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Research Article

Atomic Solutions of Partial Differential Equations via Tensor Product Theory of Banach Spaces

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Abstract: This work purposes to establish a novel analytical method to obtain the atomic solutions of partial differential equations. This will be carried out by employing the tensor product theory in Banach spaces coupled with some properties of atoms operators. Some illustrative examples will be provided to validate our findings.

***Keywords*:** Banach space, tensor product, atomic solution, partial differential equations

MSC: 35E05

1. Introduction

It is known that in the investigation of differential equations (DEs) is possibly the area of mathematics which has more applications to the real world problems. There are plenty of examples of their use in economics, physics, chemistry, and biology [1-3]. Ordinary differential equations (ODEs) handle derivatives of certain functions that include a single variable, whereas partial differential equations (PDEs) can be employed to establish problems that involve functions of more than one variable. Thus, one-dimensional dynamical systems can be often modeled by using ODEs, while multi-dimensional systems can be modeled by using PDEs. Generally, PDEs are much more complicated to be solved analytically than ODEs [4, 5]. There are no generally applicable methods to solve all PDEs of a given order, even numerically, and classes for which we have general analytic methods of solution are quite restricted [5, 6]. Thus, we have to study fairly small classes of PDEs individually.

Due to its relative simpleness and the substantial physical significance in dealing with several challenges, the separation of variables method is one of the most popular schemes for handling specific types of PDEs. In general, this method reduces the linear PDE into a set of ODEs in which the superposition principle might be conjured by constructing specific linear combinations of the fundamental set of solutions to establish extra solutions. However, the procedure of separation of variables does have some limitations, it works in very special cases involving equations with high degree of symmetry. In addition, to be able to apply this method, many constraints need to be imposed on the coefficients expressions along with the companion initial and boundary conditions. Such method is estabished on the basis of the idea that the equation's solution is separated into a product of functions each in terms of one variable.

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That is, the final solution might be represented as the product of several functions, each of them only relies on one independent variable.

Some of the analytical and numerical techniques of solving PDEs are Fourier transform, Laplace transform, Green's functions, finite difference method, finite volume method, and finite element method, see [9, 10]. Recently, in 2010, R. Khalil [11] introduced a novel approach to handle DEs for both ordinary and fractional orders. The new technique is proposed on the basis of the tensor product theory of Banach spaces, and it can be utilized in order to obtain the so-called atomic solutions of the DE under study. In [13], Al-Khaleel et al. investigated the atomic solution of the renowned third- order Gardner's equation in its fractional case, which is commonly named by the KdV-mKdV in some contexts. This kind of equations plays a key role in modeling several asymmetric and symmetric problems. In this paper, motivated by the atomic solution approach in [11-13], we obtain an atomic solution of the form $u(x, y) = P(x)Q(y)$ to the following PDE

$$\nabla^2 u(x,y) = u_{xx}(x,y) + u_{yy}(x,y) = f(x)g(y),$$

where u is an unknown function and f, g are two known functions. Clearly, Equation (2) is inseparable as we cannot move the x-terms to one side and the y-terms to the other. Hence, the separation of variables method be no longer operational. Before we introduce our main result for atomic solutions of PDEs with ordinary orders, we commence with some definitions and theorems reported in [14-18].

2. Atoms operators

In this section, we introduce some preliminaries related to the main result of this paper. For tensor (direct) product of linear spaces, there are many ways to introduce this concept one of them is as follows:

Definition 1 A tensor product of two linear spaces X and Y (denoted by $X \otimes Y$) over the field k, which is either R or C, is a linear space defined by a bilinear map $\phi: X \times Y \to X \otimes Y$ in which for any bilinear map $\psi: X \times Y \to V$ and any linear space V, $\exists$ a unique linear map $\pi: X \otimes Y \to V$ in which $\psi = \pi \circ \phi$. This is also called the universal property. The tensor product elements are called tensors and we can express them as $\phi(x, y) = x \otimes y$.

Theorem 1 (i) We have $X \otimes Y$, $\forall$ two linear spaces X and Y.

(ii) Any two tensor products of two linear spaces X and Y are isomorphic.

(iii) If $x_1, x_2, \cdots, x_n$ are basis for X and $y_1, y_2, \cdots, y_n$ are basis for Y, then $\{x_i \otimes y_j\}_{1 \le i \le n,\, 1 \le j \le m}$ are basis for $X \otimes Y$.

The algebraic relations of the elements of the tensor product $X \otimes Y$ are outlined by the clear several operators' algebraic laws. Hence, we have the following relations:

$$(i)\ x \otimes 0 = 0 \otimes x = 0,$$

$$(ii)\ \lambda \otimes (x + y) = \lambda x \otimes y + x \otimes \lambda y,$$

$$(iii)\ x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2,$$

$$(iv)\ (x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y.$$

Definition 2 Suppose that X and Y are Banach spaces and X^* is the dual space of X. The operator $T: X^* \to Y$, outlined by

$$T(x^*) = x^*(x)y = \langle x, x^* \rangle y,$$

is bounded one rank linear operator, for $x \in X$ and $y \in Y$. This can be written as $x \otimes y$ for T. This operator is called atoms.

Atoms are typically utilized to generate a superior approach in Banach spaces [19, 20]. Also, they are considered among the fundamental components in the tensor product theory. One of the most important results that needed in this work can be presented in the next theorem [21] which guarantees that the sum of two atoms is an atom.

Theorem 2 Suppose $x_1 \otimes y_1$ and $x_2 \otimes y_2$ are two nonzero atoms in $X \otimes Y$ in which

$$x_1 \otimes y_1 + x_2 \otimes y_2 = x_3 \otimes y_3$$

Then either $x_1 = x_2 = x_3$ or $y_1 = y_2 = y_3$.

The previous discussion leads us to the next interesting theorem that lies at the heart of functional analysis as well as approximation theory and guarantee that every continuous function of several variables can be written as a sum of products of continuous separated functions.

Theorem 3 Suppose $C(I)$, $C(J)$ and $C(I \times J)$ are spaces of continuous functions on the intervals I, J and $I \times J$ respectively, where I and J are two compact intervals. Then every $f \in C(I \times J)$ might be expressed as

$$f(x, y) = \sum_{i=1}^{+\infty} u_i(x) v_i(y). \tag{1}$$

where $u_i(x) \in C(I)$ and $v_i(y) \in C(J)$.

3. General scheme for atomic solution method

Consider the following general two-dimensional non-homogeneous linear Laplace's PDE:

$$\nabla^2 u(x, y) = u_{xx}(x, y) + u_{yy}(x, y) = f(x)g(y), \tag{2}$$

where u is an unknown function and f, g are given. The following conditions are imposed on u:

$$u(0,0) = 1,\ u_x(0,0) = 1 \text{ and } u_y(0,0) = 1 \tag{3}$$

According to Theorem 1, we start our approach with assuming that

$$u(x, y) = P(x)Q(y). \tag{4}$$

Now, we substitute (4) into the main PDE (2). Hence,

$$P''(x)Q(y) + P(x)Q''(y) = f(x)g(y). \tag{5}$$

Clearly, each term of (5) is just a product of two functions one of them is pure in x and the other is pure in y. Therefore, in tensor product form, (5) can be presented as

$$P''(x) \otimes Q(y) + P(x) \otimes Q''(y) = f(x) \otimes g(y). \tag{6}$$

This implies that the sum of two atoms is an atom. By Theorem 3, one might get one of the two situations:

$$(i)\ P''(x) = P(x) = f(x),$$

$$(ii)\ Q(y) = Q''(y) = g(y). \tag{7}$$

Case (i): This case has the following three situations:

$$(a)\ P''(x) = P(x),$$

$$(b)\ P''(x) = f(x),$$

$$(c)\ P(x) = f(x).$$

Without loss of generality and from (3), we can suppose that

$$P(0) = Q(0) = P'(0) = Q'(0) = 1. \tag{8}$$

Now, from (a) and conditions (8), we get

$$P(x) = e^x. \tag{9}$$

Now, from both (9) and situation (c), we have $f(x) = e^x$ which clearly satisfies the argument of situation (b). Thus, an atomic solution can be obtained for case (i) provided that $f(x) = e^x$ otherwise no available atomic solution.

The next step is to substitute $P(x) = f(x) = e^x$ into (5) which implies

$$Q''(y) + Q(y) = g(y). \tag{10}$$

Clearly, (10) is a 2nd-linear non-homogeneous ODEs with constant coefficientsand its general solution has the form $Q(y) = Q_h(y) + Q_p(y)$ where $Q_h(y) = \sin(y) + \cos(y)$ is the complementary solution that can be obtained by considering the companion homogeneous equation, namely, $Q''(y) + Q(y) = 0$ together with $Q(0) = 1$ and $Q'(0) = 1$ (8). While the particular solution $Q_p(y)$ might be achieved by the variation of parameters method, which canbe carried out in the following manner

$$Q_p(y) = \sin(y)\int \frac{-\cos(y)g(y)}{W[\sin(y), \cos(y)]}dy + \cos(y)\int \frac{\sin(y)g(y)}{W[\sin(y), \cos(y)]}dy$$

$$= \sin(y)\int \cos(y)g(y)dy - \cos(y)\int \sin(y)g(y)dy, \tag{11}$$

where $W[\sin(y), \cos(y)]$ is the Wronskian of $\sin(y)$ and $\cos(y)$. Therefore, the general solution to (10) can be expressed as

$$Q(y) = Q_h(y) + Q_p(y)$$

$$= \sin(y) + \cos(y) + \sin(y)\int \cos(y)g(y)dy - \cos(y)\int \sin(y)g(y)dy. \tag{12}$$

Hence, the first atomic solution with respect to case (i) can be obtained by considering (4), (9) and (12) as follows:

$$u_1(x, y) = e^x\left(\sin(y) + \cos(y) + \sin(y)\int \cos(y)g(y)dy - \cos(y)\int \sin(y)g(y)dy\right). \tag{13}$$

Similarly, for case (ii), one can handle the three situations, namely,

$$Q''(y) = Q(y),$$

$$Q''(y) = g(y),$$

$$Q(y) = g(y);$$

to obtain the second atomic solution that is

$$u_2(x, y) = e^y \left(\sin(x) + \cos(x) + \sin(x)\int \cos(x) f(x)dx - \cos(x)\int \sin(x) f(x)dx\right). \tag{14}$$

4. Applications

In this section, we utilize the new method of atomic solutions to drive two examples for solving non-linear PDEs for which separation of variables method does not work.

Example 1 Consider the following non-linear 2-D PDE

$$u_{xx}u_y + u_{yy}u_x = u_y u, \tag{15}$$

where $u(x, y)$ is the unknown function and subjected to the following conditions:

$$u(0,0) = 1,\ u_x(0,0) = 1,\ \text{and } u_y(0,0) = 1. \tag{16}$$

By substituting $u(x, y) = P(x)Q(y)$ into (15) we get,

$$P''(x)Q'(y) + P'(x)Q''(y) = P(x)Q'(y). \tag{17}$$

Therefore, in the form of tensor product, (17) becomes

$$P''(x) \otimes Q'(y) + P'(x) \otimes Q''(y) = P(x) \otimes Q'(y), \tag{18}$$

By Theorem 3, we have one of the following two situations:

$$(i)\ P''(x) = P'(x) = P(x),$$

$$(ii)\ Q''(y) = Q'(y). \tag{19}$$

Hence, for case (i), we have the following three situations:

$$(a)\ P''(x) = P'(x),$$

$$(b)\ P''(x) = P(x),$$

$$(c)\ P'(x) = P(x).$$

Without loss of generality and from (16), we can suppose that

$$P(0)=Q(0)=P'(0)=Q'(0)=1. \tag{20}$$

Now, from situation (*a*) and the two conditions $P(0) = 1$ and $P'(0) = 1$, we get

$$P(x)=e^x. \tag{21}$$

Also, by considering (20), both situations (*b*) and (*c*) give the same result in (21). Therefore, an atomic solution exists with respect to case (*i*). Now, we proceed by substituting (21) into (17) which implies $Q''(y) = 0$. But, by (20), we have $Q(0) = 1$ and $Q'(0) = 1$ so,

$$Q(y)=y+1. \tag{22}$$

Hence, referring to (21) and (22), the first atomic solution with respect to case (*i*) is

$$u_1=(x,y)=e^x(y+1). \tag{23}$$

Now, we again consider (19) in order to handle case (*ii*). Clearly, for this particular case, we have only one situation that is $Q''(y) = Q'(y)$. Hence, having $Q(0) = 1$ and $Q'(0) = 1$ from (20) yields

$$Q(x)=e^y. \tag{24}$$

Our next step is to substitute (24) into (17) which yields

$$P''(x)+P'(x)-P(x)=0. \tag{25}$$

Assuming the conditions in (20), the second-order ODE (25) has the following solution

$$P(x)=\frac{1}{2\sqrt{5}}\left(\left(\sqrt{5}+3\right)e^{\frac{-1+\sqrt{5}}{2}x}+\left(\sqrt{5}-3\right)e^{\frac{-1-\sqrt{5}}{2}x}\right). \tag{26}$$

Hence, referring to (24) and (26), the second atomic solution with respect to case (*ii*) is

$$u_2(x,y)=\frac{e^y}{2\sqrt{5}}\left(\left(\sqrt{5}+3\right)e^{\frac{-1+\sqrt{5}}{2}x}+\left(\sqrt{5}-3\right)e^{\frac{-1-\sqrt{5}}{2}x}\right) \tag{27}$$

The two atomic solutions $u_1(x, y)$ (23) and $u_2(x, y)$ (27) of problem (15) are exhibited respectively in Figures 1 and 2.
Example 2 Consider the following non-linear 2-D PDE

$$u_{xx}u_y+u_xu_{yy}=u_xu_y, \tag{28}$$

where $u(x, y)$ is the unknown function and subjected to the following conditions:

$$u(0,0)=1,\ u_x(0,0)=1,\ \text{and } u_y(0,0)=1. \tag{29}$$

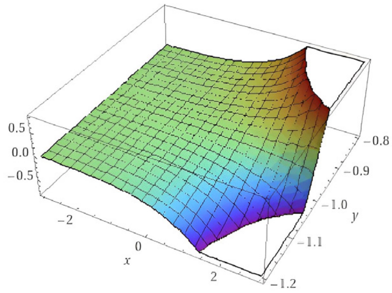

Figure 1. The first atomic solution $u_1(x, y)$ (23) of problem (15)

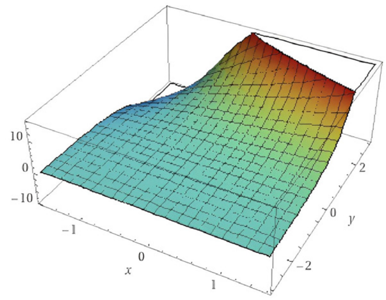

Figure 2. The second atomic solution $u_2(x, y)$ (27) of problem (15)

By substituting $u(x, y) = P(x)Q(y)$ into (28) we get,

$$P''(x)Q'(y) + P'(x)Q''(y) = P'(x)Q'(y). \tag{30}$$

Therefore, in tensor product form, (30) becomes

$$P''(x) \otimes Q'(y) + P'(x) \otimes Q''(y) = P'(x) \otimes Q'(y), \tag{31}$$

By Theorem 3, the two resultant situations are

$$(i)\ P''(x) = P'(x),$$

$$(ii)\ Q''(y) = Q'(y). \tag{32}$$

Without loss of generality and from (16), we can suppose that

$$P(0) = Q(0) = P'(0) = Q'(0) = 1. \tag{33}$$

First, form case (i) and assuming (33), we have

$$P(x) = e^x. \tag{34}$$

Now, substituting (34) into the main equation (30) yields $Q''(y) = 0$ and hence

$$Q(y) = y. \tag{35}$$

where the conditions $Q(0) = 1$ and $Q'(0) = 1$ are given by (33). Hence, the first atomic solution corresponding to case (i) can be obtained by considering (34) and (35) as follows:

$$u_1(x, y) = ye^x. \tag{36}$$

Similarly, from case (ii), one can find out the second atomic solution that is

$$u_2(x, y) = xe^y. \tag{37}$$

In order to see how the solutions of problem (28) look like, we plot its solutions $u_1(x, y)$ and $u_2(x, y)$ in Figure 3 and Figure 4, respectively.

5. Conclusions

This paper has successfully introduced a new analytical scheme for handling nonlinear and non-homogeneous PDEs via atomic solutions method. The tensor product theory of Banach spaces coupled with some properties of atoms operators have been utilized for achieving such a notion. Some other kinds of PDEs are left to the future for further consideration.

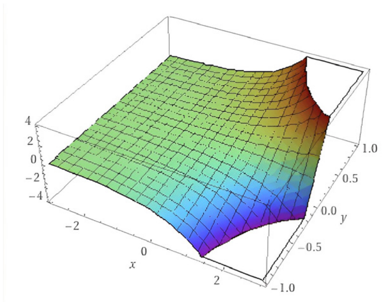

Figure 3. The first atomic solution $u_1(x, y)$ (36) of problem (28)

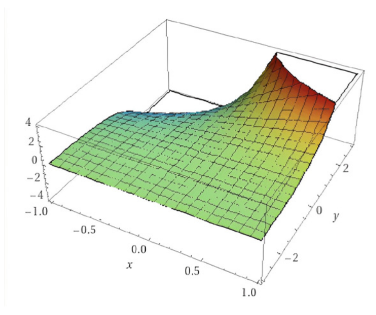

Figure 4. The second atomic solution $u_2(x, y)$ (37) of problem (28)

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Ethical approval

This material is the authors' own original work, which has not been previously published elsewhere.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Authors' contributions

W.G.A. and I.M.B. conceived of the presented idea. M.A.H. developed the theory and performed the computations. B.A. and A.A. verified the analytical methods. W.G.A. encouraged I.M.B. to investigate the generated atomic solution and supervised of this work. All authors discussed the results and contributed to the final manuscript.

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