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Research Article

Finite-Time Stability to Fractional Delay Cauchy Problem of Hilfer Type

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Abstract: The purpose of this work is to look at the solution's representation for the non-homogeneous fractional time-delay Cauchy problem of Hilfer type in terms of the cosine and sine fractional delayed matrices of two parameters. The solutions are determined using the constant variation approach. Following that, finite-time stability under moderate circumstances is examined. At last, an example is offered to show how the theoretical results may be used.

***Keywords*:** Hilfer derivative, fractional delay Cauchy problem, finite time stability, fractional delayed matrix cosine and sine

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1. Introduction

Ordinary differentiation and integration of arbitrary order, which may be non-integer, are generalized in fractional calculus. Many studies have focused on differential equations of fractional order [1-3]. Many definitions of fractional integrals and derivatives may be found in the literature, ranging from the most well-known Riemann-Liouville (R-L) and Caputo-type fractional derivatives to others like the Hadamard fractional derivative, the Katugampola fractional derivative, the Hilfer fractional derivative, and so on.

Recent advances in engineering, mathematics, physics, bioengineering, and other applied sciences have led to the application of fractional differential equations in a number of these fields, see for example [4-7].

Hilfer [8] first proposed the Hilfer fractional derivative in 2000. It was a generalization of the R-L and Caputo fractional derivatives. Such derivative interpolates between the R-L and Caputo derivative in some sense. Recently, it has become increasingly important to study differential equations' Hilfer fractional derivative. In recent years, the Hilfer derivative of differential equations has seen significant progress. A new representation formula and several advantages of the Hilfer derivative have been presented by Kamocki [9].

A new fractional derivative with respect to another function, the so-called ψ-Hilfer fractional derivative, has been introduced and discussed by Sousa and Oliveira [10]. Such derivatives, have been used in many contributions, see [11-13] and the papers cited therein.

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The differential equations of fractional delay are equations that contain fractional order derivatives and time-delays. The phrase time-delay can be used to explain the history of a previous condition. Time-delay has applications in biology, control theory, engineering, population dynamics, physics, and other fields [14-22]. As a result, information regarding fractional differential delay equations have disseminated through research and investigations. Stability throughout time in contrast to exponential/asymptotic stability, finite-time stability requires a finite number of time periods. In 1961, Dorato [23] introduced the concept of finite-time stability.

In 2008, Khusainov et al. [24] introduced a novel solution representation for the equation

$$\begin{aligned} &w''(r) + A^2 w(r-s) = 0, \quad s > 0, \quad r \in [0, T], \\ &w(r) = \psi(r), \quad w'(r) = \psi'(r), \quad r \in [-s, 0] \end{aligned} \tag{1}$$

where $w \in \mathbb{R}^n, s$ denotes the time-delay, ψ is a vector function that can be differentiated twice continuously, $T > 0$ is a constant integer, and A is a nonsingular $m \times m$ matrix. The sine delayed matrix polynomial of a $2k+1$ degree (sins Ar) and the cosine delayed matrix polynomial of a $2k$ degree (coss Ar) are the foundations of their approach. For more information about cosine and sine delayed matrix polynomials, we advise the reader to read [24]. Liang et al. [25] adopted the method of cosine and sine delayed matrix to study the stability of the system (1).

Liang et al. [26] have given the concepts of cosine and sine fractional delayed matrices to resolve the equation to a fractional linear system in pure delay

$$\begin{cases} {}^c\mathbb{D}^{\sigma}_{-s^+}\left({}^c\mathbb{D}^{\sigma}_{-s^+} w\right)(r) = -A^2 w(r-s), A \in \mathbb{R}^{n\times n}, r \geq 0, > 0 \\ w(r) = \psi(r), \quad w'(r) = \psi(r), \quad -s \leq r \leq 0, \end{cases}$$

where ${}^c\mathbb{D}^{\sigma}_{-s^+}$ indicates the order's fractional derivative of Caputo $< 0\sigma \leq 1$ and $\psi \in C^1([-s,0], \mathbb{R}^n)$. They also discussed its finite-time stability results on $L = [0,T]$, $T > 0$.

Using the R-L fractional derivative instead of Caputo derivative, Mahmudov [27] discovered an explicit formulation for a solution to the aforementioned problems.

In our previous work [28], we represented the solution of the following non-homogeneous system of a Hilfer type

$$\begin{cases} \left(\mathbb{D}^{\alpha,\beta}_{-s} + y\right)(r) = By(r-s) + h(r), & r \in [0,T], s > 0 \\ y(r) = \phi(r), & \phi(r) \in \mathbb{R}^n, -s < r \leq 0 \\ \lim\limits_{r \to s^+}\left(\mathbb{I}^{1-\gamma}_{-s} + y\right)(r) = b, & b \in \mathbb{R}^n \end{cases}$$

where $h(r) \in C([0,T], \mathbb{R}^n), \mathbb{D}^{\alpha,\beta}_{-s^+} y$ is the Hilfer fractional derivative of type $\beta \in [0,1]$ and order $0 < \alpha < 1$, $\mathbb{I}^{1-\gamma}_{-s^+} y$ denotes $(1-\gamma)$-order of R-L fractional integral with $\gamma = \alpha + \beta - \alpha\beta, T = k\tau, k \in \mathbb{N}, s$ is a fixed moment and $\phi \in C((-\tau, 0], \mathbb{R}^n)$ is an arbitrary R-L differentiable function. In addition, we discussed the finite-time stability results under appropriate conditions.

Motivated by the previous research, we attempt to solve the following linear non-homogeneous fractional differential delay equations of the Hilfer type

$$\begin{cases} \mathbb{D}^{\alpha,\beta}_{-\tau+}\left(\mathbb{D}^{\alpha,\beta}_{-\tau+} y\right)(x) = -B^2 y(x-\tau) + h(x) & B \in \mathbb{R}^{n\times n}, x \in [0,T], \tau > 0 \\ y(x) = \phi(x), & \phi(x) \in \mathbb{R}^n, -r < x \leq 0 \\ \lim\limits_{x\to\tau^+}\left(\mathbb{I}^{1-\gamma}_{-\tau+} y\right)(x) = b_1, & b_1 \in \mathbb{R}^n \\ \lim\limits_{x\to\tau^+}\mathbb{I}^{1-\gamma}_{-\tau}\left(\mathbb{D}^{\alpha,\beta}_{-\tau+} y\right)(x) = Bb_2, & b_2 \in \mathbb{R}^n \end{cases} \tag{2}$$

where $h(x) \in C([0,T], \mathbb{R}^n), \mathbb{D}^{\alpha,\beta}_{-\tau^+} y$ denotes the Hilfer fractional derivative with type $\beta \in [0,1]$ and of order $0 < \alpha < 1, \mathbb{I}^{\gamma}_{-\tau^+}$ denotes γ-order of R-L fractional integral, $\mathbb{D}^{\gamma+\alpha}_{-\tau^+}\phi$ is the $\gamma + \alpha$-order of R-L fractional derivative to the initial function $\phi(x), b_1, b_2 \in \mathbb{R}^n$ are constants vectors, B is nonsingular matrix, and $T = j\tau, j \in \mathbb{N}$ and τ is a fixed moment. That much is clear to observe $0 < \gamma = \alpha + \beta - \alpha\beta < 1$, $\gamma \geq \alpha$ and $\gamma \geq \beta$.

The rest portions of the document are systemized as the following: Section 2 outlines specific fractional derivatives and presents the concepts of cosine and sine fractional delayed matrices, which are utilized to solve the fractional differential system. It also introduces basic lemmas utilized in later theories. Section 3 depicts the solutions of linear non-homogeneous fractional differential delay equations of the Hilfer type. Section 4 investigates finite-time stability findings. Section 5 provides a numerical example to demonstrate the key results. Finally, the manuscript is concluded with a conclusion section.

2. Basic definitions

This section is devoted to introducing some basic concepts and definitions needed to apply our idea.

Definition 2.1 [29]. For a function $w:[b,\infty)\to\mathbb{R}$, the left-sided fractional integral of order $\sigma > 0$ is introduced by

$$(\mathbb{I}_{b^+}^{\sigma} w)(r)=\frac{1}{\Gamma(\sigma)}\int_b^r (r-t)^{\sigma-1}w(t)dt,$$

where the Euler gamma-function is represented by $\Gamma(\sigma)$.

Definition 2.2 [29]. Let $m\in\mathbb{N}$ and $m-1\le\sigma<m$, for a function $w:[b,\infty)\to\mathbb{R}$. The R-L fractional derivative of order σ is introduced by

$$(\mathbb{D}_{b^+}^{\sigma} w)(r)=(\mathbb{D}_{b^+}^{m}\mathbb{I}_{b^+}^{m-\sigma} w)(r)=\frac{1}{\Gamma(m-\sigma)}\frac{d^m}{dr^m}\int_b^r (r-t)^{m-\sigma-1}w(t)dt.$$

Lemma 2.1 [30]. Let $\alpha > 0$, $0 < \beta < 1$, $\lambda > 0$ and $0 < \gamma < 1$. Then,

$$\mathbb{D}_{a^+}^{\gamma}\mathbb{I}_{a^+}^{\gamma}g(r)=g(r),$$

$$\mathbb{D}_{a^+}^{\beta}\mathbb{I}_{a^+}^{\gamma}g(r)=\begin{cases}\mathbb{I}_{a^+}^{\gamma-\beta}g(r),\gamma>\beta,\\ \mathbb{D}_{a^+}^{\beta-\gamma}g(r),\gamma<\beta,\end{cases}$$

$$\mathbb{I}_{a^+}^{\gamma}\mathbb{D}_{a^+}^{\gamma}g(r)=g(r)-\frac{(r-a)^{\gamma-1}}{\Gamma(\gamma)}\mathbb{I}_{a^+}^{1-\gamma}g(a),$$

$$\mathbb{I}_{a^+}^{\gamma}\mathbb{D}_{a^+}^{\beta}g(r)=\begin{cases}\mathbb{I}_{a^+}^{\gamma-\beta}g(r)-\dfrac{(r-a)^{\beta-1}}{\Gamma(\beta)}\mathbb{I}_{a^+}^{1-\gamma}g(a),\gamma>\beta,\\ \mathbb{D}_{a^+}^{\beta-\gamma}g(r)-\dfrac{(r-a)^{\gamma-1}}{\Gamma(\gamma)}\mathbb{I}_{a^+}^{1-\beta}g(a),\gamma<\beta,\end{cases}$$

$$\mathbb{I}_{a^+}^{\alpha}(r-a)^{\lambda-1}=\frac{\Gamma(\lambda)}{\Gamma(\lambda+\alpha)}(r-a)^{\lambda+\alpha-1},$$

$$\mathbb{D}_{a^+}^{\gamma}(r-a)^{\lambda-1}=\begin{cases}\dfrac{\Gamma(\lambda)}{\Gamma(\lambda-\gamma)}(r-a)^{\lambda-\gamma-1}, & \lambda\ne\gamma,\\ 0, & \lambda=\gamma.\end{cases}$$

Definition 2.3 [31]. Let $m\in\mathbb{N}$ with $m-1<\alpha<m$ and $\beta\in[0,1]$. The Hilfer fractional derivative with type β and of order α is given as

$$\mathbb{D}_{a^+}^{\alpha,\beta}f(x)=\mathbb{I}_{a^+}^{\beta(m-\alpha)}\mathbb{D}_{a^+}^{m}\mathbb{I}_{a^+}^{(1-\beta)(m-\alpha)}f(x),\quad x>a.$$

The following characteristic is simple to demonstrate.

Lemma 2.2. Let $0<\alpha<1$, $\beta\in[0,1]$, $\lambda>0$ and $\gamma=\alpha+\beta-\alpha\beta$. Then,

$$\mathbb{I}_{a^+}^{\alpha}\mathbb{D}_{a^+}^{\alpha,\beta}f(x)=\mathbb{I}_{a^+}^{\gamma}\mathbb{D}_{a^+}^{\gamma}f(x)=f(x)-\frac{(x-a)^{\gamma-1}}{\Gamma(\gamma)}\mathbb{I}_{a^+}^{1-\gamma}f(a),$$

$$\mathbb{D}_{a^+}^{\alpha,\beta}(x-a)^{\lambda-1}=\begin{cases}\dfrac{\Gamma(\lambda)}{\Gamma(\lambda-\alpha)}(x-a)^{\lambda-\alpha-1}, & \lambda\neq\gamma,\\ 0, & \lambda=\gamma.\end{cases}$$

Definition 2.4 [32]. The system described by (2) that meets the initial requirement is finite-time stability in terms of $\{0, L, \sigma, \eta\}$ if and only if $\|\phi\|<\sigma$ for all $r\ \epsilon\ L$ implies $\|w(r)\|<\eta$ for all $r\ \epsilon\ L,$ where L denotes time interval $L\subset\mathbb{R},\ \sigma,\eta$ are real positive numbers and $\phi(r)(-\tau<r\le 0)$ is the initial function of observation.

Definition 2.5 [33]. The special function $\mathbb{E}_{a,b}$ is called Mittag-Leffler function of two parameters which relays on two parameters $a,b\in\mathbb{C}$. It is defined by the next series with strictly positive real part of a:

$$\mathbb{E}_{a,b}(y)=\sum_{k=0}^{\infty}\frac{y^k}{\Gamma(ak+b)}.$$

Definition 2.6 [34]. The two parameters fractional delayed matrix Mittag-Leffler $\mathbb{E}_{h,\eta}^{Ar^{\sigma}}:\mathbb{R}\to\mathbb{R}^{n\times n}$ is defined by

$$\mathbb{E}_{h,\eta}^{Ar^{\sigma}}=\begin{cases}\Theta, & -\infty<r\le -h,\\ I\dfrac{(r+h)^{\eta-1}}{\Gamma(\eta)}, & -h<r\le 0,\\ I\dfrac{(r+h)^{\eta-1}}{\Gamma(\eta)}+A\dfrac{r^{\sigma+\eta-1}}{\Gamma(\sigma+\eta)}+\cdots & \\ +A^j\dfrac{(r-(j-1)h)^{j\sigma+\eta-1}}{\Gamma(j\sigma+\eta)}, & (j-1)h<r\le jh\end{cases}$$

where $j\in\mathbb{N}_0$, and I and Θ are the unit and null the matrices, respectively.

Next, we present the concept of cosine and sine fractional delayed matrices.

Definition 2.7 [27]. Two parameters fractional delayed matrix cosine $\cos_{h,\sigma,\eta}(Ar^{\sigma}):\mathbb{R}\to\mathbb{R}^{n\times n}$ is defined by

$$\cos_{h,\sigma,\eta}(Ar^{\sigma})=\begin{cases}\Theta, & -\infty<r\le -h,\\ I\dfrac{(r+h)^{\eta-1}}{\Gamma(\eta)}, & -h<r\le 0,\\ I\dfrac{(r+h)^{\eta-1}}{\Gamma(\eta)}-A^2\dfrac{r^{2\sigma+\eta-1}}{\Gamma(2\sigma+\eta)}+\cdots & \\ +(-1)^jA^{2j}\dfrac{(r-(j-1)h)^{2j\sigma+\eta-1}}{\Gamma(2j\sigma+\eta)}, & (j-1)h<r\le jh.\end{cases}$$

Definition 2.8 [27]. Two parameters fractional delayed matrix sine $\sin_{h,\sigma,\eta}(Ar^{\sigma}):\mathbb{R}\to\mathbb{R}^{n\times n}$ is represented as

$$\sin_{h,\sigma,\eta}(Ar^{\sigma})=\begin{cases}\Theta, & -\infty<r\le -h,\\ A\dfrac{(r+h)^{\sigma+\eta-1}}{\Gamma(\sigma+\eta)}, & -h<r\le 0,\\ A\dfrac{(r+h)^{\sigma+\eta-1}}{\Gamma(\sigma+\eta)}-A^3\dfrac{r^{3\sigma+\eta-1}}{\Gamma(3\sigma+\eta)}+\cdots & \\ +(-1)^jA^{2j+1}\dfrac{(r-(j-1)h)^{(2j+1)\sigma+\eta-1}}{\Gamma((2j+1)\sigma+\eta)}, & (j-1)h<r\le jh.\end{cases}$$

The next Lemmas can be formulated similar to Lemma 2.10 and Lemma 2.11 in [26].

Lemma 2.3. Let $j\in\mathbb{N}_0, r\in[(j-1)h, jh], \alpha\in(0,1)$ and $\beta\ \epsilon\ [0,1]$, with $\alpha+\beta\ge 1$. Then,

$$
\begin{aligned}
\left\|\cos_{h,\alpha,\gamma} Ar^{\alpha}\right\| &\leq \left(r+h\right)^{\gamma-1} \mathbb{E}_{2\alpha,\gamma}\left(\|A\|^2\left(r+h\right)^{2\alpha}\right)\\
&\leq h^{\gamma-1}\mathbb{E}_{2\alpha,\gamma}\left(\|A\|^2\left(r+h\right)^{2\alpha}\right),\\
\left\|\sin_{h,\alpha,\gamma} Ar^{\alpha}\right\| &\leq \|A\|\left(r+h\right)^{\alpha+\gamma-1} \mathbb{E}_{2\alpha,\alpha+\gamma}\left(\|A\|^2\left(r+h\right)^{2\alpha}\right)\\
&\leq \|A\| h^{\alpha+\gamma-1}\mathbb{E}_{2\alpha,\alpha+\gamma}\left(\|A\|^2 r^{2\alpha}\right).
\end{aligned}
$$

Lemma 2.4 [26]. Let $j \in \mathbb{N}_0, r \in [(j-1)h, jh], \alpha \in (0,1), \beta \in [0,1]$ with $\alpha+\beta \geq 1$ and $g:(-h,0] \to \mathbb{R}^n$ is continuous function. Then,

$$
\left\|\int_{-h}^{0} \sin_{h,\alpha,\gamma} A\left(r-h-s\right)^{\alpha} g\left(s\right)ds\right\| \leq \|A\|\mathbb{E}_{2\alpha,\alpha+\gamma}\left(\|A\|^2 x^{2\alpha}\right)\int_{-h}^{0}\left(r-s\right)^{\alpha+\gamma-1}\left\|g\left(s\right)\right\|ds.
$$

3. Representation of the solutions

The current section is presented to derive the representation of solutions of (2) using $\cos_{\tau,\alpha,\gamma} Bx^{\alpha}$ and $\sin_{\tau,\alpha,\gamma} Bx^{\alpha}$, First, some properties of two parameters fractional delayed matrices cosine and sine are stated and proved which are necessary to establish the solution of (2).

Lemma 3.1. For the delayed matrix $\cos_{\tau,\alpha,\gamma} Bx^{\alpha}$ and the delayed matrix $\sin_{\tau,\alpha,\gamma} Bx^{\alpha}$, one has

$$
\begin{aligned}
\mathbb{I}_{-\tau^+}^{\gamma-\alpha} \cos_{\tau,\alpha,\alpha}(Bx^{\alpha}) &= \cos_{\tau,\alpha,\gamma}(Bx^{\alpha}),\\
\mathbb{I}_{-\tau^+}^{\gamma-\alpha} \sin_{\tau,\alpha,\alpha}(Bx^{\alpha}) &= \sin_{\tau,\alpha,\gamma}(Bx^{\alpha}).
\end{aligned}
$$

Proof. Without losing generality, suppose $(k-1)\tau < x \leq k\tau, k \in \mathbb{N}_0$. Using Definition 2.1 gives

$$
\begin{aligned}
\mathbb{I}_{-\tau^+}^{\gamma-\alpha} \cos_{\tau,\alpha,\alpha}(Bx^{\alpha}) &= \frac{1}{\Gamma(\gamma-\alpha)}\int_{-\tau}^{x}(x-t)^{\gamma-\alpha-1}\cos_{\tau,\alpha,\alpha}(Bt^{\alpha})dt\\
&= \frac{1}{\Gamma(\gamma-\alpha)}\left[\int_{-\tau}^{0}(x-t)^{\gamma-\alpha-1}\left(I\frac{(x+\tau)^{\alpha-1}}{\Gamma(\alpha)}\right)dt + \int_{0}^{\tau}(x-t)^{\gamma-\alpha-1}\left(I\frac{(x+\tau)^{\alpha-1}}{\Gamma(\alpha)} - B^2\frac{(x)^{3\alpha-1}}{\Gamma(3\alpha)}\right)dt\right.\\
&\left. +\cdots+\int_{(k-1)\tau}^{x}(x-t)^{\gamma-\alpha-1}\left(I\frac{(x+\tau)^{\alpha-1}}{\Gamma(\alpha)} - B^2\frac{(x)^{3\alpha-1}}{\Gamma(3\alpha)}+\cdots+(-1)^k B^{2k}\frac{(x-(k-1)\tau)^{(2k+1)\alpha-1}}{\Gamma((2k+1)\alpha)}\right)dt\right].
\end{aligned}
$$

Reformulating the previous integrals and using Lemma 2.1 leads to

$$
\begin{aligned}
&\mathbb{I}_{-\tau^+}^{\gamma-\alpha} \cos_{\tau,\alpha,\alpha}(Bx^{\alpha})\\
&= \frac{1}{\Gamma(\gamma-\alpha)}\left[\int_{-\tau}^{x}(x-t)^{\gamma-\alpha-1}\left(I\frac{(x+\tau)^{\alpha-1}}{\Gamma(\alpha)}\right)dt + \int_{0}^{x}(x-t)^{\gamma-\alpha-1}\left(-B^2\frac{(x)^{3\alpha-1}}{\Gamma(3\alpha)}\right)dt+\cdots\right.\\
&\left. +\int_{(k-1)\tau}^{x}(x-t)^{\gamma-\alpha-1}\left((-1)^k B^{2k}\frac{(x-(k-1)\tau)^{(2k+1)\alpha-1}}{\Gamma((2k+1)\alpha)}\right)dt\right]\\
&= \mathbb{I}_{-\tau^+}^{\gamma-\alpha}\left(I\frac{(x+\tau)^{\alpha-1}}{\Gamma(\alpha)}\right) + \mathbb{I}_{0^+}^{\gamma-\alpha}\left(-B^2\frac{(x)^{3\alpha-1}}{\Gamma(3\alpha)}\right)+\cdots+\mathbb{I}_{(k-1)\tau^+}^{\gamma-\alpha}\left((-1)^k B^{2k}\frac{(x-(k-1)\tau)^{(2k+1)\alpha-1}}{\Gamma((2k+1)\alpha)}\right)\\
&= I\frac{(x+\tau)^{\gamma-1}}{\Gamma(\gamma)} - B^2\frac{(x)^{2\alpha+\gamma-1}}{\Gamma(2\alpha+\gamma)}+\cdots+(-1)^k B^{2k}\frac{(x-(k-1)\tau)^{2k\alpha+\gamma-1}}{\Gamma(2k\alpha+\gamma)}\\
&= \cos_{\tau,\alpha,\gamma}(Bx^{\alpha}).
\end{aligned}
$$

By following same steps the second statement will be proved.

Lemma 3.2. For the delayed matrix $\cos_{\tau,\alpha,\gamma}(Bx^{\alpha})$ and the delayed matrix $\sin_{\tau,\alpha,\gamma}(Bx^{\alpha})$, one has

$$\mathbb{I}^{\gamma-\alpha}_{-\tau^+}\mathbb{D}^{\gamma}_{-\tau^+}\cos_{\tau,\alpha,\gamma}(Bx^{\alpha}) = B\sin_{\tau,\alpha,\gamma}(B(x-\tau)^{\alpha}), \tag{3}$$
$$\mathbb{I}^{\gamma-\alpha}_{-\tau^+}\mathbb{D}^{\gamma}_{-\tau^+}\sin_{\tau,\alpha,\gamma}(Bx^{\alpha}) = B\cos_{\tau,\alpha,\gamma}(Bx^{\alpha}). \tag{4}$$

Proof. By investigating the proof to each interval, we obtain

For any $-\infty < x \le -\tau$. Obviously (3) and (4) hold.

For any $-\tau < x \le 0$. In view of the last statement in Lemma 2.1, we find

$$\mathbb{I}^{\gamma-\alpha}_{-\tau^+}\mathbb{D}^{\gamma}_{-\tau^+}\left[\cos_{\tau,\alpha,\gamma}(Bx^{\alpha})\right] = \mathbb{I}^{\gamma-\alpha}_{-\tau^+}\mathbb{D}^{\gamma}_{-\tau^+}\left[I\frac{(x+\tau)^{\gamma-1}}{\Gamma(\gamma)}\right] = \mathbb{I}^{\gamma-\alpha}_{-\tau^+}[\Theta] = \Theta$$

and by using last two statements in Lemma 2.1, we get

$$\begin{aligned}\mathbb{I}^{\gamma-\alpha}_{-\tau^+}\mathbb{D}^{\gamma}_{-\tau^+}\left[\cos_{\tau,\alpha,\gamma}(Bx^{\alpha})\right] &= \mathbb{I}^{\gamma-\alpha}_{-\tau^+}\mathbb{D}^{\gamma}_{-\tau^+}\left[B\frac{(x+\tau)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\right]\\ &= \mathbb{I}^{\gamma-\alpha}_{-\tau^+}\left[B\frac{(x+\tau)^{\alpha-1}}{\Gamma(\alpha)}\right] = B\frac{(x+\tau)^{\gamma-1}}{\Gamma(\gamma)}.\end{aligned}$$

Thus (3) and (4) hold for $x \in (-\tau,0]$.

For any $(k-1)\tau < x \le k\tau$. With applying the rules of Lemma 2.1 and Lemma 3.1, we have

$$\begin{aligned}&\mathbb{I}^{\gamma-\alpha}_{-\tau^+}\mathbb{D}^{\gamma}_{-\tau^+}\cos_{\tau,\alpha,\gamma}(Bx^{\alpha})\\ &= \mathbb{I}^{\gamma-\alpha}_{-\tau^+}\mathbb{D}^{\gamma}_{-\tau^+}\left[I\frac{(x+\tau)^{\gamma-1}}{\Gamma(\gamma)} - B^2\frac{x^{2\alpha+\gamma-1}}{\Gamma(2\alpha+\gamma)} + \cdots + (-1)^k B^{2k}\frac{(x-(k-1)\tau)^{2k\alpha+\gamma-1}}{\Gamma(2k\alpha+\gamma)}\right]\\ &= \mathbb{I}^{\gamma-\alpha}_{-\tau^+}\left[\Theta - B^2\frac{x^{2\alpha-1}}{\Gamma(2\alpha)} + \cdots + (-1)^k B^{2k}\frac{(x-(k-1)\tau)^{2k\alpha-1}}{\Gamma(2k\alpha)}\right]\\ &= \mathbb{I}^{\gamma-\alpha}_{-\tau^+}\left[-B\sin_{\tau,\alpha,\alpha}(B(x-\tau)^{\alpha})\right] = -B\sin_{\tau,\alpha,\gamma}(B(x-\tau)^{\alpha})\end{aligned}$$

and

$$\begin{aligned}&\mathbb{I}^{\gamma-\alpha}_{-\tau^+}\mathbb{D}^{\gamma}_{-\tau^+}\sin_{\tau,\alpha,\gamma}(Bx^{\alpha})\\ &= \mathbb{I}^{\gamma-\alpha}_{-\tau^+}\mathbb{D}^{\gamma}_{-\tau^+}\left[B\frac{(x+\tau)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} - B^3\frac{x^{3\alpha+\gamma-1}}{\Gamma(3\alpha+\gamma)} + \cdots + (-1)^k B^{2k+1}\frac{(x-(k-1)\tau)^{(2k+1)\alpha+\gamma-1}}{\Gamma((2k+1)\alpha+\gamma)}\right]\\ &= \mathbb{I}^{\gamma-\alpha}_{-\tau^+}\left[B\frac{(x+\tau)^{\alpha-1}}{\Gamma(\alpha)} - B^3\frac{x^{3\alpha-1}}{\Gamma(3\alpha)} + \cdots + (-1)^k B^{2k+1}\frac{(x-(k-1)\tau)^{(2k+1)\alpha-1}}{\Gamma((2k+1)\alpha)}\right]\\ &= \mathbb{I}^{\gamma-\alpha}_{-\tau^+}\left[B\cos_{\tau,\alpha,\alpha}(Bx^{\alpha})\right] = B\cos_{\tau,\alpha,\gamma}(Bx^{\alpha}).\end{aligned}$$

These end the proof.

Theorem 3.1. Let $\cos_{\tau,\alpha,\gamma}:\mathbb{R}\to\mathbb{R}^{n\times n}$ be the two parameters fractional delayed matrix cosine defined in Definition 2.7. Then,

$$\mathbb{D}^{\alpha,\beta}_{-\tau^+}(\mathbb{D}^{\alpha,\beta}_{-\tau^+}\cos_{\tau,\alpha,\gamma}(Bt^{\alpha}))(x) = -B^2\cos_{\tau,\alpha,\gamma}(B(x-\tau)^{\alpha}). \tag{5}$$

Proof. Without losing generality, suppose $(k-1)\tau < x \le k\tau,\ k\in\mathbb{N}_0$. Using Lemma 2.2, we have

$$
\begin{aligned}
&\mathbb{D}^{\alpha,\beta}_{-\tau^+}(\mathbb{D}^{\alpha,\beta}_{-\tau^+}\cos_{\tau,\alpha,\gamma}(Bt^\alpha))(x)\\
&=\mathbb{D}^{\alpha,\beta}_{-\tau^+}\left(\mathbb{D}^{\alpha,\beta}_{-\tau^+}\left[I\frac{(t+\tau)^{\gamma-1}}{\Gamma(\gamma)}-B^2\frac{t^{2\alpha+\gamma-1}}{\Gamma(2\alpha+\gamma)}+B^4\frac{(t-\tau)^{4\alpha+\gamma-1}}{\Gamma(4\alpha+\gamma)}-\cdots+(-1)^kB^{2k}\frac{(t-(k-1)\tau)^{2k\alpha+\gamma-1}}{\Gamma(2k\alpha+\gamma)}\right]\right)(x)\\
&=\mathbb{D}^{\alpha,\beta}_{-\tau^+}\left(\Theta-B^2\frac{t^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}+B^4\frac{(t-\tau)^{3\alpha+\gamma-1}}{\Gamma(3\alpha+\gamma)}-\cdots+(-1)^kB^{2k}\frac{(t-(k-1)\tau)^{(2k-1)\alpha+\gamma-1}}{\Gamma((2k-1)\alpha+\gamma)}\right)(x)\\
&=-B^2\frac{x^{\gamma-1}}{\Gamma(\gamma)}+B^4\frac{(x-\tau)^{2\alpha+\gamma-1}}{\Gamma(2\alpha+\gamma)}-\cdots+(-1)^kB^{2k}\frac{(x-(k-1)\tau)^{(2k-2)\alpha+\gamma-1}}{\Gamma((2k-2)\alpha+\gamma)}\\
&=-B^2\left[I\frac{x^{\gamma-1}}{\Gamma(\gamma)}-B^2\frac{(x-\tau)^{2\alpha+\gamma-1}}{\Gamma(2\alpha+\gamma)}-\cdots+(-1)^kB^{2k-2}\frac{(x-(k-1)\tau)^{(2k-2)\alpha+\gamma-1}}{\Gamma((2k-2)\alpha+\gamma)}\right]\\
&=-B^2\cos_{\tau,\alpha,\gamma}(B(x-\tau)^\alpha).
\end{aligned}
$$

The evidence is now complete.

Theorem 3.2. Let $\sin_{\tau,\alpha,\gamma}(Bx^\alpha):\mathbb{R}\to\mathbb{R}^{n\times n}$ be the two parameters fractional delayed matrix sine defined in Definition 2.8. Then,

$$
\mathbb{D}^{\alpha,\beta}_{-\tau^+}(\mathbb{D}^{\alpha,\beta}_{-\tau^+}\sin_{\tau,\alpha,\gamma}(Bt^\alpha))(x)=-B^2\sin_{\tau,\alpha,\gamma}(B(x-\tau)^\alpha). \tag{6}
$$

Proof. This theorem can be proven as previous one.

The solution of homogeneous equation corresponding to (2) can be deduced by the following theorem.

Theorem 3.3. The solution's representation $y\in C((-\tau,T],\mathbb{R}^n)$ of (2) when $h(x)=0$ is formulated by

$$
\begin{aligned}
y(x)=&\,b_1\cos_{\tau,\alpha,\gamma}(Bx^\alpha)+b_2\sin_{\tau,\alpha,\gamma}(Bx^\alpha)\\
&+\int_{-\tau}^{0}\sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^\alpha\}(\mathbb{D}^{\gamma+\alpha}_{-\tau^+}\phi)(s)ds.
\end{aligned}
$$

Proof. In view of Theorems 3.1 and 3.2, the homogeneous solution of (2) when $h(x)=0$ should be in the form

$$
\begin{aligned}
(x)=&\,c_1\cos_{\tau,\alpha,\gamma}(Bx^\alpha)+c_2\sin_{\tau,\alpha,\gamma}(Bx^\alpha)\\
&+\int_{-\tau}^{0}\sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^\alpha\}g(s)ds,
\end{aligned}
\tag{7}
$$

where $c_1,c_2\in\mathbb{R}^n$ are unknown constants vectors and $g(\cdot)$ is an unidentified function that must be determined.

When $-\tau<x\le 0$ and from the second condition in (2), we have

$$
\begin{aligned}
b_1&=\lim_{x\to-\tau^+}(\mathbb{I}^{1-\gamma}_{-\tau^+}y)(x)\\
&=\lim_{x\to-\tau^+}\left(\frac{1}{\Gamma(1-\gamma)}\int_{-\tau}^{x}(x-t)^{-\gamma}\left[c_1\cos_{\tau,\alpha,\gamma}(Bt^\alpha)+c_2\sin_{\tau,\alpha,\gamma}(Bt^\alpha)+\int_{-\tau}^{0}\sin_{\tau,\alpha,\gamma}\{B(t-\tau-s)^\alpha\}g(s)ds\right]dt\right).
\end{aligned}
$$

The limit $x\to-\tau^+$ implies to $t\to-\tau^+$, by Definition 2.8, we get that $\sin_{\tau,\alpha,\gamma}\{B(t-\tau-s)^\alpha\}=\Theta$ for $-\tau<s\le 0$. Hence, by using Lemma 2.5 in [35], we have

$$
\begin{aligned}
b_1&=\lim_{x\to-\tau^+}\left(\frac{1}{\Gamma(1-\gamma)}\int_{-\tau}^{x}(x-t)^{-\gamma}\left[c_1I\frac{(t+\tau)^{\gamma-1}}{\Gamma(\gamma)}+c_2B\frac{(t+\tau)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\right]dt\right)\\
&=\lim_{x\to-\tau^+}\left(\frac{c_1}{\Gamma(1-\gamma)\Gamma(\gamma)}\mathbb{B}[1-\gamma,\gamma]\right)+\lim_{x\to-\tau^+}\left(\frac{c_2B(x+\tau)^\alpha}{\Gamma(1-\gamma)\Gamma(\alpha+\gamma)}\mathbb{B}[1-\gamma,\alpha+\gamma]\right)\\
&=c_1.
\end{aligned}
$$

Now, according to the third condition in (2) with using Lemma 2.2, we find that

$$\begin{aligned}
Bb_2 &= \lim_{x\to-\tau^+} \mathbb{I}^{1-\gamma}_{-\tau^+}(\mathbb{D}^{\alpha,\beta}_{-\tau^+} y)(x) \\
&= \lim_{x\to-\tau^+}\left(\frac{1}{\Gamma(1-\gamma)}\int_{-\tau}^{x}(x-t)^{-\gamma}\mathbb{D}^{\alpha,\beta}_{-\tau^+}\left[Ic_1\frac{(t+\tau)^{\gamma-1}}{\Gamma(\gamma)}+Bc_2\frac{(t+\tau)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\right]dt\right) \\
&= \lim_{x\to-\tau^+}\left(\frac{Bc_2}{\Gamma(1-\gamma)\Gamma(\gamma)}\int_{-\tau}^{x}(x-t)^{-\gamma}(t+\tau)^{\gamma-1}dt\right) \\
&= \lim_{x\to-\tau^+}\left(\frac{Bc_2}{\Gamma(1-\gamma)\Gamma(\gamma)}\mathbb{B}[1-\gamma,\gamma]\right)=Bc_2.
\end{aligned}$$

Since B is non singular matrix hence it has an inverse. So $c_2 = b_2$. It indicates that equation (7) has the form

$$\begin{aligned}
y(x) &= b_1\cos_{\tau,\alpha,\gamma}(Bx^\alpha)+b_2\sin_{\tau,\alpha,\gamma}(Bx^\alpha) \\
&\quad+\int_{-\tau}^{0}\sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^\alpha\}g(s)ds.
\end{aligned}$$

When $-\tau < x \le 0$. Two subintervals will be created from this interval.

Interval I when $-\tau < s \le x$. In this case, we have $-\tau < x - \tau - s \le x$, which implies that

$$\sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^\alpha\} = B\frac{(x-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}, \qquad -\tau < s \le x.$$

Interval II when $x \le s \le 0$. In this case, we have $x - \tau \le x - \tau - s \le -\tau$, which implies that

$$\sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^\alpha\} = \Theta, \qquad x \le s \le 0.$$

Then, according to our problem (2) when $h(x) = 0$ with two cases above, rewriting the previous integral leads to

$$\begin{aligned}
\phi(x) &= b_1 I\frac{(x+\tau)^{\gamma-1}}{\Gamma(\gamma)}+b_2B\frac{(x+\tau)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}+\int_{-\tau}^{x}\frac{(x-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}g(s)ds \\
&= b_1 I\frac{(x+\tau)^{\gamma-1}}{\Gamma(\gamma)}+b_2B\frac{(x+\tau)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}+\mathbb{I}^{\alpha+\gamma}_{-\tau^+}g(x).
\end{aligned}$$

By operating the fractional derivative $\mathbb{D}^{\gamma+\alpha}_{-\tau^+}$ both sides of the aforementioned equation with using Lemma 2.1 and Lemma 2.5 in [35], we can get

$$(\mathbb{D}^{\gamma+\alpha}_{-\tau^+}\phi)(x) = g(x).$$

This completes the proof.

Theorem 3.4. The representation of particular solution $y_p \in C((-\tau, T], \mathbb{R}^n)$ of (2) verifying condition $y_p(x) = 0$ if $x \in [-\tau, 0]$ is formulated by

$$y_p(x) = \int_0^x \sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^\alpha\}(\mathbb{D}^{\gamma-\alpha}_{0^+}h(s))ds. \tag{8}$$

Proof. The solution, according to the technique of variation of constants, should be in the form

$$y_p(x) = \int_0^x \sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^\alpha\}a(s)ds \tag{9}$$

where $a(s)$, $0 \le s \le x$ is an unknown vector function and $y_p(0) = 0$ for $k\tau < x \le (k+1)\tau$ and $k \in \mathbb{N}_0$. According to (2), we obtain

$$\left(\mathbb{D}^{\alpha,\beta}_{-\tau^+}\mathbb{D}^{\alpha,\beta}_{-\tau^+}y_p\right)(x) = -B^2\int_0^{x-\tau}\sin_{\tau,\alpha,\gamma}\left\{B\left(x-2\tau-s\right)^{\alpha}\right\}a(s)\,ds + h(x). \tag{10}$$

Operating two Hilfer derivative operators on (9) with noting that $y_p(x) = 0$ if $x \in [-\tau,0]$. Then,

$$\begin{aligned}(\mathbb{D}^{\alpha,\beta}_{-\tau^+}\mathbb{D}^{\alpha,\beta}_{-\tau^+}y_p)(x) &= \mathbb{D}^{\alpha,\beta}_{0^+}\mathbb{D}^{\alpha,\beta}_{0^+}\int_0^x \sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^{\alpha}\}a(s)ds \\ &= \mathbb{D}^{\alpha,\beta}_{0^+}\mathbb{I}^{\gamma-\alpha}_{0^+}\mathbb{D}^{\gamma}_{0^+}\int_0^x \sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^{\alpha}\}a(s)ds.\end{aligned}$$

In the first step of proof of Lemma 7 in [27], he proved that

$$\mathbb{D}^{\gamma}_{0^+}\int_0^x \sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^{\alpha}\}a(s)ds = \int_0^x a(s)\mathbb{D}^{\gamma}_{0^+}\sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^{\alpha}\}ds,$$

which can be used with the results in Lemma 3.2 to get

$$\begin{aligned}&(\mathbb{D}^{\alpha,\beta}_{-\tau^+}\mathbb{D}^{\alpha,\beta}_{-\tau^+}y_p)(x) = \mathbb{D}^{\alpha,\beta}_{0^+}\mathbb{I}^{\gamma-\alpha}_{0^+}\int_0^x \mathbb{D}^{\gamma}_{0^+}\sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^{\alpha}\}a(s)ds \\ &= \mathbb{D}^{\alpha,\beta}_{0^+}\left\{\frac{1}{\Gamma(\gamma-\alpha)}\int_0^x (x-t)^{\gamma-\alpha-1}\left[\int_0^t \mathbb{D}^{\gamma}_{0^+}\sin_{\tau,\alpha,\gamma}\{B(t-\tau-s)^{\alpha}\}a(s)ds\right]dt\right\} \\ &= \mathbb{D}^{\alpha,\beta}_{0^+}\left\{\frac{1}{\Gamma(\gamma-\alpha)}\int_0^x a(s)\left[\int_s^x (x-t)^{\gamma-\alpha-1}\mathbb{D}^{\gamma}_{0^+}\sin_{\tau,\alpha,\gamma}\{B(t-\tau-s)^{\alpha}\}dt\right]ds\right\}.\end{aligned}$$

Since $0 \le s \le x$, the integral from s to x could be rewritten as

$$\int_0^x (x-t)^{\gamma-\alpha-1}\mathbb{D}^{\gamma}_{0^+}\sin_{\tau,\alpha,\gamma}\{B(t-\tau-s)^{\alpha}\}dt - \int_0^s (x-t)^{\gamma-\alpha-1}\mathbb{D}^{\gamma}_{0^+}\sin_{\tau,\alpha,\gamma}\{B(t-\tau-s)^{\alpha}\}dt.$$

While proving Lemma 3.2, we found that

$$\mathbb{D}^{\gamma}_{0^+}\sin_{\tau,\alpha,\gamma}\{B(t-\tau-s)^{\alpha} = B\cos_{\tau,\alpha,\alpha}\{B(t-\tau-s)^{\alpha}.$$

When $t \in (0, s)$, we find that $-\tau - s < t - \tau - s < -\tau$ and so

$$\cos_{\tau,\alpha,\alpha}\{B(t-\tau-s)^{\alpha} = \Theta,$$

which implies that the second integral is identically zero. Therefore, we have

$$\begin{aligned}(\mathbb{D}^{\alpha,\beta}_{-\tau^+}\mathbb{D}^{\alpha,\beta}_{-\tau^+}y_p)(x) &= \mathbb{D}^{\alpha,\beta}_{0^+}\int_0^x a(s)\mathbb{I}^{\gamma-\alpha}_{0^+}\mathbb{D}^{\gamma}_{0^+}\sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^{\alpha}\}ds \\ &= \mathbb{D}^{\alpha,\beta}_{0^+}\int_0^x B\cos_{\tau,\alpha,\gamma}\{B(x-\tau-s)^{\alpha}\}a(s)ds \\ &= \mathbb{I}^{\gamma-\alpha}_{0^+}\mathbb{D}^{\gamma}_{0^+}\int_0^x B\cos_{\tau,\alpha,\gamma}\{B(x-\tau-s)^{\alpha}\}a(s)ds \\ &= \mathbb{I}^{\gamma-\alpha}_{0^+}\frac{d}{dx}\left\{\frac{1}{\Gamma(1-\gamma)}\int_0^x (x-t)^{-\gamma}\left[\int_0^t \cos_{\tau,\alpha,\gamma}\{B(t-\tau-s)^{\alpha}\}a(s)ds\right]dt\right\} \\ &= \mathbb{I}^{\gamma-\alpha}_{0^+}\frac{d}{dx}\left\{\frac{1}{\Gamma(1-\gamma)}\int_0^x a(s)\left[\int_s^x (x-t)^{-\gamma}\cos_{\tau,\alpha,\gamma}\{B(t-\tau-s)^{\alpha}\}dt\right]ds\right\}.\end{aligned}$$

Dividing the intervals of double integral leads to

$$
\begin{aligned}
(\mathbb{D}^{\alpha,\beta}_{-\tau^+}\mathbb{D}^{\alpha,\beta}_{-\tau^+}y_p)(x) = \mathbb{I}^{\gamma-\alpha}_{0^+}\frac{d}{dx}\Bigg\{ &\int_0^x a(s)\mathbb{I}^{1-\gamma}_{s^+}\left(I\frac{(x-s)^{\gamma-1}}{\Gamma(\gamma)}\right)ds \\
&+\int_0^{x-\tau} a(s)\mathbb{I}^{1-\gamma}_{s+\tau^+}\left(-B^2\frac{(x-\tau-s)^{2\alpha+\gamma-1}}{\Gamma(2\alpha+\gamma)}\right)ds \\
&+\cdots+\int_0^{x-k\tau} a(s)\mathbb{I}^{1-\gamma}_{s+k\tau^+}\left((-1)^k B^{2k}\frac{(x-k\tau-s)^{2k\alpha+\gamma-1}}{\Gamma(2k\alpha+\gamma)}\right)ds\Bigg\}.
\end{aligned}
$$

Applying Lemma 2.1 to compute the fractional integrals

$$
\begin{aligned}
(\mathbb{D}^{\alpha,\beta}_{-\tau^+}\mathbb{D}^{\alpha,\beta}_{-\tau^+}y_p)(x) = \mathbb{I}^{\gamma-\alpha}_{0^+}\frac{d}{dx}\Bigg\{ &\int_0^x a(s)ds - B^2\int_0^{x-\tau} a(s)\frac{(x-\tau-s)^{2\alpha}}{\Gamma(2\alpha+1)}ds \\
&+\cdots+(-1)^k B^{2k}\int_0^{x-k\tau} a(s)\frac{(x-k\tau-s)^{2k\alpha}}{\Gamma(2k\alpha+1)}ds\Bigg\}.
\end{aligned}
$$

Using Leibnitz rule for differentiation under the integral, we get

$$
\begin{aligned}
(\mathbb{D}^{\alpha,\beta}_{-\tau^+}\mathbb{D}^{\alpha,\beta}_{-\tau^+}y_p)(x) &= \mathbb{I}^{\gamma-\alpha}_{0^+}\left\{a(x) - B^2\left[I\int_0^{x-\tau} a(s)\frac{(x-\tau-s)^{2\alpha-1}}{\Gamma(2\alpha)}ds + \cdots + (-1)^k B^{2k-2}\int_0^{x-k\tau} a(s)\frac{(x-k\tau-s)^{2k\alpha-1}}{\Gamma(2k\alpha)}ds\right]\right\} \\
&= \mathbb{I}^{\gamma-\alpha}_{0^+}\left[a(x) - B^2\int_0^{x-\tau}\sin_{\tau,\alpha,\alpha}\{B(x-2\tau-s)^\alpha\}a(s)ds\right] \\
&= \mathbb{I}^{\gamma-\alpha}_{0^+}a(x) - B^2\frac{1}{\Gamma(\gamma-\alpha)}\int_0^x (x-t)^{\gamma-\alpha-1}\left[\int_0^{t-\tau}\sin_{\tau,\alpha,\alpha}\{B(t-2\tau-s)^\alpha\}a(s)ds\right]dt \\
&= \mathbb{I}^{\gamma-\alpha}_{0^+}a(x) - B^2\frac{1}{\Gamma(\gamma-\alpha)}\int_0^{x-\tau} a(s)\left[\int_{s+\tau}^x (x-t)^{\gamma-\alpha-1}\sin_{\tau,\alpha,\alpha}\{B(t-2\tau-s)^\alpha\}dt\right]ds \\
&= \mathbb{I}^{\gamma-\alpha}_{0^+}a(x) - B^2\int_0^{x-\tau}\mathbb{I}^{\gamma-\alpha}_{s+\tau^+}\sin_{\tau,\alpha,\alpha}\{B(x-2\tau-s)^\alpha\}a(s)ds \\
&= \mathbb{I}^{\gamma-\alpha}_{0^+}a(x) - B^2\int_0^{x-\tau}\sin_{\tau,\alpha,\gamma}\{B(x-2\tau-s)^\alpha\}a(s)ds.
\end{aligned}
$$

Equating the last line with (10) leads to

$$
a(x) = \mathbb{D}^{\gamma-\alpha}_{0^+}h(x).
$$

The proof is completed.

Combining the results of the previous two theorems, we can establish the general solution of the problem (2) as will be stated in the following theorem.

Theorem 3.5. The solution's representation $y \in C((-\tau, T], \mathbb{R}^n)$ of (2) is formulated as

$$
\begin{aligned}
y(x) &= b_1\cos_{\tau,\alpha,\gamma}(Bx^\alpha) + b_2\sin_{\tau,\alpha,\gamma}(Bx^\alpha) \\
&+ \int_{-\tau}^0 \sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^\alpha\}(\mathbb{D}^{\gamma+\alpha}_{-\tau^+}\phi)(s)ds \\
&+ \int_0^x \sin_{\tau,\alpha,\gamma}\{B(x-\tau-s)^\alpha\}(\mathbb{D}^{\gamma-\alpha}_{0^+}h(s))ds. \qquad (11)
\end{aligned}
$$

4. Finite-time stability results

In the current part, sufficient conditions are presented to guarantee finite-time stability by using the two parameters of fractional delayed matrices, cosine, sine, and Mittag-Leffler. Before proving the next theorems, we display the following hypotheses.

(R_1) The functions $(\mathbb{D}^{\gamma-\alpha}_{0^+}h)(x) \in C([0,T], \mathbb{R}^n)$ and $(\mathbb{D}^{\gamma+\alpha}_{-\tau^+}\phi)(x) \in C((-\tau, 0], \mathbb{R}^n)$.

(R_2) There is a positive function $\psi(x) \in C([0,T], \mathbb{R}_+)$ that says $|\mathbb{D}^{\gamma-\alpha}_{0^+}h(x)| \le \psi(x)$ with $\psi = \|\psi\| = \sup_{x\in[0,T]}\{\psi(x)\}$.

(R$_3$) There is a positive function $\varphi(x) \in C((-\tau, 0], \mathbb{R}^+)$ that says $\left\|(\mathbb{D}_{-\tau^+}^{\gamma+\alpha}\phi)(x)\right\| \le \varphi(x)$ with $\varphi = \|\varphi\| = \sup_{x\in(-\tau,0]}\{\varphi(x)\}$.

(R$_4$) There is a positive function $\eta(x) \in L^q([0,T], \mathbb{R}^+)$ with $p > 1$, $q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$ such that $\left\|\mathbb{D}_{0^+}^{\gamma-\alpha}h(x)\right\| \le \eta(x)$ and

$$J(x) = \left(\int_0^x (\eta(t))^q dt\right)^{\frac{1}{q}} < \infty$$

with $J = \sup_{0\le x\le T} J(x)$.

(R$_5$) There is a positive constant A that says

$$A = \left(\int_{-\tau}^0 \left\|(\mathbb{D}_{-\tau^+}^{\gamma+\alpha}\phi)(s)\right\|^q ds\right)^{\frac{1}{q}} < \infty.$$

Theorem 4.1. Suppose that hypotheses (R$_1$) − (R$_3$) hold. If $\alpha < (1-\beta)/(2-\beta), \|\phi(x)\| < \delta$ and

$$\mathbb{E}_{2\alpha,\alpha+\gamma}(\|B\|^2 x^{2\alpha}) < \frac{\alpha+\gamma}{\|B\|}\left[\frac{\varepsilon - \|b_1\|\tau^{\gamma-1}\mathbb{E}_{2\alpha,\gamma}(\|B\|^2 x^{2\alpha})}{(\alpha+\gamma)\|b_2\|\tau^{\alpha+\gamma-1}+\tau^{\alpha+\gamma}\varphi+T^{\alpha+\gamma}\psi}\right]$$

for all $x \in [0,T]$. Then, the problem (2) is finite-time stability in relation to $\{0, [0,T], \tau, \delta, \varepsilon\}$.

Proof. In view of the expression (11), which is the solution of (2) with using the results obtained in Lemmas 2.3 and 2.4 and norm properties, we have

$$\begin{aligned}
\|y(x)\| &\le \|b_1\|\tau^{\gamma-1}\mathbb{E}_{2\alpha,\gamma}\left(\|B\|^2 x^{2\alpha}\right) + \|B\|\mathbb{E}_{2\alpha,\alpha+\gamma}\left(\|B\|^2 x^{2\alpha}\right) \\
&\times\left\{\|b_2\|\tau^{\alpha+\gamma-1} + \|\varphi(x)\|\int_{-\tau}^0 (x-s)^{\alpha+\gamma-1} ds + \|\psi(c)\|\int_0^x (x-s)^{\alpha+\gamma-1} ds\right\} \\
&= \|b_1\|\tau^{\gamma-1}\mathbb{E}_{2\alpha,\gamma}\left(\|B\|^2 x^{2\alpha}\right) + \left(\|B\|^2 x^{2\alpha}\right) \\
&\times\left\{\|b_2\|\tau^{\alpha+\gamma-1} + \frac{(x+\tau)-x^{\alpha+\gamma}}{\alpha+\gamma}\|\varphi(x)\| + \frac{x^{\alpha+\gamma}}{\alpha+\gamma}\|\psi(x)\|\right\}.
\end{aligned}$$

Obviously, the function $x \mapsto (x+\tau)^{\alpha+\gamma} - x^{\alpha+\gamma}$ is decreasing on $[0,T]$ for all $\tau > 0$ and $0 < \alpha + \gamma < 1$, which implies that $(x+\tau)^{\alpha+\gamma} - x^{\alpha+\gamma} \le \tau^{\alpha+\gamma}$.

Hence,

$$\begin{aligned}
\|y(x)\| &\le \|b_1\|\tau^{\gamma-1}\mathbb{E}_{2\alpha,\gamma}(\|B\|^2 x^{2\alpha}) + \|B\|\mathbb{E}_{2\alpha,\alpha+\gamma}(\|B\|^2 x^{2\alpha}) \\
&\times\left\{\|b_2\|\tau^{\alpha+\gamma-1} + \frac{\tau^{\alpha+\gamma}}{\alpha+\gamma}\varphi + \frac{T^{\alpha+\gamma}}{\alpha+\gamma}\psi\right\} < \varepsilon, \qquad x \in [0,T].
\end{aligned}$$

The proof is completed.

Theorem 4.2. Suppose that the hypotheses (R$_1$), (R$_4$), and (R$_5$) hold. If $\alpha < (1-\beta)/(2-\beta), \alpha+\gamma > 1-\frac{1}{p}=\frac{1}{q}$, $\|\phi(x)\| < \delta$ and

$$\|b_1\|\tau^{\gamma-1}\mathbb{E}_{2\alpha,\gamma}\left(\|B\|^2 x^{2\alpha}\right) + \|B\|\mathbb{E}_{2\alpha,\alpha+\gamma}\left(\|B\|^2 x^{2\alpha}\right)\left\{\|b_2\|\tau^{\alpha+\gamma-1} + \frac{A\tau^{\alpha+\gamma-\frac{1}{q}} + JT^{\alpha+\gamma-\frac{1}{q}}}{\left(1-p(1-\alpha-\gamma)\right)^{\frac{1}{p}}}\right\} < \varepsilon$$

for all $x \in [0,T]$. Then, the problem (2) is finite-time stability in relation to $\{0, [0,T], \tau, \delta, \varepsilon\}$.

Proof. Using norm properties, we have

$$
\begin{aligned}
&\left\|y(x)\right\| \le \left\|b_1\right\| \tau^{\gamma-1} \mathbb{E}_{2\alpha,\gamma}\left(\left\|B\right\|^2 x^{2\alpha}\right) + \left\|B\right\| \mathbb{E}_{2\alpha,\alpha+\gamma}\left(\left\|B\right\|^2 x^{2\alpha}\right) \\
&\times\left\{\left\|b_2\right\| \tau^{\alpha+\gamma-1} + \int_{-\tau}^{0} (x-s)^{\alpha+\gamma-1} \left\|\left(\mathbb{D}_{-\tau^+}^{\gamma+\alpha}\phi\right)(s)\right\| ds + \int_{0}^{x} (x-s)^{\alpha+\gamma-1} \left\|\left(\mathbb{D}_{0^+}^{\gamma-\alpha}h\right)(s)\right\| ds\right\} \\
&\le \left\|b_1\right\| \tau^{\gamma-1} \mathbb{E}_{2\alpha,\gamma}\left(\left\|B\right\|^2 x^{2\alpha}\right) + \left\|B\right\| \mathbb{E}_{2\alpha,\alpha+\gamma}\left(\left\|B\right\|^2 x^{2\alpha}\right) \\
&\times\left\{\left\|b_2\right\| \tau^{\alpha+\gamma-1} + \left(\int_{-\tau}^{x} (x-s)^{p(\alpha+\gamma-1)} ds\right)^{\frac{1}{p}} \left(\int_{-\tau}^{x} \left\|\left(\left(\mathbb{D}_{-\tau^+}^{\gamma+\alpha}\phi\right)(s)\right)\right\|^q ds\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{-\tau}^{x} (x-s)^{p(\alpha+\gamma-1)} ds\right)^{\frac{1}{p}} \left(\int_{-\tau}^{x} \left\|\left(\left(\mathbb{D}_{-\tau^+}^{\gamma+\alpha}\phi\right)(s)\right)\right\|^q ds\right)^{\frac{1}{q}}\right\} \\
&\le \left\|b_1\right\| \tau^{\gamma-1} \mathbb{E}_{2\alpha,\gamma}\left(\left\|B\right\|^2 x^{2\alpha}\right) + \left\|B\right\| \mathbb{E}_{2\alpha,\alpha+\gamma}\left(\left\|B\right\|^2 x^{2\alpha}\right) \\
&\times\left\{\left\|b_2\right\| \tau^{\alpha+\gamma-1} + A\left(\frac{(x+\tau)^{1-p(1-\alpha-\gamma)}}{1-p(1-\alpha-\gamma)}\right)^{\frac{1}{p}} + J(x)\left(\frac{x^{1-p(1-\alpha-\gamma)}}{1-p(1-\alpha-\gamma)}\right)\right\}.
\end{aligned}
$$

Since $p > 1$ and $\alpha + \gamma < 1$, then $p(\alpha + \gamma - 1) + 1 < 1$ and so as above, we get

$$
\begin{aligned}
&\left\|y(x)\right\| \le \left\|b_1\right\| \tau^{\gamma-1} \mathbb{E}_{2\alpha,\gamma}\left(\left\|B\right\|^2 x^{2\alpha}\right) + \left\|B\right\| \mathbb{E}_{2\alpha,\alpha+\gamma}\left(\left\|B\right\|^2 x^{2\alpha}\right) \\
&\times\left\{\left\|b_2\right\| \tau^{\alpha+\gamma-1} + A\left(\frac{\tau^{1-p(1-\alpha-\gamma)}}{1-p(1-\alpha-\gamma)}\right)^{\frac{1}{p}} + J\left(\frac{x^{1-p(1-\alpha-\gamma)}}{1-p(1-\alpha-\gamma)}\right)^{\frac{1}{p}}\right\} < \varepsilon.
\end{aligned}
$$

The proof is over now.

5. A numerical example

In this part, we provide an illustration to demonstrate the validity of our theoretical findings. In this case, we use $|y| = \sum_{i=1}^{n} |y_n|$ and

$$\left\|B\right\| = \max_{1 \le j \le n} \sum_{i=1}^{n} \left|b_{ij}\right|$$

where $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ is a vector and $B \in \mathbb{R}^{n \times n}$ with elements $b_{ij} \in \mathbb{R}$, which are a rectilinear norm or ℓ_1-norm and matrix norm respectively.

Example 5.1. Consider the problem

$$\begin{cases} \mathbb{D}_{-3^+}^{\frac{1}{4},\frac{1}{8}}(\mathbb{D}_{-3^+}^{\frac{1}{4},\frac{1}{8}}y)(x) = -B^2 y(x-3) + h(x), & x \in [0,9], \\ y(x) = \phi(x) = \left(\frac{1}{6}(x+3)^2, \frac{1}{6}(x+3)^2\right)^T, & -3 < x \le 0, \\ \lim_{x \to -3^+} (\mathbb{I}_{-3^+}^{1-\gamma} y)(x) = b_1, \\ \lim_{x \to -3^+} \mathbb{I}_{-3^+}^{1-\gamma} (\mathbb{D}_{-3^+}^{\frac{1}{4},\frac{1}{8}} y)(x) = Bb_2 \end{cases}$$

where $\alpha = 0.25$, $\beta = 0.125$, $\gamma = 0.3438$, $\tau = 3$, $T = 9$, $k = 3$, $p = 2$, $q = 2$, $y(x) = (y_1(x), y_2(x))T$, and

$$B = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.03 \end{pmatrix}, \quad h(x) = \frac{1}{100}\begin{pmatrix} x^3 - 2x^{2\gamma} \\ x^2 - x^{\gamma} \end{pmatrix}.$$

Obviously, $\|B\| = 0.1, \alpha = 0.25 < (1-\beta)/(2-\beta) = 0.4667$, and $\alpha + \gamma = 0.5938 > 1/q = 1/2$. By carrying out Mathematica software, we can find that

$$\sup_{0 \le x \le 9} \mathbb{E}_{2\alpha,\gamma}\left(\|B\|^2 x^{2\alpha}\right) = \mathbb{E}_{0.5,0.3438}\left((0.1)^2 \times 9^{0.5}\right) \sim 0.413374,$$
$$\sup_{0 \le x \le 9} \mathbb{E}_{2\alpha,\alpha+\gamma}\left(\|B\|^2 x^{2\alpha}\right) = \mathbb{E}_{0.5,0.5938}\left((0.1)^2 \times 9^{0.5}\right) \sim 0.697507.$$

The function

$$\phi(x) = \left(\frac{1}{6}(x+3)^2, \frac{1}{6}(x+3)^2\right)^T$$

is continuous on (−3,0] and

$$\|\phi\| = \sup_{-3 < x \le 0} \frac{1}{3}(x+3)^2 = 3,$$

which concludes that $\delta = 3$. According to Lemma 2.1, we obtain

$$\mathbb{I}_{-3^+}^{1-\gamma}\phi(x) = \frac{1}{3}\left(\frac{(x+3)^{3-\gamma}}{\Gamma(4-\gamma)}, \frac{(x+3)^{3-\gamma}}{\Gamma(4-\gamma)}\right)^T.$$

Since $y(x) = \phi(x)$ for all $-3 < x \le 0$, then

$$b_1 = \lim_{x \to -3^+} \mathbb{I}_{-3^+}^{1-\gamma} y(x) = \lim_{x \to -3^+} \mathbb{I}_{-3^+}^{1-\gamma} \phi(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$Bb_2 = \lim_{x \to -3^+} \mathbb{I}_{-3^+}^{1-\gamma} (\mathbb{D}_{-3^+}^{\frac{1}{4},\frac{1}{8}} y)(x) = \lim_{x \to -3^+} \mathbb{I}_{-3^+}^{1-\gamma} (\mathbb{D}_{-3^+}^{\frac{1}{4},\frac{1}{8}} \phi)(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$b_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which implies that $\|b_1\| = 0$ and $\|b_2\| = 0$ Also, we can calculate the following

$$(\mathbb{D}_{-3^+}^{\gamma+\alpha}\phi)(x)=\frac{d}{dx}\mathbb{I}_{-3^+}^{1-\gamma}\phi(x)=\frac{1}{3}\frac{(x+3)^{2-\gamma}}{\Gamma(3-\gamma)}\begin{pmatrix}1\\1\end{pmatrix},$$
$$\left\|(\mathbb{D}_{-3^+}^{\gamma+\alpha}\phi)(x)\right\|=\frac{2}{3}\frac{(x+3)^{2-\gamma}}{\Gamma(3-\gamma)},$$

which leads to $(\mathbb{D}_{-3^+}^{\gamma+\alpha}\phi)(x)\in C((-3,0],\mathbb{R}^2)$ and the hypothesis (R_1) is satisfied. Also, we can find $\psi(x)=\frac{1}{100}\left(\frac{6x^{3-\gamma+\alpha}}{\Gamma(4-\gamma+\alpha)}+\frac{2\Gamma(2\gamma+1)x^{\gamma+\alpha}}{\Gamma(1+\gamma+\alpha)}+\frac{2x^{2-\gamma+\alpha}}{\Gamma(3-\gamma+\alpha)}+\frac{\Gamma(\gamma+1)x^{\alpha}}{\Gamma(1+\alpha)}\right)$, $\mathbb{D}_{0^+}^{\gamma-\alpha}h(x)$ as

$$\mathbb{D}_{0^+}^{\gamma-\alpha}h(x)=\frac{1}{100}\begin{pmatrix}\frac{6x^{3-\gamma+\alpha}}{\Gamma(4-\gamma+\alpha)}-\frac{2\Gamma(2\gamma+1)x^{\gamma+\alpha}}{\Gamma(1+\gamma+\alpha)}\\ \frac{2x^{2-\gamma+\alpha}}{\Gamma(3-\gamma+\alpha)}-\frac{\Gamma(\gamma+1)x^{\alpha}}{\Gamma(1+\alpha)}\end{pmatrix},$$

which implies that

$$\begin{aligned}\left\|\mathbb{D}_{0^+}^{\gamma-\alpha}h(x)\right\|&=\frac{1}{100}\left|\frac{6x^{3-\gamma+\alpha}}{\Gamma(4-\gamma+\alpha)}-\frac{2\Gamma(2\gamma+1)x^{\gamma+\alpha}}{\Gamma(1+\gamma+\alpha)}\right|+\frac{1}{100}\left|\frac{2x^{2-\gamma+\alpha}}{\Gamma(3-\gamma+\alpha)}-\frac{\Gamma(\gamma+1)x^{\alpha}}{\Gamma(1+\alpha)}\right|\\&\le\frac{1}{100}\left(\frac{6x^{3-\gamma+\alpha}}{\Gamma(4-\gamma+\alpha)}+\frac{2\Gamma(2\gamma+1)x^{\gamma+\alpha}}{\Gamma(1+\gamma+\alpha)}+\frac{2x^{2-\gamma+\alpha}}{\Gamma(3-\gamma+\alpha)}+\frac{\Gamma(\gamma+1)x^{\alpha}}{\Gamma(1+\alpha)}\right).\end{aligned}$$

This leads to $\mathbb{D}_{0^+}^{\gamma-\alpha}h(x)\in C((-3,0],\mathbb{R}^2)$ and the hypothesis (R_1) is satisfied.

Application to Theorem 4.1. According to assumption (R_2), we can take

$$\psi(x)=\frac{1}{100}\left(\frac{6x^{3-\gamma+\alpha}}{\Gamma(4-\gamma+\alpha)}+\frac{2\Gamma(2\gamma+1)x^{\gamma+\alpha}}{\Gamma(1+\gamma+\alpha)}+\frac{2x^{2-\gamma+\alpha}}{\Gamma(3-\gamma+\alpha)}+\frac{\Gamma(\gamma+1)x^{\alpha}}{\Gamma(1+\alpha)}\right),$$

which concludes that $\psi(x)\in C([0,9],\mathbb{R}^+)$ and

$$\psi=\sup_{0\le x\le 9}\psi(x)\sim 7.47538.$$

According to assumption (R_3), we can take

$$\varphi(x)=\frac{2}{3}\frac{(x+3)^{2-\gamma}}{\Gamma(3-\gamma)},$$

which concludes that

$$\varphi=\sup_{-3<x\le 0}\varphi(x)=\frac{2}{3}\frac{3^{2-\gamma}}{\Gamma(3-\gamma)}\sim 2.75583.$$

In view of our calculations and the results of Theorem 4.1, we must take ϵ 3.85869, which concludes that the system given by (2) is finite-time stable with respect to $\{0, [0, 9], 3, 3.86\}$.

Application to Theorem 4.2. In view of hypothesis (R_4), we can take

$$\eta(x)=\frac{1}{100}\left(\frac{6x^{3-\gamma+\alpha}}{\Gamma(4-\gamma+\alpha)}+\frac{2\Gamma(2\gamma+1)x^{\gamma+\alpha}}{\Gamma(1+\gamma+\alpha)}+\frac{2x^{2-\gamma+\alpha}}{\Gamma(3-\gamma+\alpha)}+\frac{\Gamma(\gamma+1)x^{\alpha}}{\Gamma(1+\alpha)}\right),$$

which concludes that $\eta(x) \in L^2([0,9], \mathbb{R}^+)$ and

$$J(x) = \left(\int_0^x (\eta(t))^2 dt \right)^{\frac{1}{2}}$$
$$= \frac{1}{100} \left(\int_0^x \left(\frac{6t^{3-\gamma+\alpha}}{\Gamma(4-\gamma+\alpha)} + \frac{2\Gamma(2\gamma+1)t^{\gamma+\alpha}}{\Gamma(1+\gamma+\alpha)} + \frac{2t^{2-\gamma+\alpha}}{\Gamma(3-\gamma+\alpha)} + \frac{\Gamma(\gamma+1)t^{\alpha}}{\Gamma(1+\alpha)} \right)^2 dt \right)^{\frac{1}{2}}.$$

This leads to $J \sim 8.79481$. In view of hypothesis (R_5), we find that

$$A = \left(\int_{-3}^0 \left\| (\mathbb{D}_{-3^+}^{\gamma+\alpha} \phi)(s) \right\|^2 ds \right)^{\frac{1}{2}}$$
$$\leq \frac{2}{3\Gamma(3-\gamma)} \left(\int_{-3}^0 (s+3)^{4-2\gamma} ds \right)^{\frac{1}{2}}$$
$$= \frac{2(3)^{\frac{3}{2}-\gamma}}{(5-2\gamma)^{\frac{1}{2}} \Gamma(3-\gamma)} \sim 2.29852.$$

In view of our calculations and the results of Theorem 4.2, we have to take $\epsilon > 2.15116$, which concludes that the system given by (2) is finite-time stable with respect to $\{0, [0, 9], 3, 2.16\}$.

Table 1. Finite-time stability results of problem in Example 5.1

Theorem	α	γ	T	τ	δ	$\|y(x)\|$	ϵ	Finite-time stability
4.1	0.25	0.3438	9	3	3	3.85869	3.86	Yes
4.2	0.25	0.3438	9	3	3	2.15116	2.16 (optimal)	Yes

6. Conclusion

In Section 3, the representation of solutions for the fractional delay Cauchy problem of Hilfer type was derived using two parameters of fractional delayed matrices, cosine, and sine. Finite-time stability results are established under appropriate conditions in Section 4. According to these results and applying the values in Example 5.1, we can take ϵ = 3.86 in Theorem 4.1 and in Theorem 4.2. Comparing the values of ϵ in Table 1, we find an optimal threshold ϵ = 2.16, such that $\|y(x)\|$ does not exceed it on [0, 9].

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Conflict of interest

The authors declare no conflict of interest.

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