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Research Article

Controllability of Hilfer Fractional Semilinear Integro-Differential Equation of Order $\alpha \in (0, 1)$, $\beta \in [0, 1]$

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Abstract: In this paper, the Hilfer fractional differential equation is studied. Firstly, we have used the Laplace transform and semigroup theory to find the mild solution of the system. Then, the exact controllability of the proposed system is established using the Arzela-Ascoli theorem and the Schauder fixed point theorem. To illustrate the developed theory, we have provided an example at the end.

***Keywords*:** Hilfer fractional derivative, semigroup theory, exact controllability, fixed point theorem

MSC: 35R11, 35R09, 93C05, 37L05

1. Introduction

Fractional order differential equations have been gaining significant attention among scientists due to their numerous applications in various fields, including engineering, medicine, and other disciplines. To explore further insights into fractional differential systems and their practical uses, interested readers can consult the references [1-4]. The Hilfer fractional derivative (HFD), a generalized form that combines Riemann-Liouville and Caputo fractional derivatives, was introduced by Hilfer [1]. This derivative plays a crucial role in fractional calculus and provides a broader perspective on fractional differential equations (FDEs). In a study conducted by Furati et al. [5], they delved into non-linear systems and investigated the existence of solutions and stability analysis for initial value problems (IVPs) of non-linear FDEs that employed the HFD. Overall, the research in fractional order differential equations and the development of the HFD have opened up new avenues for understanding and utilizing these mathematical tools in solving real-world problems across various scientific domains.

Numerous researchers have addressed the topic of exact and approximate controllability in systems involving the HFD, as evident from references [6-11]. In recent contributions, Lv et al. [12] presented a study on the approximate controllability of Hilfer FDEs with orders in the range of $1 < \alpha < 2$. They employed the Banach contraction principle as a key mathematical tool in establishing the results. Singh [13] investigated the issue of exact controllability in non-dense domains for systems governed by the Hilfer FDEs. To do so, he utilized the concept of the measure of noncompactness, which is a valuable tool in functional analysis. Raja et al. [14] conducted research on the approximate controllability of Caputo-type integro-differential systems with fractional orders in the range of $1 < \alpha < 2$. Furthermore, Kavitha et al. [15] explored the controllability aspects of Hilfer FDEs with infinite delay, and they utilized measures of noncompactness in their investigation. In recent studies, various researchers have explored the existence of solutions and approximate controllability in different types of fractional control systems. Kavitha et al. [16] applied Dhage's fixed point theorem

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to investigate the approximate controllability of non-densely defined Sobolev-type Hilfer fractional neutral Volterra-Fredholm integro-differential systems. Ma et al. [17] formulated the mild solution and necessary conditions for approximate controllability of hemivariational inequalities in Sobolev-type Hilfer fractional neutral stochastic evolution systems. Sousa et al. [18] explored the reachability of both linear and non-linear systems using the concept of the ψ-Hilfer pseudo-fractional derivative in g-calculus, utilizing the Mittag-Leffler functions. Johnson et al. [19] studied the existence of solutions and optimal controllability of Hilfer fractional stochastic integro-differential systems with infinite delay. Kumar et al. [20] discussed the existence and uniqueness of generalized Caputo-type IVPs with delay using fixed point theory. Selvam et al. [21] employed the ψ-Caputo fractional derivative to derive the existence and uniqueness of control for linear and non-linear fractional dynamical systems with distributed delay in control. Vijayakumar et al. [22] examined two different types of essential conditions for determining the approximate controllability of Hilfer fractional semilinear control systems. Dineshkumar et al. [23] presented a formulation combining stochastic analysis theory, fractional calculus, multivalued maps, and Karlin's fixed point technique to establish the approximate controllability of a non-linear Hilfer stochastic system. Kavitha et al. [24] demonstrated the approximate controllability of Hilfer fractional control systems with time delays using the sequential approach. Selvam et al. [25] derived controllability results for fractional dynamical systems with ψ-Caputo fractional derivatives by utilizing Grammian matrices, Mittag-Leffler functions, and a fixed-point approach. These studies collectively contribute to the advancement of understanding and exploring various aspects of approximate controllability in fractional systems.

The previously mentioned research papers have inspired us to investigate the exact controllability of Hilfer FDEs with non-linear integro-differential functions of the following form,

$$\begin{aligned} \mathrm{D}_{0^+}^{\alpha,\beta} x(t) &= Ax(t) + Bu(t) + t^n f\left(t, x(t), \int_0^l z(t,s,x(s))ds\right), t \in (0,l], n \in \mathbb{Z}^+ \\ J_{0^+}^{(1-\alpha)(1-\beta)} x(t)|_{t=0} &= x_0 \end{aligned} \tag{1}$$

where $\mathrm{D}_{0^+}^{\alpha,\beta}$ denotes the HFD of order $\alpha \in (0,1)$, $\beta \in [0,1]$. The state $x(\cdot) \in X$ equipped with the sup-norm $\| x \| = sup_{t\in V} \| x(t) \|$, where X and U are Banach spaced and $V = [0, l]$. The control function $u(\cdot) \in L^{\epsilon}(V,U)$, $\epsilon > \frac{1}{\beta}$, which denotes Banach space of admissible controls. $A: D(A) \subset X \to X$ is a densely defined closed linear operator on X and $B : L^{\epsilon}(V,U) \to L^{\epsilon}(V,X)$ is a bounded linear operator. The non-linear term $f: V \times X \times X \to X$ is a given function, f and z are appropriate functions which satisfy some assumptions.

The primary objective of this research paper is to establish the exact controllability of a Hilfer fractional integro-differential equation with an order in the range of $0 < \alpha < 1$. This equation involves a non-linear function with an integral term. In contrast, a similar integro-differential equation with Caputo fractional order $1 < \alpha < 2$ was considered in [14], where the author derived an approximate controllability result.

In our work, we have adapted the approach presented in [26] to obtain the mild solution of the Hilfer fractional integro-differential equation (1). To achieve this, we have utilized semigroup theory and Laplace transform techniques. The Schauder fixed-point approach has been employed to establish the final controllability result. Notably, we have relaxed the growth condition imposed on the integral term in the non-linear function by employing a simpler Lipschitz condition. Instead of employing other various techniques like the measure of non-compactness, the sequential method, the Grammian approach, etc., our approach revolves around using fixed point methods.

To illustrate the effectiveness of the established theory, we have included a practical example in the paper. The concepts and ideas presented in this research can be extended and applied to establish the exact controllability of a broader class of dynamical systems.

This article is structured as follows: we compile all pertinent definitions and prior findings in Section 2. Section 3 discusses the controllability results. Section 4 provides an example of the developed theory.

2. Preliminaries

Here, we present some pre-defined lemmas and definitions.

Definition 1. [4] Riemann-Liouville fractional integral $J_{0^+}^{\alpha} g(t)$ for $g \in L^1((0,l),R)$ is given as

$$J_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau) d\tau, t > 0, \alpha > 0.$$

Definition 2. [4] Riemann-Liouville fractional derivative ${}^L D_{0^+}^{\alpha} g(t)$ for $g \in AC^n(I,R)$ is given as

$${}^L D_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} g(\tau) d\tau,\ t > 0, n = [\alpha]+1.$$

Definition 3. [4] Caputo fractional derivative ${}^c D_{0^+}^{\alpha} g(t)$ for $g \in AC^n(I,R)$ is given as

$${}^c D_{0^+}^{\alpha} g(t) = {}^L D_{0^+}^{\alpha} \left[g(t) - \sum_{m=0}^{n-1} \frac{t^m}{m!} g^{(m)}(0) \right], t > 0, n = [\alpha]+1.$$

Definition 4. [1] HFD $D_{0^+}^{\alpha,\beta} g(t)$ of order α and type β is given as

$$D_{0^+}^{\alpha,\beta} g(t) = \left(J_{0^+}^{\beta(n-\alpha)} \frac{d^n}{dt^n} \left(I_{0+}^{(1-\beta)(n-\alpha)} g \right) \right)(t) = \left(J_{0^+}^{\beta(n-\alpha)L} D_{0^+}^{\alpha+\beta n-\alpha\beta} g \right)(t),$$

where $\alpha \in (n-1, n]$, $\beta \in [0,1]$.

Remark 2.1.

(a) For $\beta = 0$, $D_{0^+}^{\alpha,0}$ represents Reimann-Liouville fractional derivative: $D_{0^+}^{\alpha,0} = {}^L D_{0^+}^{\alpha}$.

(b) For $\beta = 1$, $D_{0^+}^{\alpha,1}$ represents Caputo fractional derivative: $D_{0^+}^{\alpha,1} = {}^C D_{0^+}^{\alpha}$.

The following assumptions are taken into account for the subsequent analysis:

(A1) The operator A, generates a C_0 semigroup $\{T(t)\}_{t \geq 0}$ on X with

$$\| T(t) \| \leq M e^{\omega t},$$

where M and ω are introduced in the Hille-Yosida condition as constants.

(A2) For $t > 0$, $T(t)$ exhibits continuity in the uniform operator topology. So, we can define

$$sup_{t \in [0,l]} \| T(t) \| \leq M_0.$$

We present the Wright function to determine integral solution of the problem (1).

$$\tilde{W}_\beta(\varphi) = \sum_{m=1}^{\infty} \frac{(-\varphi)^{m-1}}{(m-1)! \Gamma(1-\beta m)}, \quad 0 < \beta < 1, \quad \varphi \in \mathbb{C},$$

that holds the conditions

$$\int^{\infty} \varphi^{\mu} \tilde{W}_\beta(\varphi) d\varphi = \frac{\Gamma(1+\mu)}{\Gamma(1+\beta\mu)}, \quad \varphi \geq 0.$$

Definition 5. The function $x: V \to X$ is known to be an integral solution of equation (1) if

(a) $x: V \to X$ is continuous,

(b) $J_{0^+}^{\beta} x(t) \in D(A), \forall t \in [0, l]$,

(c) [5] for $t \in (0, l]$, $x(t)$ holds

$$x(t) = \frac{x_0}{\Gamma(\alpha(1-\beta)+\beta)} t^{(\alpha-1)(1-\beta)} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [Ax(s) + Bu(s) + s^n f(s, x(s), (Zx)(s))] ds \tag{2}$$

where $(Zx)(t) = \int_0^l z(t, s, x(s)) ds$.

Lemma 2.2. If $x(t)$ satisfies equation (2) then it can also be written as

$$x(t) = \mathfrak{T}_{\alpha,\beta}(t) x_0 + \int_0^t (t-s)^{\beta-1} \mathfrak{S}_{\beta}(t-s) [Bu(s) + s^n f(s, x(s), (Zx)(s))] ds \tag{3}$$

where $\mathfrak{P}_{\beta}(t) = t^{\beta-1} \mathfrak{S}_{\beta}(t)$, $\mathfrak{S}_{\beta}(t) = \int_0^{\infty} \beta \varphi \tilde{W}_{\beta}(\varphi) T(t^{\beta} \varphi) d\varphi$ and $\mathfrak{T}_{\alpha,\beta}(t) = \mathrm{J}_{0^+}^{\alpha(1-\beta)} \mathfrak{P}_{\beta}(t)$.

Proof. Let us define Laplace transformation for $r > 0$ as

$$\kappa(r) = \int_0^{\infty} e^{-rs} x(s) ds \text{ and } \chi(r) = \int_0^{\infty} e^{-rs} [Bu(s) + s^n f(s, x(s), (Zx)(s))] ds. \tag{4}$$

Taking Laplace transformation of equation (2) and applying equation (4), we will get

$$\begin{aligned} \kappa(r) &= r^{(1-\alpha)(1-\beta)-1} x_0 + \frac{A}{r^{\beta}} \kappa(r) + \frac{1}{r^{\beta}} \chi(r) \\ &= r^{\alpha(\beta-1)} (r^{\beta} I - A)^{-1} x_0 + (r^{\beta} I - A)^{-1} \chi(r) \\ &= r^{\alpha(\beta-1)} \int_0^{\infty} e^{-r^{\beta} s} T(s) ds \, x_0 + \int_0^{\infty} e^{-r^{\beta} s} T(s) \chi(r) ds \end{aligned} \tag{5}$$

where I denotes identity operator, and provided that the integrals in (5) exist. Let Laplace of $\zeta_{\beta}(\varphi) = \frac{\beta}{\varphi^{\beta+1}} \tilde{W}_{\beta}(\varphi^{-\beta})$ is provided by

$$\int_0^{\infty} e^{-r\varphi} \zeta_{\beta}(\varphi) d\varphi = e^{-r^{\beta}}. \tag{6}$$

Using (6), we have

$$\begin{aligned} \int_0^{\infty} e^{-r^{\beta} s} T(s) x_0 ds &= \int_0^{\infty} \beta t^{\beta-1} e^{-(rt)^{\beta}} T(t^{\beta}) x_0 dt \\ &= \int_0^{\infty} \int_0^{\infty} \beta t^{\beta-1} \zeta_{\beta}(\varphi) e^{-(rt\varphi)} T(t^{\beta}) x_0 d\varphi dt \\ &= \int_0^{\infty} e^{-(rt)} \left[\int_0^{\infty} \beta \frac{t^{\beta-1}}{\varphi^{\beta}} \zeta_{\beta}(\varphi) T\left(\frac{t^{\beta}}{\varphi^{\beta}} \right) x_0 d\varphi \right] dt \\ &= \int_0^{\infty} e^{-(rt)} \left[t^{\beta-1} \mathfrak{S}_{\beta}(t) x_0 \right] dt. \end{aligned} \tag{7}$$

$$\begin{aligned} \int_0^{\infty} e^{-r^{\beta} s} T(s) \chi(r) ds &= \int_0^{\infty} \beta t^{\beta-1} e^{-(rt)^{\beta}} T(t^{\beta}) e^{-rs} [Bu(s) + s^n f(s, x(s), (Zx)(s))] ds dt \\ &= \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \beta t^{\beta-1} \zeta_{\beta}(\varphi) e^{-(rt\varphi)} T(t^{\beta}) e^{-rs} [Bu(s) + s^n f(s, x(s), (Zx)(s))] d\varphi ds dt \\ &= \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \beta \frac{t^{\beta-1}}{\varphi^{\beta}} \zeta_{\beta}(\varphi) e^{-r(t+s)} T\left(\frac{t^{\beta}}{\varphi^{\beta}} \right) [Bu(s) + s^n f(s, x(s), (Zx)(s))] d\varphi ds dt \end{aligned} \tag{8}$$

$$
= \int_0^{\infty}\int_0^{t}\int_0^{\infty} \beta \frac{(t-s)^{\beta-1}}{\varphi^{\beta}} \zeta_{\beta}(\varphi) e^{-(rt)} T\left(\frac{(t-s)^{\beta}}{\varphi^{\beta}}\right)[Bu(s)+s^{n} f(s,x(s),(Zx)(s))]d\varphi ds dt
$$

$$
= \int_0^{\infty} e^{-(rt)} \left[\int_0^{t} (t-s)^{\beta-1} \mathfrak{S}_{\beta}(t-s)[Bu(s)+s^{n} f(s,x(s),(Zx)(s))]ds\right] dt. \tag{9}
$$

Therefore, from equation (5), (7) and (9), for $t \in (0, l]$ we get

$$
x(t) = \left(L^{-1}(r^{\alpha(\beta-1)} * \mathfrak{P}_{\beta})\right)(t)x_0 + \int_0^{t} (t-s)^{\beta-1} \mathfrak{S}_{\beta}(t-s)[Bu(s)+s^{n} f(s,x(s),(Zx)(s))]ds
$$

$$
x(t) = J_{0^+}^{\alpha(\beta-1)} \mathfrak{P}_{\beta}(t)x_0 + \int_0^{t} (t-s)^{\beta-1} \mathfrak{S}_{\beta}(t-s)[Bu(s)+s^{n} f(s,x(s),(Zx)(s))]ds
$$

$$
x(t) = \mathfrak{T}_{\alpha,\beta}(t)x_0 + \int_0^{t} (t-s)^{\beta-1} \mathfrak{S}_{\beta}(t-s)[Bu(s)+s^{n} f(s,x(s),(Zx)(s))]ds.
$$

This concludes the proof.

Using Lemma 2.2, the mild solution of (1) is defined as follows.

Definition 6. The function $x \in C(V, X)$ is known to be the mild solution of the Cauchy problem (1) if it satisfies

$$
x(t) = \mathfrak{T}_{\alpha,\beta}(t)x_0 + \int_0^{t} (t-s)^{\beta-1} \mathfrak{S}_{\beta}(t-s)[Bu(s)+s^{n} f(s,x(s),(Zx)(s))]ds, \; t \in V. \tag{10}
$$

Remark 2.3. [6, 8, 26, 27]

(a) $\{\mathfrak{T}_{\alpha,\beta}(t)\}$ and $\{\mathfrak{P}_{\beta}(t)\}$ are strongly continuous linear operator, then for any fixed $t > 0$ and for any $x \in X$, we have

$$
\| \mathfrak{P}_{\beta}(t)x \| \le \frac{M_0 t^{\beta-1}}{\Gamma(\beta)} \| x \| \text{ and } \| \mathfrak{T}_{\alpha,\beta}(t)x \| \le \frac{M_0 t^{\gamma-1}}{\Gamma(\alpha(1-\beta)+\beta)} \| x \| \tag{11}
$$

where $\gamma = \alpha + \beta - \alpha\beta$.

(b) $\{\ _{\beta}(t)\}$ exhibits continuity in uniform topology and for $t \in V$ and $x \in X$,

$$
\| \mathfrak{S}_{\beta}(t)x \| \le \frac{M_0}{\Gamma(\beta)} \| x \|. \tag{12}
$$

The symbol $x(t) = x_t(x_0,u)$ denotes the state value of the system (1) associated to the control u at the time t. In specific, the final state with control u at time l is known to be the state $x(l) = x_l(x_0, u)$ of the system (1).

Definition 7. The FDE (1) is known to be exact controllable in X on the interval V if for any terminal state $x_l \in X$; $\exists$ a control $u \in L^{\epsilon}(V, U)$ s.t the solution $x(t)$ of (1) satisfies $J_{0^+}^{(1-\alpha)(1-\beta)} x(t)|_{t=0} = x_0$ and $x(l) = x_l$.

3. Controllability results

To identify the key findings, we consider the following hypotheses:

(H1) The function $f: V \times X \times X \to X$ is Carathéodory, $\exists$ a positive function $q \in L^{\epsilon}(V, \mathbb{R}^+)$ for some $\epsilon \in (1, \infty)$ s.t $\| f(t,x,y) \| \le q(t), \forall x, y \in X$ and $t \in V$ where $\| q \|_{L^{\epsilon^2}(V,\mathbb{R}^+)} = M_1$.

(H2) The function $z: D(z) \times X \to X$ is continuous and $\exists \mathscr{L}_1 > 0$ s.t $\| z(t,s,x) - z(t,s,y) \| \le \mathscr{L}_1 \| x - y \|$, $\forall (t,s) \in D(z) = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le l\}$ and $x, y \in X$.

(H3) The linear operator $W : L^{\epsilon}(V, U) \to X$ is bounded and defined as

$$
Wu = \int_0^{l} (l-s)^{\beta-1} \mathfrak{S}_{\beta}(l-s)Bu(s)ds \tag{13}
$$

has an induced inverse operator $W^{-1} \in L^{\epsilon}(V,U)/\ker W, \exists G_0$ and $G_1 \geq 0$ such that $\| B \| \leq G_0, \| W^{-1} \| \leq G_1$.

Theorem 3.1. If the hypotheses (H1-H3) holds, then the FDE (1) is exact controllable on V.

Proof. To analyze exact controllability of the FDE (1), we will define control function $u_x(t)$ as

$$u_x(t) = W^{-1}[x(l) - \mathfrak{T}_{\alpha,\beta}(l)x_0 - \int_0^l (l-s)^{\beta-1} \mathfrak{S}_\beta(l-s)s^n f(s,x(s),(Zx)(s))ds](t). \tag{14}$$

Let the operator $\Omega : C[V,X] \to C[V,X]$ is defined as

$$(\Omega x)(t) = \mathfrak{T}_{\alpha,\beta}(t)x_0 + \int_0^t (t-s)^{\beta-1} \mathfrak{S}_\beta(t-s)[Bu(s) + s^n f(s,x(s),(Zx)(s))]ds, \ \ t \in V. \tag{15}$$

The next three steps in our proof establish that using the control defined in equation (14), the operator has a fixed point in $C[V,X]$.

Step 1: In $C[V,X]$, we define a set

$$Y_\lambda = \{x(\cdot) \in C[V,X] : \| x \| \leq \lambda\} \tag{16}$$

such that $\Omega(Y_\lambda) \subseteq Y_\lambda$, where $\lambda > 0$.

We show that Ω maps Y_λ into Y_λ. For this, first let $x \in Y_\lambda$ and then, for $t \in V$ and $\frac{1}{\epsilon} < \beta \Rightarrow (\epsilon\beta - 1) > 0$. From Holders inequality, hypothesis (H1) and Remark 2.3(a), we have

$$\begin{aligned}
\| u(s) \|_{L^\epsilon} &= \| \mathrm{W}^{-1}[x(l) - \mathfrak{T}_{\alpha,\beta}(l)x_0 - \int_0^l (l-\Upsilon)^{\beta-1}\mathfrak{S}_\beta(l-\Upsilon)\Upsilon^n f(\Upsilon, x(\Upsilon),(Zx)(\Upsilon))d\Upsilon](s) \| \\
&\leq G_1 \left[\| x(l) \| + \| \mathfrak{T}_{\alpha,\beta}(l)x_0 \| + \int_0^l (l-\Upsilon)^{\beta-1}\Upsilon^n \| \mathfrak{S}_\beta(l-\Upsilon) f(\Upsilon, x(\Upsilon),(Zx)(\Upsilon))d\Upsilon \| \right] \\
&\leq G_1 \left[\| x(l) \| + \frac{M_0 l^{\gamma-1} \| x_0 \|}{\Gamma(\alpha(1-\beta)\beta)} + \frac{M_0}{\Gamma(\beta)} \left(\int_0^l (l-\Upsilon)^{\frac{(\beta-1)\epsilon}{\epsilon-1}} d\Upsilon \right)^{\frac{\epsilon-1}{\epsilon}} \left(\int_0^l (\Upsilon^n q(\Upsilon))^\epsilon d\Upsilon \right)^{\frac{1}{\epsilon}} \right] \\
&\leq G_1 \left[\| x(l) \| + \frac{M_0 l^{\gamma-1} \| x_0 \|}{\Gamma(\alpha(1-\beta)\beta)} + \frac{M_0}{\Gamma(\beta)} \left(\frac{\epsilon-1}{\epsilon\beta-1} \right)^{\frac{\epsilon-1}{\epsilon}} l^{\frac{\epsilon\beta-1}{\epsilon}} \left(\int_0^l \Upsilon^{\frac{n\epsilon^2}{\epsilon-1}} d\Upsilon \right)^{\frac{\epsilon-1}{\epsilon^2}} \left(\int_0^l (q(\Upsilon))^{\epsilon^2} d\Upsilon \right)^{\frac{1}{\epsilon^2}} \right] \\
&\leq G_1 \left[\| x(l) \| + \frac{M_0 l^{\gamma-1} \| x_0 \|}{\Gamma(\alpha(1-\beta)\beta)} + \frac{M_0 M_1}{\Gamma(\beta)} \left(\frac{\epsilon-1}{\epsilon\beta-1} \right)^{\frac{\epsilon-1}{\epsilon}} \left(\frac{\epsilon-1}{n\epsilon^2+\epsilon-1} \right)^{\frac{\epsilon-1}{\epsilon^2}} l^{n+\beta-\frac{1}{\epsilon^2}} \right].
\end{aligned}$$

Now, we have

$$M_2 := G_1 \left[\| x(l) \| + \frac{M_0 l^{\gamma-1} \| x_0 \|}{\Gamma(\alpha(1-\beta)\beta)} + \frac{M_0 M_1}{\Gamma(\beta)} \left(\frac{\epsilon-1}{\epsilon\beta-1} \right)^{\frac{\epsilon-1}{\epsilon}} \left(\frac{\epsilon-1}{n\epsilon^2+\epsilon-1} \right)^{\frac{\epsilon-1}{\epsilon^2}} l^{n+\beta-\frac{1}{\epsilon^2}} \right]. \tag{17}$$

Now, we show that $\Omega x \in Y_\lambda$.

$$\begin{aligned}
\| (\Omega x)(t) \| &\leq sup_{t\in(0,l]} \left[t^{1-\gamma} \| \mathfrak{T}_{\alpha,\beta}(l)x_0 \| + t^{1-\gamma} \| \int_0^t (t-s)^{\beta-1}\mathfrak{S}_\beta(t-s)[Bu(s)+s^n f(s,x(s),(Zx)(s))]ds \| \right] \\
&\leq \frac{M_0 \| x_0 \|}{\Gamma(\alpha(1-\beta)\beta)} + l^{1-\gamma} \left[\frac{M_0 M_2 G_0}{\Gamma(\beta)} \left(\frac{\epsilon-1}{\epsilon\beta-1} \right)^{\frac{\epsilon-1}{\epsilon}} l^{\frac{\epsilon\beta-1}{\epsilon}} + \frac{M_0 M_1}{\Gamma(\beta)} \left(\frac{\epsilon-1}{\epsilon\beta-1} \right)^{\frac{\epsilon-1}{\epsilon}} \left(\frac{\epsilon-1}{n\epsilon^2+\epsilon-1} \right)^{\frac{\epsilon-1}{\epsilon^2}} l^{n+\beta-\frac{1}{\epsilon^2}} \right].
\end{aligned} \tag{18}$$

Hence, $\| (\Omega x)(t) \| \leq \lambda$. Therefore, Ω maps Y_λ to Y_λ.

Step 2: Next, we shall prove that Ω maps Y_λ into a precompact subset of Y_λ.

Firstly, we need to prove that $\mathcal{K}(t) = \{(\Omega x)(t) : x(\cdot) \in Y_\lambda\}$ is relatively compact in X, $\forall t \in V$. At $t = 0$, the case is trivial. So, let $t \in (0, l]$ be fixed and $\forall \tau \in (0, t)$, $\xi > 0$ and $x \in Y_\lambda$, we define

$$\begin{aligned}(\Omega^{\tau,\xi} x)(t) \leq\, & \beta J_{0^+}^{\alpha(\beta-1)} t^{\beta-1} T(\tau^\beta \xi) \int_\xi^\infty \varphi \tilde{W}_\beta(\varphi) \left[T(t^\beta \varphi) - T(\tau^\beta \xi) \right] d\varphi x_0 \\ & + \beta T(\tau^\beta \xi) \int_0^{t-\tau} \int_\xi^\infty (t-s)^{\beta-1} \varphi \tilde{W}_\beta(\varphi) \left[T((t-s)^\beta \varphi) - T(\tau^\beta \xi) \right] Bu(s) d\varphi ds \\ & + \beta T(\tau^\beta \xi) \int_0^{t-\tau} \int_\xi^\infty (t-s)^{\beta-1} \varphi \tilde{W}_\beta(\varphi) \left[T((t-s)^\beta \varphi) - T(\tau^\beta \xi) \right] s^n f(s, x(s), (Zx)(s)) d\varphi ds \\ & := T(\tau^\beta \xi) x(t, \tau).\end{aligned}$$

From the compactness of $T(\tau^\beta \xi)$ and boundedness of $x(t, \tau)$ on Y_λ, the set $\mathcal{K}_\tau(t) = \{(\Omega^{\tau,\xi} x)(t) : x(\cdot) \in Y_\lambda\}$ is relatively compact in X.

Further, $\forall x \in Y_\lambda$, we have

$$\begin{aligned}\| (\Omega x)(t) - (\Omega^{\tau,\xi} x)(t) \| \leq\, & \beta \int_0^t \int_0^\xi (t-s)^{\beta-1} \varphi \tilde{W}_\beta(\varphi) \| T((t-s)^\beta \varphi) Bu(s) \| d\varphi ds \\ & + \beta \int_{t-\tau}^t \int_\xi^\infty (t-s)^{\beta-1} \varphi \tilde{W}_\beta(\varphi) \| T((t-s)^\beta \varphi) Bu(s) \| d\varphi ds \\ & + \beta \int_0^t \int_0^\xi (t-s)^{\beta-1} \varphi \tilde{W}_\beta(\varphi) \| T((t-s)^\beta \varphi) s^n f(s, x(s), (Zx)(s)) \| d\varphi ds \\ & + \beta \int_{t-\tau}^t \int_\xi^\infty (t-s)^{\beta-1} \varphi \tilde{W}_\beta(\varphi) \| T((t-s)^\beta \varphi) s^n f(s, x(s), (Zx)(s)) \| d\varphi ds \\ \leq\, & \beta M_0 M_2 G_0 \left(\frac{\epsilon - 1}{\epsilon \beta - 1} \right)^{\frac{\epsilon - 1}{\epsilon}} \left[l^{\frac{\epsilon \beta - 1}{\epsilon}} \int_0^\xi \varphi \tilde{W}_\beta(\varphi) d\varphi + \frac{\tau^{\frac{\epsilon \beta - 1}{\epsilon}}}{\Gamma(\beta)} \right] \\ & + \beta M_0 M_1 \left(\frac{\epsilon - 1}{\epsilon \beta - 1} \right)^{\frac{\epsilon - 1}{\epsilon}} \left(\frac{\epsilon - 1}{n\epsilon^2 + \epsilon - 1} \right)^{\frac{\epsilon - 1}{\epsilon^2}} \left[l^{n + \beta - \frac{1}{\epsilon^2}} \int_0^\xi \varphi \tilde{W}_\beta(\varphi) d\varphi + \frac{\tau^{n + \beta - \frac{1}{\epsilon^2}}}{\Gamma(\beta)} \right].\end{aligned}$$

This implies that for $t \in (0, l]$, there exist precompact sets $\mathcal{K}_\tau(t)$ arbitrarily close to $\mathcal{K}(t)$. As $\mathcal{K}(t)$ is compact at $t = 0$, therefore it is precompact in X, $\forall t \in (0, l]$.

Step 3: Now, we will verify that $\mathcal{K}(t) = \{(\Omega x)(t) : x(\cdot) \in Y_\lambda\}$ is an equicontinuous family of function on [0, l]. For any $x \in Y_\lambda$ and $0 < t_1' < t_2' < l$, we have

$$\begin{aligned}\| (\Omega x)(t_2') - (\Omega x)(t_1') \| \leq\, & \| \mathfrak{T}_{\alpha,\beta}(t_2') x_0 - \mathfrak{T}_{\alpha,\beta}(t_1') x_0 \| + \int_{t_1'}^{t_2'} (t_2' - s)^{\beta-1} \| \mathfrak{S}_\beta(t_2' - s) Bu(s) \| ds \\ & + \int_{t_1'}^{t_2'} (t_2' - s)^{\beta-1} s^n \| \mathfrak{S}_\beta(t_2' - s) f(s, x(s), (Zx)(s)) \| ds \\ & + \int_0^{t_1'} [(t_2' - s)^{\beta-1} - (t_1' - s)^{\beta-1}] \| \mathfrak{S}_\beta(t_2' - s) Bu(s) \| ds \\ & + \int_0^{t_1'} (t_1' - s)^{\beta-1} \| [\mathfrak{S}_\beta(t_2' - s) - \mathfrak{S}_\beta(t_1' - s)] Bu(s) \| ds \\ & + \int_0^{t_1'} (t_1' - s)^{\beta-1} s^n \| [\mathfrak{S}_\beta(t_2' - s) - \mathfrak{S}_\beta(t_1' - s)] f(s, x(s), (Zx)(s)) \| ds \\ & + \int_0^{t_1'} [(t_2' - s)^{\beta-1} - (t_1' - s)^{\beta-1}]\, s^n \| \mathfrak{S}_\beta(t_2' - s) f(s, x(s), (Zx)(s)) \| ds \\ \leq\, & \sum_{i=0}^{6} \mathscr{I}_i.\end{aligned}$$

Next, we have to show that $\mathscr{I}_0, \mathscr{I}_1, \mathscr{I}_2, \mathscr{I}_3, \mathscr{I}_4, \mathscr{I}_5$ and $\mathscr{I}_6$ ends to zero independent of $x \in Y_\lambda$ as $t_2' \to t_1'$.

$$\mathscr{I}_0 \leq \| \mathfrak{T}_{\alpha,\beta}(t_2') - \mathfrak{T}_{\alpha,\beta}(t_1') \| \| x_0 \|.$$

From Remark 2.3(a), $\mathfrak{T}_{\alpha,\beta}(t)$ is strongly continuous, therefore

$$\| \mathfrak{T}_{\alpha,\beta}(t_2') - \mathfrak{T}_{\alpha,\beta}(t_1') \| \; \| x_0 \| \to 0 \text{ as } t_2' \to t_1'.$$

$$\mathscr{I}_1 \leq \frac{M_0 M_2 G_0}{\Gamma(\beta)} \left(\frac{\epsilon - 1}{\epsilon\beta - 1} \right)^{\frac{\epsilon-1}{\epsilon}} (t_2' - t_1')^{\frac{\epsilon\beta-1}{\epsilon}}$$

$$\mathscr{I}_2 \leq \frac{M_0 M_1}{\Gamma(\beta)} \left(\frac{\epsilon - 1}{\epsilon\beta - 1} \right)^{\frac{\epsilon-1}{\epsilon}} \left(\frac{\epsilon - 1}{n\epsilon^2 + \epsilon - 1} \right)^{\frac{\epsilon-1}{\epsilon^2}} (t_2' - t_1')^{\frac{\epsilon\beta-1}{\epsilon}} \left[(t_2')^{\frac{n\epsilon^2+\epsilon-1}{\epsilon-1}} - (t_1')^{\frac{n\epsilon^2+\epsilon-1}{\epsilon-1}} \right]^{\frac{\epsilon-1}{\epsilon^2}}$$

$$\mathscr{I}_3 \leq \frac{M_0 M_2 G_0}{\Gamma(\beta)} \left[\int_0^{t_1'} [(t_2' - s)^{\beta-1} - (t_1' - s)^{\beta-1}]^{\frac{\epsilon}{\epsilon-1}} ds \right]^{\frac{\epsilon-1}{\epsilon}}$$

$$\mathscr{I}_4 \leq M_2 G_0 \left(\frac{\epsilon - 1}{\epsilon\beta - 1} \right)^{\frac{\epsilon-1}{\epsilon}} (t_1')^{\frac{\epsilon\beta-1}{\epsilon}} \quad sup_{t\in(0,l]} \| [\mathfrak{S}_\beta(t_2' - s) - \mathfrak{S}_\beta(t_1' - s)] \|$$

$$\mathscr{I}_5 \leq M_1 \left(\frac{\epsilon - 1}{\epsilon\beta - 1} \right)^{\frac{\epsilon-1}{\epsilon}} \left(\frac{\epsilon - 1}{n\epsilon^2 + \epsilon - 1} \right)^{\frac{\epsilon-1}{\epsilon^2}} (t_1')^{n+\beta-\frac{1}{\epsilon^2}} sup_{t\in(0,l]} \| [\mathfrak{S}_\beta(t_2' - s) - \mathfrak{S}_\beta(t_1' - s)] \|$$

$$\mathscr{I}_6 \leq \frac{M_0 M_1}{\Gamma(\beta)} \left(\frac{\epsilon - 1}{n\epsilon^2 + \epsilon - 1} \right)^{\frac{\epsilon-1}{\epsilon^2}} t_1'^{\frac{n\epsilon^2-1}{\epsilon^2}} \left[\int_0^{t_1'} [(t_2' - s)^{\beta-1} - (t_1' - s)^{\beta-1}]^{\frac{\epsilon}{\epsilon-1}} ds \right]^{\frac{\epsilon-1}{\epsilon}}.$$

It is clear that $\mathscr{I}_1$ and $\mathscr{I}_2$ as $t_2' \to t_1'$. By Lagrange mean value theorem $\mathscr{I}_3$ and $\mathscr{I}_6$ as $t_2' \to t_1'$ and from Remark 2.3(b), $\mathfrak{S}_\beta(t)$ is continuous for $t > 0$ in the uniform operator topology, we get $\mathscr{I}_4$ and $\mathscr{I}_5 \to 0$. Therefore, $\| (\Omega x)(t_2') - (\Omega x)(t_1') \| \to 0$ implies that $\Omega(Y_\lambda)$ is bounded and equicontinuous. By Ascoli-Arzela theorem, $\Omega(Y_\lambda)$ is precompact in $C[V, X]$, where Ω is continuous on $C[V, X]$. Therefore, Ω is completely continuous operator on $C[V, X]$. Since Ω has a fixed point in (Y_λ) according to Schauder's fixed point theorem, it follows that the integral solution of our problem (1) is driven by the control $u(t)$ from the initial state x_0 to the final state x_l in time l. Hence, the system (1) is exact controllable on [0, l].

4. Example

In order to implement our established result, we considered an example. Take the following Hilfer fractional integro-differential system:

$$D_{0^+}^{\frac{3}{4},\beta} x(t,y) = \frac{\partial^2}{\partial y^2} x(t,y) + v(t,y) + t^3 sin\left(x(t,y) + \int_0^1 K_1 e^{-x(s,y)} ds \right),$$

$$J_{0^+}^{\frac{1}{4}(1-\beta)} x(t,y)|_{t=0} = x_0(y), \quad t \in (0,1], y \in [0,\pi]$$
$$x(t,0) = x(t,\pi) = 0, \quad t \in (0,1] \tag{19}$$

where, $D_{0^+}^{\frac{3}{4},\beta}$ denotes HFD whose order is $\alpha = \dfrac{3}{4}$ and $\beta \in [0,1]$.

Consider $U = X = L^2[0,\pi]$ and the linear operator A is defined as $Ax = x''$ with the domain $D(A) = \{x(t,\cdot) \in L^2[0,\pi]$

$: x'' \in L^2[0,\pi],\ x(t,0) = x(t,\pi) = 0\}$. Then,

$$Ax = -\sum_{m=1}^{\infty} m^2 \langle x, \phi_m \rangle \phi_m, \quad x \in D(A),$$

where $\phi_m(y) = \sqrt{\frac{2}{\pi}} sin(my)$ are the eigen functions corresponding to eigen vectors $-m^2$ and $\{\phi_m\}_{m=1}$ forms an orthonormal basis of X.

The operator A generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ in X which can be defined as

$$T(t)x = \sum_{m=1}^{\infty} e^{-m^2 t} \langle x, \phi_m \rangle \phi_m.$$

Comparing values with our defined system, $x(t)(y) = x(t, y)$ and the bounded linear operator $B: L^2(V, U) \to L^2(V, X)$ is defined as $(Bu)(t)(y) = v(t, y)$ where $v: [0,1] \times [0,\pi] \to \mathbb{R}$. The non-linear function

$$f\left(t, x(t), \int_0^l g(t, s, x(s)) ds\right) = sin\left(x(t, y) + \int_0^1 K_1 e^{-x(s,y)} ds\right)$$

where $(Zx)(t)(y) = \int_0^1 K_1 e^{-x(s,y)} ds.$

Therefore, the hypotheses (H1) and (H2) are satisfied. Also, we can define linear operator

$$Wu = \int_0^l (l-s)^{\beta-1} \mathfrak{S}_\beta (l-s) Bu(s) ds$$

induced with inverse operator W^{-1}. Hence, all the hypotheses are satisfied and using theorem we can conclude that equation (19) is exact controllable on the interval [0,1].

5. Conclusion

In this manuscript, we focus on investigating the exact controllability of Hilfer FDEs with non-linear integro-differential functions, considering an order in the range of $0 < \alpha < 1$ within a Banach space. To obtain the mild solution of the system, we utilize semigroup theory and Laplace transformation techniques.

In the controllability analysis, we introduce specific hypotheses and formulate our main result using the Schauder fixed point theorem. The advantage of our approach lies in avoiding the need for complex growth conditions on the integral term in the non-linear function. Instead, we employ a fixed point approach.

Furthermore, we present an illustrative example to showcase the practical application of the established theory. The concepts and ideas developed in this paper can be extended to establish the exact controllability of a broader class of dynamical systems. Additionally, our results have the potential to be extended and applied to systems with impulsive and non-local conditions.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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