lim_{k\to\infty}\phi\left(\|Sz_k - Tz_k\|\right) = \lim_{k\to\infty}\phi\left(\|Sy_k - Ty_k\|\right) = \lim_{k\to\infty}\phi\left(\|Sx_k - Tx_k\|\right) = 0.$$

As $\{Sx_k\}_{k=0}^{\infty}$ converges to p and we have assumed that $\{S_1 a_k\}_{k=0}^{\infty}$ converges to q, so $\lim_{k\to\infty} Sx_k = p$ and $\lim_{k\to\infty} S_1 a_k = q$. Using the above, (31) becomes

$$\|p-q\| \le \frac{3\varepsilon_1 + 3\varepsilon_2}{1-\delta}.$$

For $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, one gets

$$\|p-q\| \le \frac{6\varepsilon}{1-\delta}.$$

The following example has been provided to check out the effectiveness of Theorem 3.3.

Example 3.1. Let S, T, S_1 and $T_1 : [0,1] \to [0,1]$ be defined as $S_1 x = \frac{3x}{4}, T_1 x = \frac{2+x}{4}, Sx = \frac{x}{2}$, and $Tx = \frac{-x+1}{4}$. Here, $T_1([0,1]) \subset S_1([0,1]), T([0,1]) \subset S([0,1])$, and S_1, S are injectives. Clearly, (S, T) has a coincidence point $z = \frac{1}{3}$, i.e., $T\frac{1}{3} = S\frac{1}{3} = \frac{1}{6} = p$, and also $z_1 = 1$ is a coincidence point of (S_1, T_1), i.e., $T_1 1 = S_1 1 = \frac{3}{4} = q$. One can easily see $\sup_{x\in[0,1]} |Tx - T_1 x| = \frac{3}{4} = \varepsilon_1$ and $\sup_{x\in[0,1]} |Sx - S_1 x| = \frac{1}{4} = \varepsilon_2$. Here, $\varepsilon_1 = \max\{\varepsilon_1, \varepsilon_2\} = \left\{\frac{3}{4}, \frac{1}{4}\right\} = \frac{3}{4}$ and $|p-q| = \frac{7}{12}$. Clearly, we can verify the validity of Theorem 3.3.

4. Numerical examples

To check out the credibility and strength of the proposed iteration scheme, we have considered five different non-linear and transcendental equations given in Examples (4.1)-(4.5). The iteration scheme (10) has been denoted by NIS (New Iterative Scheme). We have obtained the results after fifteen iterations (i.e., k = 15). In Tables 1-6, we have considered three different types of sequences mentioned as follows:

(i) For Tables 1 and 2, we have verified the results for $\alpha_k = \beta_k = \gamma_k = \frac{1}{3}, \lambda_k = \zeta_k = \frac{2}{3}$, and $m = \frac{1}{10}$, i.e., for comparison of the iteration scheme NIS with various iteration schemes, we have considered $\{\gamma_k\}_{k=0}^{\infty}, \{\beta_k\}_{k=0}^{\infty}$, and $\{\alpha_k\}_{k=0}^{\infty}$ as constant sequences in $\left(0, \frac{1}{2}\right)$.

(ii) For Tables 3 and 4, we have checked the results for $\alpha_k = \beta_k = \gamma_k = \frac{4}{k+5}, \lambda_k = \zeta_k = \frac{1}{n+5}$, and $m = \frac{1}{10}$, i.e., for comparison of the iteration scheme NIS with various iteration schemes, we have considered $\{\gamma_k\}_{k=0}^{\infty}, \{\beta_k\}_{k=0}^{\infty}$, and $\{\alpha_k\}_{k=0}^{\infty}$ as non-constant decreasing sequences in (0,1).

(iii) For Tables 5 and 6, we have evaluated the results for $\alpha_k = \beta_k = \gamma_k = \frac{9}{10}, \lambda_k = \zeta_k = \frac{1}{10}$, and $m = \frac{1}{10}$, i.e., for comparison of the iteration scheme NIS with various iteration schemes, we have considered $\{\gamma_k\}_{k=0}^{\infty}, \{\beta_k\}_{k=0}^{\infty}$, and $\{\alpha_k\}_{k=0}^{\infty}$ as constant sequences in $\left(\frac{1}{2}, 1\right)$.

Mathematica 12.0 has been used for the computations. $a \times 10^{\pm b}$ is represented by $a(\pm b)$. The term E.C. stands for error between two consecutive iterations.

Example 4.1. Consider the equation

$$x^2 - 3x - 10 = 0.$$

Let $X = [5,7] \subset \mathbb{R}$ be equipped with usual metric. Define $T, S : [5,7] \to [25,49]$ with a coincidence point 5 by $Tx = 3x + 10$ and $Sx = x^2$. It is clear that $T([5,7]) \subseteq S([5,7])$ and S([5, 7]) is a complete subset of [25, 49]. Let $x_0 = 7$

be the initial guess.

Example 4.2. Consider the Legendre polynomial

$$\frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x = 0.$$

Let $X = [0.2, 0.9] \subset \mathbb{R}$. Define $T, S : [0.2, 0.9] \to [0.56, 51.03]$ with a coincidence point 0.538469 by $Tx = 63x^5 + 15x$ and $Sx = 70x^3$. Let $x_0 = 0.3$ be the initial guess.

Example 4.3. Consider the transcendental equation

$$\sin x - 5x + 1 = 0.$$

Let $X = [0, 2] \subset \mathbb{R}$. Define $T, S : [0, 2] \to [0, 10]$ with a coincidence point 0.249356 by $Sx = 5x$ and $Tx = 1 + \sin x$. Here, $T([0,2]) \subseteq S([0,2])$. Let $x_0 = 1.5$ be the initial guess.

Example 4.4. Consider the non-linear equation

$$3x^2 - e^x = 0.$$

Let $X = [0, 2] \subset \mathbb{R}$. Define $T, S : [0, 2] \to [0, 12]$ with a coincidence point 0.91008 by $Tx = e^x$ and $Sx = 3x^2$. Let $x_0 = 1.5$ be the initial guess.

Example 4.5. Consider another transcendental equation as

$$e^x - \sin x - 2 = 0.$$

Let $X = [0, 2] \subset \mathbb{R}$. Define $T, S : [0, 2] \to [0, 7.38906]$ with a coincidence point 1.054127 by $Tx = 2 + \sin x$ and $Sx = e^x$. Here, $T([0,2]) \subseteq S([0,2])$. Let $x_0 = 2$ be the initial guess.

The comparative study of errors between consecutive iterations corresponding to Tables 5 and 6 has been shown in Figures 1-5.

Table 1. Comparison of iteration schemes on Examples (4.1)-(4.5) with $k = 15, \alpha_k = \beta_k = \gamma_k = \frac{1}{3}, \lambda_k = \zeta_k = \frac{2}{3}$, and $m = \frac{1}{10}$

Examples	E.C.	J-Mann (1)	J-Ishikawa (2)	J-Noor (3)	J-SP [16]	J-CR [12]	J-Agarwal [17]
(4.1)	$\lvert x_{k+1} - x_k\rvert$	9.3(−3)	6.7(−3)	6.5(−3)	7.6(−6)	1.2(−10)	5.8(−9)
	$\lvert Sx_{k+1} - Sx_k\rvert$	9.4(−2)	6.7(−2)	6.5(−2)	7.6(−5)	1.2(−9)	5.8(−8)
	$\lvert Tx_{k+1} - Tx_k\rvert$	2.8(−2)	2.0(−2)	1.9(−2)	2.3(−5)	3.5(−10)	1.7(−8)
(4.2)	$\lvert x_{k+1} - x_k\rvert$	3.1(−3)	2.4(−3)	2.2(−3)	2.5(−4)	1.9(−5)	8.5(−5)
	$\lvert Sx_{k+1} - Sx_k\rvert$	1.7(−1)	1.4(−1)	1.3(−1)	1.5(−2)	1.2(−3)	5.1(−3)
	$\lvert Tx_{k+1} - Tx_k\rvert$	1.1(−1)	9.1(−2)	8.7(−2)	1.0(−2)	7.9(−4)	3.5(−3)
(4.3)	$\lvert x_{k+1} - x_k\rvert$	2.8(−3)	2.2(−3)	2.1(−3)	5.3(−7)	3.2(−14)	3.3(−12)
	$\lvert Sx_{k+1} - Sx_k\rvert$	1.4(−2)	1.1(−2)	1.0(−2)	2.7(−6)	1.6(−13)	1.6(−11)
	$\lvert Tx_{k+1} - Tx_k\rvert$	2.7(−3)	2.1(−3)	2.1(−3)	5.1(−7)	3.1(−14)	3.2(−12)
(4.4)	$\lvert x_{k+1} - x_k\rvert$	6.8(−3)	4.7(−3)	4.5(−3)	4.3(−5)	7.4(−8)	1.3(−6)
	$\lvert Sx_{k+1} - Sx_k\rvert$	3.9(−2)	2.6(−2)	2.5(−2)	2.3(−4)	4.1(−7)	7.2(−6)
	$\lvert Tx_{k+1} - Tx_k\rvert$	1.8(−2)	1.2(−2)	1.1(−2)	1.1(−4)	1.8(−7)	3.3(−6)
(4.5)	$\lvert x_{k+1} - x_k\rvert$	2.4(−3)	2.1(−3)	2.1(−3)	3.4(−7)	7.8(−16)	9.3(−13)
	$\lvert Sx_{k+1} - Sx_k\rvert$	7.0(−3)	6.0(−3)	6.0(−3)	9.7(−7)	2.2(−15)	2.7(−13)
	$\lvert Tx_{k+1} - Tx_k\rvert$	1.1(−3)	1.0(−3)	1.0(−3)	1.7(−7)	3.9(−16)	4.6(−14)

Table 2. Comparison of iteration schemes on Examples (4.1)-(4.5) with $k=15, \alpha_k=\beta_k=\gamma_k=\frac{1}{3}, \lambda_k=\zeta_k=\frac{2}{3}$, and $m=\frac{1}{10}$

Examples	E.C.	J-Sahu (4)	J-Khan (5)	J-Khan Special (6)	J-Abbas [18]	NIS
	$\lvert x_{k+1}-x_k\rvert$	4.0(−10)	3.3(−10)	1.0(−16)	1.1(−13)	**9.5(−23)**
(4.1)	$\lvert Sx_{k+1}-Sx_k\rvert$	4.0(−9)	3.3(−9)	1.0(−15)	1.1(−12)	**9.5(−22)**
	$\lvert Tx_{k+1}-Tx_k\rvert$	1.2(−9)	1.0(−9)	3.1(−16)	3.2(−13)	**2.8(−22)**
	$\lvert x_{k+1}-x_k\rvert$	3.1(−5)	2.7(−5)	4.3(−7)	2.1(−6)	**5.3(−9)**
(4.2)	$\lvert Sx_{k+1}-Sx_k\rvert$	1.9(−3)	1.7(−3)	2.6(−5)	1.3(−4)	**3.2(−7)**
	$\lvert Tx_{k+1}-Tx_k\rvert$	1.3(−3)	1.1(−3)	1.8(−5)	8.8(−5)	**2.2(−7)**
	$\lvert x_{k+1}-x_k\rvert$	1.5(−13)	1.1(−13)	7.9(−23)	3.9(−18)	**7.9(−31)**
(4.3)	$\lvert Sx_{k+1}-Sx_k\rvert$	7.3(−13)	5.4(−13)	3.9(−22)	2.0(−17)	**3.9(−30)**
	$\lvert Tx_{k+1}-Tx_k\rvert$	1.4(−13)	1.1(−13)	7.6(−23)	3.8(−18)	**7.6(−31)**
	$\lvert x_{k+1}-x_k\rvert$	1.8(−7)	1.6(−7)	1.4(−11)	6.9(−10)	**1.4(−15)**
(4.4)	$\lvert Sx_{k+1}-Sx_k\rvert$	1.0(−6)	8.5(−7)	7.6(−11)	3.8(−9)	**7.5(−15)**
	$\lvert Tx_{k+1}-Tx_k\rvert$	4.5(−7)	3.9(−7)	3.4(−11)	1.7(−9)	**3.4(−15)**
	$\lvert x_{k+1}-x_k\rvert$	7.0(−15)	1.8(−15)	3.7(−25)	9.8(−20)	**7.5(−34)**
(4.5)	$\lvert Sx_{k+1}-Sx_k\rvert$	2.0(−14)	5.1(−14)	1.1(−24)	2.8(−19)	**2.1(−33)**
	$\lvert Tx_{k+1}-Tx_k\rvert$	3.5(−15)	8.8(−16)	1.8(−25)	4.8(−20)	**3.7(−34)**

Table 3. Comparison of iteration schemes on Examples (4.1)-(4.5) with $k=15, \alpha_k=\beta_k=\gamma_k=\frac{4}{k+5}, \lambda_k=\zeta_k=\frac{1}{k+5}$, and $m=\frac{1}{10}$

Examples	E.C.	J-Mann (1)	J-Ishikawa (2)	J-Noor (3)	J-SP [16]	J-CR [12]	J-Agarwal [17]
	$\lvert x_{k+1}-x_k\rvert$	1.9(−3)	7.5(−4)	6.2(−4)	2.1(−7)	1.5(−11)	2.2(−9)
(4.1)	$\lvert Sx_{k+1}-Sx_k\rvert$	1.9(−2)	7.5(−3)	6.2(−3)	2.1(−6)	1.5(−10)	2.2(−8)
	$\lvert Tx_{k+1}-Tx_k\rvert$	5.7(−3)	2.2(−3)	1.9(−3)	6.2(−7)	4.6(−11)	6.6(−9)
	$\lvert x_{k+1}-x_k\rvert$	1.3(−3)	6.4(−4)	4.7(−4)	5.6(−5)	8.3(−6)	5.6(−5)
(4.2)	$\lvert Sx_{k+1}-Sx_k\rvert$	7.3(−2)	3.8(−2)	2.8(−2)	3.4(−3)	5.1(−4)	3.4(−3)
	$\lvert Tx_{k+1}-Tx_k\rvert$	4.9(−2)	2.6(−2)	1.9(−2)	2.3(−3)	3.4(−4)	2.3(−3)
	$\lvert x_{k+1}-x_k\rvert$	3.9(−4)	1.9(−4)	1.8(−4)	5.5(−9)	2.7(−15)	1.1(−12)
(4.3)	$\lvert Sx_{k+1}-Sx_k\rvert$	1.9(−3)	9.6(−4)	8.9(−4)	2.8(−8)	1.3(−14)	5.4(−12)
	$\lvert Tx_{k+1}-Tx_k\rvert$	3.8(−4)	1.9(−4)	1.7(−4)	5.4(−9)	2.6(−15)	1.1(−12)
	$\lvert x_{k+1}-x_k\rvert$	2.1(−3)	7.9(−4)	5.7(−4)	3.5(−6)	1.7(−8)	6.3(−7)
(4.4)	$\lvert Sx_{k+1}-Sx_k\rvert$	1.2(−2)	4.3(−3)	3.1(−3)	1.9(−5)	9.2(−8)	3.4(−6)
	$\lvert Tx_{k+1}-Tx_k\rvert$	5.3(−3)	2.0(−3)	1.4(−3)	8.6(−6)	4.2(−8)	1.6(−6)
	$\lvert x_{k+1}-x_k\rvert$	2.6(−4)	1.9(−4)	1.8(−4)	2.2(−9)	1.5(−16)	7.3(−14)
(4.5)	$\lvert Sx_{k+1}-Sx_k\rvert$	7.5(−4)	5.3(−4)	5.2(−4)	6.4(−9)	4.2(−16)	2.1(−13)
	$\lvert Tx_{k+1}-Tx_k\rvert$	1.3(−4)	9.2(−5)	8.9(−5)	1.1(−9)	7.2(−17)	3.6(−14)

Table 4. Comparison of iteration schemes on Examples (4.1)-(4.5) with $k = 15, \alpha_k = \beta_k = \gamma_k = \frac{4}{k+5}, \lambda_k = \zeta_k = \frac{1}{k+5}$, and $m = \frac{1}{10}$

Examples	E.C.	J-Sahu (4)	J-Khan (5)	J-Khan Special (6)	J-Abbas [18]	NIS
(4.1)	$\|x_{k+1} - x_k\|$	1.4(−10)	4.2(−5)	5.8(−12)	7.4(−14)	**9.5(−23)**
	$\|Sx_{k+1} - Sx_k\|$	1.4−9)	4.2(−4)	5.8(−11)	7.4(−13)	**9.5(−22)**
	$\|Tx_{k+1} - Tx_k\|$	4.1(−10)	1.3(−4)	1.8(−11)	2.2(−13)	**2.8(−22)**
(4.2)	$\|x_{k+1} - x_k\|$	2.0(−5)	2.4(−4)	5.4(−6)	1.9(−6)	**5.3(−9)**
	$\|Sx_{k+1} - Sx_k\|$	1.2(−3)	1.4(−2)	3.3(−4)	1.2(−4)	**3.2(−7)**
	$\|Tx_{k+1} - Tx_k\|$	8.4(−4)	9.8(−3)	2.2(−4)	7.9(−5)	**2.2(−7)**
(4.3)	$\|x_{k+1} - x_k\|$	3.9(−14)	4.0(−6)	8.4(−16)	2.5(−18)	**7.9(−31)**
	$\|Sx_{k+1} - Sx_k\|$	2.0(−13)	2.0(−5)	4.2(−15)	1.3(−17)	**3.9(−30)**
	$\|Tx_{k+1} - Tx_k\|$	3.8(−14)	3.9(−6)	8.2(−16)	2.5(−18)	**7.6(−31)**
(4.4)	$\|x_{k+1} - x_k\|$	8.6(−8)	1.1(−4)	8.1(−9)	5.4(−10)	**1.4(−15)**
	$\|Sx_{k+1} - Sx_k\|$	4.7(−7)	6.4(−4)	4.4(−8)	3.0(−9)	**7.5(−15)**
	$\|Tx_{k+1} - Tx_k\|$	2.1(−7)	2.9(−4)	2.0(−8)	1.3(−9)	**3.4(−15)**
(4.5)	$\|x_{k+1} - x_k\|$	3.4(−15)	5.5(−7)	4.3(−17)	1.3(−19)	**7.5(−34)**
	$\|Sx_{k+1} - Sx_k\|$	9.7(−15)	1.6(−6)	1.2(−16)	3.7(−19)	**2.1(−33)**
	$\|Tx_{k+1} - Tx_k\|$	1.7(−15)	2.7(−7)	2.1(−17)	6.4(−20)	**3.7(−34)**

Table 5. Comparison of iteration schemes on Examples (4.1)-(4.5) with $k = 15, \alpha_k = \beta_k = \gamma_k = \frac{9}{10}, \lambda_k = \zeta_k = \frac{1}{10}, m = \frac{1}{10}$

Examples	E.C.	J-Mann (1)	J-Ishikawa (2)	J-Noor (3)	J-SP [16]	J-CR [12]	J-Agarwal [17]
(4.1)	$\|x_{k+1} - x_k\|$	4.1(−7)	5.5(−11)	1.2(−12)	6.8(−20)	3.1(−20)	8.4(−14)
	$\|Sx_{k+1} - Sx_k\|$	4.1(−6)	5.5(−10)	1.2(−11)	6.8(−19)	3.1(−19)	8.4(−13)
	$\|Tx_{k+1} - Tx_k\|$	1.2(−6)	1.7(−10)	3.7(−12)	2.1(−19)	9.2(−20)	2.5(−13)
(4.2)	$\|x_{k+1} - x_k\|$	2.4(−4)	5.6(−6)	2.4(−7)	2.1(−8)	1.9(−8)	2.4(−5)
	$\|Sx_{k+1} - Sx_k\|$	1.5(−2)	3.4(−4)	1.5(−5)	1.3(−6)	1.2(−6)	1.5(−4)
	$\|Tx_{k+1} - Tx_k\|$	1.0(−2)	2.3(−4)	1.0(−5)	8.8(−7)	8.0(−7)	9.9(−5)
(4.3)	$\|x_{k+1} - x_k\|$	2.5(−9)	3.4(−13)	3.3(−14)	4.8(−26)	8.3(−27)	2.1(−18)
	$\|Sx_{k+1} - Sx_k\|$	1.3(−8)	1.7(−12)	1.7(−13)	2.4(−25)	4.2(−26)	1.0(−17)
	$\|Tx_{k+1} - Tx_k\|$	2.4(−9)	3.3(−13)	3.2(−14)	4.7(−26)	8.1(−27)	2.0(−18)
(4.4)	$\|x_{k+1} - x_k\|$	1.6(−5)	1.2(−8)	1.3(−10)	4.6(−14)	3.3(−14)	7.1(−10)
	$\|Sx_{k+1} - Sx_k\|$	8.6(−5)	6.6(−8)	7.0(−10)	2.5(−13)	1.8(−13)	3.9(−9)
	$\|Tx_{k+1} - Tx_k\|$	3.9(−5)	3.0(−8)	3.2(−10)	1.1(−13)	8.3(−14)	1.8(−6)
(4.5)	$\|x_{k+1} - x_k\|$	5.2(−10)	1.5(−13)	2.4(−14)	1.1(−27)	9.5(−29)	7.3(−20)
	$\|Sx_{k+1} - Sx_k\|$	1.5(−10)	4.4(−13)	6.9(−14)	3.1(−27)	2.7(−28)	2.1(−19)
	$\|Tx_{k+1} - Tx_k\|$	2.6(−10)	7.5(−14)	1.2(−14)	5.3(−28)	4.7(−29)	3.6(−20)

Table 6. Comparison of iteration schemes on Examples (4.1)-(4.5) with $k = 15, \alpha_k = \beta_k = \gamma_k = \frac{9}{10}, \lambda_k = \zeta_k = \frac{1}{10}$, and $m = \frac{1}{10}$

Examples	E.C.	J-Sahu (4)	J-Khan (5)	J-Khan Special (6)	J-Abbas [18]	NIS
	$\lvert x_{k+1} - x_k \rvert$	8.1(−15)	2.9(−18)	1.3(−21)	3.0(−1 5)	**9.5(−23)**
(4.1)	$\lvert Sx_{k+1} - Sx_k \rvert$	8.1(−14)	2.9(−17)	1.3(−20)	3.0(−14)	**9.5(−22)**
	$\lvert Tx_{k+1} - Tx_k \rvert$	2.4(−14)	8.6(−18)	4.0(−21)	9.0(−15)	**2.8(−22)**
	$\lvert x_{k+1} - x_k \rvert$	1.4(−6)	3.9(−8)	1.0(−8)	9.0(−7)	**5.3(−9)**
(4.2)	$\lvert Sx_{k+1} - Sx_k \rvert$	8.3(−5)	2.4(−6)	6.1(−7)	5.5(−5)	**3.2(−7)**
	$\lvert Tx_{k+1} - Tx_k \rvert$	5.7(−5)	1.6(−6)	4.1(−7)	3.7(−5)	**2.2(−7)**
	$\lvert x_{k+1} - x_k \rvert$	3.3(−19)	1.9(−23)	4.5(−29)	2.0(−20)	**7.9(−31)**
(4.3)	$\lvert Sx_{k+1} - Sx_k \rvert$	3.3(−19)	9.7(−23)	2.2(−28)	1.0(−19)	**3.9(−30)**
	$\lvert Tx_{k+1} - Tx_k \rvert$	6.4(−20)	1.9(−23)	4.4(−29)	2.0(−20)	**7.6(−31)**
	$\lvert x_{k+1} - x_k \rvert$	1.8(−10)	2.8(−13)	6.2(−15)	8.6(−11)	**1.4(−15)**
(4.4)	$\lvert Sx_{k+1} - Sx_k \rvert$	1.0(−9)	1.6(−12)	3.4(−14)	4.7(−10)	**7.5(−15)**
	$\lvert Tx_{k+1} - Tx_k \rvert$	4.6(−10)	7.1(−13)	1.5(−14)	2.1(−10)	**3.4(−15)**
	$\lvert x_{k+1} - x_k \rvert$	2.1(−21)	3.0(−25)	2.6(−31)	6.3(−22)	**7.5(−34)**
(4.5)	$\lvert Sx_{k+1} - Sx_k \rvert$	6.1(−21)	8.5(−25)	7.5(−31)	1.8(−21)	**2.1(−33)**
	$\lvert Tx_{k+1} - Tx_k \rvert$	1.0(−21)	1.5(−25)	1.3(−31)	3.1(−22)	**3.7(−34)**

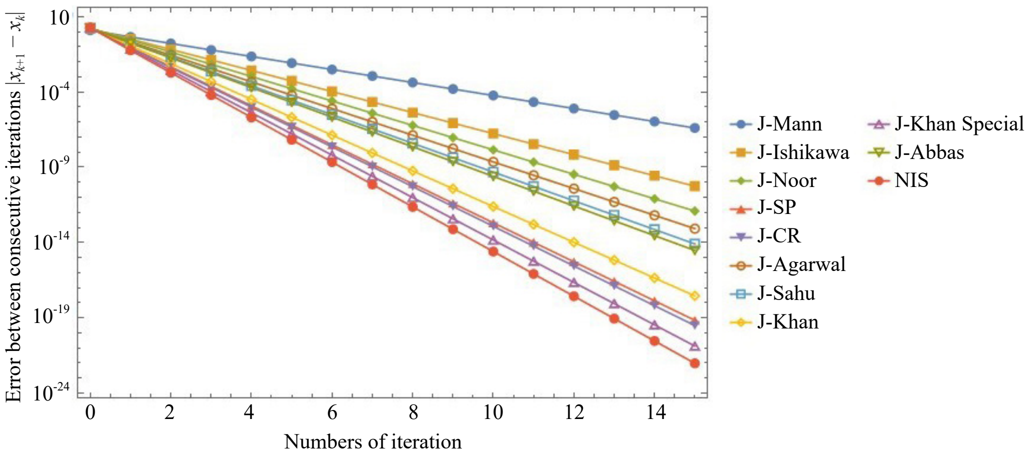

Figure 1. Comparison of rate of convergence of iteration schemes for Example 4.1 corresponding to Tables 5 and 6

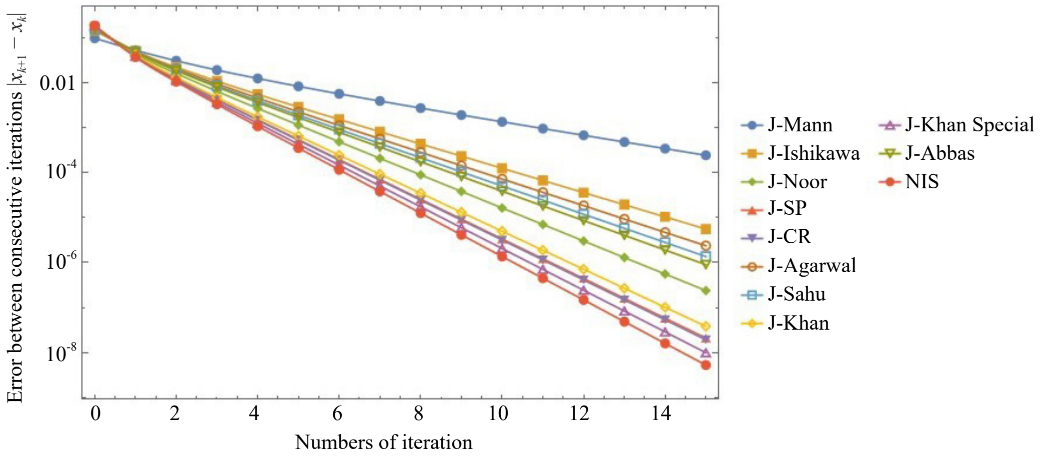

Figure 2. Comparison of rate of convergence of iteration schemes for Example 4.2 corresponding to Tables 5 and 6

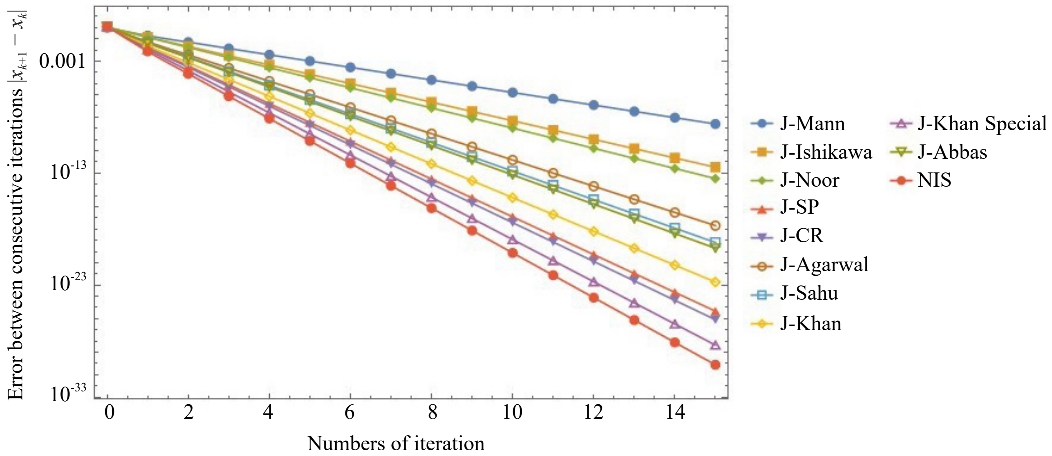

Figure 3. Comparison of rate of convergence of iteration schemes for Example 4.3 corresponding to Tables 5 and 6

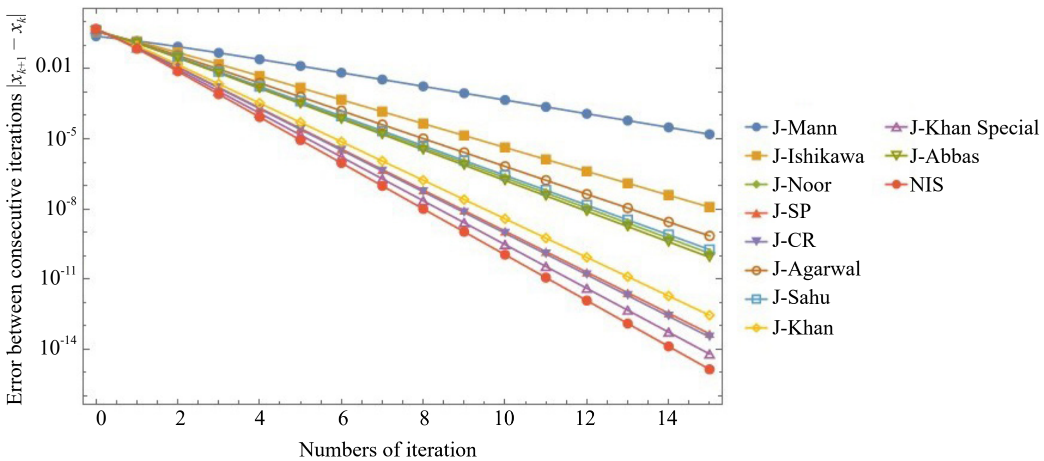

Figure 4. Comparison of rate of convergence of iteration schemes for Example 4.4 corresponding to Tables 5 and 6

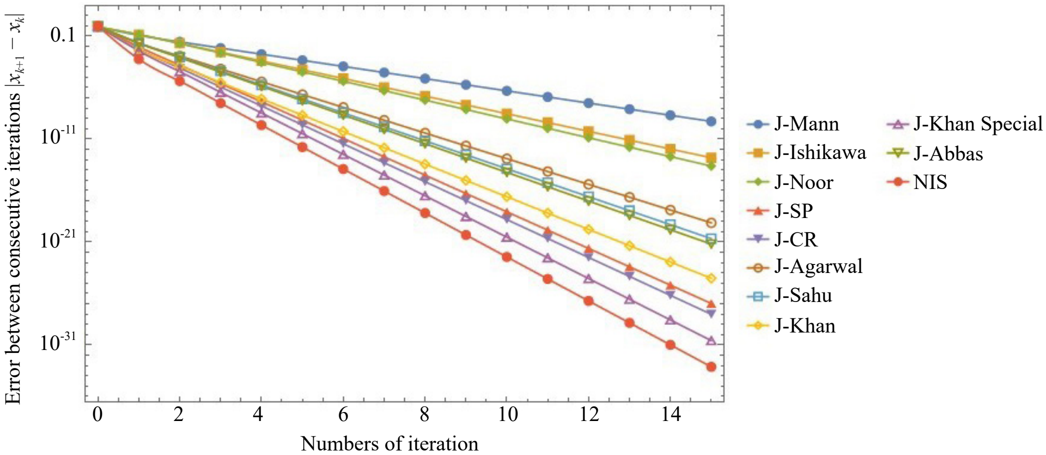

Figure 5. Comparison of rate of convergence of iteration schemes for Example 4.5 corresponding to Tables 5 and 6

5. Role of the parameter *m*

Assigning smaller values to m in the iterative scheme (10) provides better accuracy and convergence. In Table 7, we have provided the comparative analysis of the iterative scheme (10) represented by NIS, corresponding to Examples (4.1)-(4.5). The results have been considered after 15 iterations (i.e., $k = 15$).

Table 7. Comparison of iterative scheme NIS for different values of m with $k = 15$

Examples	E.C.	$m = \frac{1}{10}$	$m = \frac{1}{100}$	$m = \frac{1}{1000}$	$m = \frac{1}{10000}$	$m = 10$
(4.1)	$\|x_{k+1} - x_k\|$	9.5(−23)	7.5(−24)	5.5(−24)	5.3(−24)	1.3(−16)
	$\|Sx_{k+1} - Sx_k\|$	9.5(−22)	7.5(−23)	5.5(−23)	5.3(−23)	1.3(−15)
	$\|Tx_{k+1} - Tx_k\|$	2.8(−22)	22(−23)	1.7(−23)	1.6(−23)	3.9(−16)
(4.2)	$\|x_{k+1} - x_k\|$	5.3(−9)	3.1(−9)	2.9(−9)	2.9(−9)	4.8(−7)
	$\|Sx_{k+1} - Sx_k\|$	3.2(−7)	1.9(−7)	1.8(−7)	1.8(−7)	2.9(−5)
	$\|Tx_{k+1} - Tx_k\|$	2.2(−7)	1.3(−7)	1.2(−7)	1.2(−7)	2.0(−5)
(4.3)	$\|x_{k+1} - x_k\|$	7.9(−31)	1.2(−32)	6.8(−33)	6.4(−33)	9.7(−23)
	$\|Sx_{k+1} - Sx_k\|$	3.9(−30)	5.9(−32)	3.4(−32)	3.2(−32)	4.9(−22)
	$\|Tx_{k+1} - Tx_k\|$	7.6(−20)	1.1(−32)	6.6(−33)	6.2(−33)	9.4(−23)
(4.4)	$\|x_{k+1} - x_k\|$	1.4(−15)	3.5(−16)	3.0(−16)	3.0(−16)	1.6(−11)
	$\|Sx_{k+1} - Sx_k\|$	7.5(−15)	1.9(−15)	1.6(−15)	1.6(−15)	9.0(−11)
	$\|Tx_{k+1} - Tx_k\|$	3.4(−15)	8.7(−16)	7.5(−16)	7.3(−16)	4.1(−11)
(4.5)	$\|x_{k+1} - x_k\|$	7.5(−34)	6.5(−36)	3.5(−36)	3.3(−36)	2.8(−25)
	$\|Sx_{k+1} - Sx_k\|$	2.1(−33)	1.9(−35)	1.0(−35)	9.4(−36)	8.1(−25)
	$\|Tx_{k+1} - Tx_k\|$	3.7(−34)	3.2(−36)	1.7(−36)	1.6(−36)	1.4(−25)

Remark 5.1. It is observed that the proposed scheme gives a better approximation to the required fixed points when m is small. However, for large values of m, the proposed scheme still works but takes more iterations as compared to the smaller values of m.

6. Conclusion

We have proposed a new Jungck-type iteration scheme and studied its data dependence, stability, and strong convergence. The new iteration scheme has a higher convergence rate in comparison with various well-known iteration schemes. Moreover, numerical examples show that the new iteration scheme provides approximations of greater accuracy in comparison with the iteration schemes mentioned in Tables 1-6. Furthermore, conditions imposed on sequences $\{\gamma_k\}_{k=0}^{\infty}, \{\beta_k\}_{k=0}^{\infty}$, and $\{\alpha_k\}_{k=0}^{\infty} \subseteq [0,1]$, like $\sum_{k=0}^{\infty} \alpha_k = \infty, \sum_{k=0}^{\infty} \alpha_k \beta_k \gamma_k = \infty, \sum_{k=0}^{\infty} \alpha_k \beta_k = \infty$, and $\sum_{k=0}^{\infty} (\alpha_k + \beta_k) = \infty$ mentioned in literature for different iterative schemes, have not been used to prove the results corresponding to the proposed iterative scheme.

Conflict of interest

There is no conflict of interest in this study.

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