Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

Differential Transform Method for Solving Volterra q-Integral Equations

Altaf A. Bhat[1*], Haider Abbas Rizvi[2], Javid A. Ganie[3], Faiza A. Sulaiman[1], D. K. Jain[4]

[1]Department of General Requirements, University of Technology and Applied Sciences, Salalah, Oman
[2]Department of Mathematics and Sciences, College of Arts and Applied Sciences, Dhofar University, Salalah, Oman
[3]PG Department of Mathematics, Govt. Degree College for Boys Baramulla, Baramulla, India
[4]Department of Engineering Mathematics & Computing, Madhav Institute of Technology and Science, Gwalior, India
Email: altaf.sal@cas.edu.om, gr-hod.sal@cas.edu.om

Received: 23 February 2023; **Revised:** 13 June 2023; **Accepted:** 17 October 2023

Abstract: In this paper, Volterra q-integral equations are solved by using the method of q-differential transformation. Exact solutions of linear and nonlinear q-integral equations have been investigated. To illustrate the method, several problems are discussed for the effectiveness and performance of the proposed method.

Keywords: q-differential transform method, Volterra q-integral equations, integral transforms, q-calculus

MSC: 45D05, 35A22, 81Q40

1. Introduction

In q-calculus, we are looking for q-analogues of mathematical functions that have the original function as limits when q tends to unity [1]. The history of q-calculus (and q-hypergeometric functions) dates back to Kac et al. [2] and Jackson [3, 4], who first introduced the q in their introduction [5] in the tracks of Newton's infinite series. In recent years, interest in the subject has exploded [6]. This is, of course, due to the fact that q-analysis has proven itself extremely fruitful in quantum theory, mechanics, number theory, and the theory of relativity [7], has wide-ranging applications in areas like computer science and particle physics, and also acts as an important research tool for researchers working with analytic number theory or theoretical physics.

The differential transform scheme is a numerical method for solving systems of equations involving algebraic, differential, integral, and integro-differential differential equations. The concept of differential transformation was first proposed by Zhou [8]. The main applications of this method are to solve both linear and nonlinear initial value problems in electric circuit analysis and to give exact values of the nth derivative of an analytical function at a point in terms of known and unknown boundary conditions in a fast manner [9].

The main motive for solving integral equations is their use in mathematical models for many physical situations. Integral equations also occur as reformulations of other mathematical problems, such as partial differential equations

DOI: https://doi.org/10.37256/cm.4420232549

and ordinary differential equations [10]. In this study, the q-differential transform is introduced to solve Volterra q-integral equations. The concept of one-dimensional q-differential was first proposed and applied to solve linear and nonlinear initial value problems by Jafari et al. [11].

A general integral equation for an unknown function $y(x)$ can be written [12] as

$$f(x) = a(x)y(x) + \int_a^b K(x,t)y(t)dt, \tag{1}$$

where $f(x)$, $a(x)$, and $K(x, t)$ are given functions. The function $K(x, t)$ is called the kernel.

There are different types of integral equations:

The equation (1) is said to be of the first kind if the unknown function only appears under the integral sign, if $a(x) = 0$, otherwise of the second kind.

The equation (1) is said to be a Fredholm equation if the integration limits a and b are constants and Volterra if a and b are functions of x.

The equation (1) is said to be homogeneous if $f(x) = 0$, otherwise it is nonhomogeneous.

Moreover, the q-analogue of integral equations has been defined by Mansour [13] as

$$\phi(x) = \int_0^x k(x,t)\phi^p(t)d_qt + f(x), \quad x \in [0, a],$$

where $0 <| p |< 1$, and Fredhlom q-integral equation is defined as

$$\phi(x) = \int_0^1 k(x,t)\phi^p(t)d_qt + f(x), \quad 0 \le x \le 1$$

and

$$\phi(x) = f(x) + \int_0^1 k(x,t)\phi^p(t)d_qt, \quad 0 \le x \le 1.$$

In this paper, the q-differential transform has been employed to solve some q-integral equations of the Volterra type.

2. Preliminaries and q-notations

Some basic concepts of q-calculus are given in [3, 4, 14]. The q-derivative of a real continuous function $\zeta(x)$ is defined as

$$(D_q\zeta)(x) = \frac{d_\zeta(x)}{d_q(x)} = \frac{\zeta(qx) - \zeta(x)}{(q-1)x}, \quad x \ne 0, \quad 0 < q < 1.$$

$$(D_q\zeta)(0) = q^{-1}\zeta'(0),$$

where $\zeta'(0)$ exists.

The q-integral of real valued continuous function $\zeta(x)$ and $\psi(x)$ is defined in [15] as

$$\int_b^a \zeta(x)\psi(x)d_q(x) = \zeta(a)\int_a^c \zeta(x)\psi(x)d_q(x).$$

We find that

$$\int_a^b \zeta(x)\psi(x)d_q(x) = \zeta(a)\int_a^c \psi(x)d_q(x) + \zeta(b)\int_c^b \psi(x)d_q(x), \tag{2}$$

where $qa \le c \le qb$, this result is an analogue of du Bois-Reymond's theorem.

Definition 1. The series representation of the q-exponential function is defined as [16]

$$e_q^x = \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!}.$$

Further,

$$\frac{\delta_q}{\delta_q x} e_q^x = e_q^x.$$

If all the q-differentials of a function $\zeta(x)$ exist in some neighbourhood of a, then q-Taylor formula [17] is given as

$$\zeta(x) = \sum_{k=0}^{\infty} \frac{\delta_q^k \zeta(x,a)}{[k]_q!}.$$

Also, we have

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \ldots + q^{n-1}, \quad [n]_q! = [1]_q[2]_q \ldots [n]_q.$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!}, \qquad m, n \in \mathbb{N}_0.$$

For further details in q-calculus go through [3, 7, 14, 18].

3. Analysis of q-differential transform method

For the function $f(y)$ the differential transform of k^{th} derivative is defined by Zhou [8] as

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(y)}{dy^k}\right]_{y=y_0}. \tag{3}$$

The differential inverse transform of $F(k)$ is defined as

$$f(y) = \sum_{k=0}^{\infty} F(k)(y - y_0)^k. \tag{4}$$

From equations (3) and (4), we get

$$f(y) = \sum_{k=0}^{\infty} \frac{(y-y_0)}{k!} \frac{d \ f(y)}{dy^k}\Big|_{y=y_0}.$$

However, the basic definitions and results of the differential transform method are given in [8, 19-26].

Jafari et al. [11] has defined the q-analogue of differential transform.

Definition 2. The q-differential transform of the function $f(z, t)$ is defined as

$$F(k)=\frac{1}{[k]_q!}\left[\frac{d_q^k}{d_q t^k}f(z,t)\right]_{t=a}, \tag{5}$$

where all the q-differentials of $f(z, t)$ exist in some neighbourhood of $t = a$, and $F(k)$ is the transformed function.

Definition 3. The inverse q-differential transform of the function $F(k)$ is defined as

$$f(z,t)=\sum_{k=0}^{\infty}F(k)(t-a)^k. \tag{6}$$

From the definitions (5) and (6), we get

$$f(z,t)=\sum_{k=0}^{\infty}\frac{1}{[k]_q!}\left[\frac{d_q^k}{d_q t^k}f(z,t)\right]_{t=a}(t-a)^k.$$

The fundamental operations of q-differential transform can readily be obtained and are very useful on our approach for solving q-integral equations listed below in Table 1 [11].

Table 1. Operations of q-differential transformations

Original function	Transformed function
$f(\eta)=u(\eta)\pm v(\eta)$	$F(k)=U(k)\pm V(k)$
$f(\eta)=\alpha u(\eta)$	$F(k)=\alpha U(k)$
$f(\eta)=\frac{d_q u(\eta)}{d_q\eta}$	$F(k)=(k+1)U(k+l)$
$f(\eta)=\frac{d_q^r u(\eta)}{d_q^r\eta}$	$F(k)=[k+1]_q[K+2]_q\ldots[k+r]_qU(k+r)$
$f(\eta)=\int_{\eta_0}^{\eta}u(t)d_qt$	$F(k)=\frac{U(k-1)}{k}, k\geq 1, F(0)$
$f(\eta)=\exp_q(\lambda\eta)$	$F(k)=\frac{\lambda^k}{[k]_q!}$
$f(\eta)=\eta^m$	$F(k)=\delta(k-m)$
$f(\eta)=\sin_q(w\eta+\alpha)$	$F(k)=\frac{w^k}{[k]_q!}\sin_q(\frac{\pi k}{2}+\alpha)$
$f(\eta)=\cos_q(w\eta+\alpha)$	$F(k)=\frac{w^k}{[k]_q!}\cos_q(\frac{\pi k}{2}+\alpha)$
$f(\eta)=u(\eta)v(\eta)$	$F(k)=\sum_{n=0}^{k}U(k-n)V(n)$

Result: If $w(x, y) = x^m t^n$, then $W_k(x) = x^m\ \delta(k - n)$, where

$$\delta(k)=\begin{cases}1, & k=0\\ 0, & k\neq 0\end{cases}.$$

Proof. From (5), we have

$$W_k(x)=\frac{1}{[k]_q!}\left[\frac{\delta_q^k(x^m t^n)}{\delta_q t^k}u(x,t)\right]_{t=0}=\frac{x^m}{[k]_q!}\left[\frac{\delta_q^k(t^n)}{\delta_q t^k}u(x,t)\right]_{t=0},$$

$$=\begin{cases} x^m\dfrac{k_q!}{k_q!}=x^m, & k=n\\ x^m\dfrac{n_q(n-1)_q\ldots(n-k+1)_q}{k_q!}t^{n-k}, & k<0\\ x^m\cdot 0, \; k>0 & \end{cases}$$

$$=\left\{x^m\delta(k-n)\right..$$

4. Main result

Theorem 1. Suppose that $F(k)$ and $G(k)$ are the q-differential transformations of the function $f(y)$ and $g(y)$, respectively, then we have with the help of (2).

a) If $h(y)=\int_{y_0}^{y} g(t)f(t)d_q(t)$, then

$$H(k)=\sum_{l=0}^{k-1}G(l)\frac{F(k-l-1)}{k}+\sum_{m=0}^{n-1}G(m)\frac{F(n-m-1)}{n},\quad H(0)=0.$$

b) If $h(y)=g(y)\int_{y_0}^{y} u(t)d_q(t)$, then

$$H(k)=\sum_{l=0}^{k-1}G(l)\frac{U(k-l-1)}{k-1},\quad H(0)=0.$$

Proof. (a) For $m=0$, we have

$$\frac{\delta_q^m h(y,t)}{\delta_q y^m}=\int_{t_0}^{t}\frac{\delta_q^m}{\delta_q y^m}g(y,z)f(y,z)d_q z.$$

$$\left[\frac{\delta_q^m h(y,t)}{\delta_q y^m}\right]_{t=t_0,\,y=y_0}=0,\qquad m=0,1,2\ldots$$

Next, for $m\geq 1$, we have

$$\begin{aligned}\frac{\delta_q^m h(y,t)}{\delta_q y^m}&=\frac{\delta_q^{m-1}}{\delta_q y^{m-1}}g(y,t)f(y,t),\\ &=\sum_{k=0}^{m-1}\begin{bmatrix}m-1\\ k\end{bmatrix}\frac{\delta_q^k}{\delta_q y^k}g(y,t)\cdot\frac{\delta_q^{m-k-1}}{\delta_q y^{m-k-1}}f(y,t).\end{aligned}$$

Now, with the help of (5) and (6), we have

$$\frac{\delta_q^m h(y,t)}{\delta_q y^m} = \sum_{k=0}^{m-1} \frac{[m-1]_q!}{[m-k-1]_q![k]_q!}[k]_q!G(k)[m-k-1]_q!F(m-k-1),$$
$$= \sum_{k=0}^{m-1}[m-1]_q!G(k)F(m-k-1)$$

$$[m]_q!H(k) = [m-1]_q!\sum_{k=0}^{m-1}G(k)F(m-k-1).$$

$$H(k) = \frac{1}{m}\sum_{k=0}^{m-1}G(k)F(m-k-1).$$

(b) This can be proved on the same lines as above in *Proof* (a).

5. Applications and numerical results

In this section, we apply the method to solve some Volterra q-integral equations [13]. These results show that the method is simple and effective and leads to an exact solution.

Example 1. We consider the following Volterra q-integral equation

$$v(y) = 1 - y - \frac{y^2}{2} + \int_0^y (y-t)v(t)d_q(t), \quad 0 < y < 1.$$

Sol: By using operations from Table 1 and according to theorem as stated above, we have the following relation:

$$V(k) = \delta(k) - \delta(k-1) - \frac{\delta(k-2)}{2} + \sum_{l=0}^{k-1}\delta(l-1)\frac{V(k-l-1)}{k-l} - \sum_{l=0}^{k-1}\delta(l-l)\frac{V(k-l-1)}{k}, \quad k \geq 1, \quad V(0) = 1.$$

Consequently, we find

$$V(1) = -1, \qquad V(2) = 0,$$

$$V(3) = \frac{-1}{[3]_q!}, \qquad V(4) = 0,$$

$$V(5) = \frac{-1}{[5]_q!} \ldots$$

Therefore, from (6), the solution of the above integral equation is given by

$$v(y) = 1 - y - \frac{1}{[3]_q!}y^3 - \frac{1}{[5]_q!}y^5 - \cdots$$
$$= 1 - \sin_q h(y),$$

which is the solution of the above q-integral equation. The exact solution for different values of q and y are given in Figure 1 and exact solution for fractional order $0 < q < 1$ is given in Figure 2.

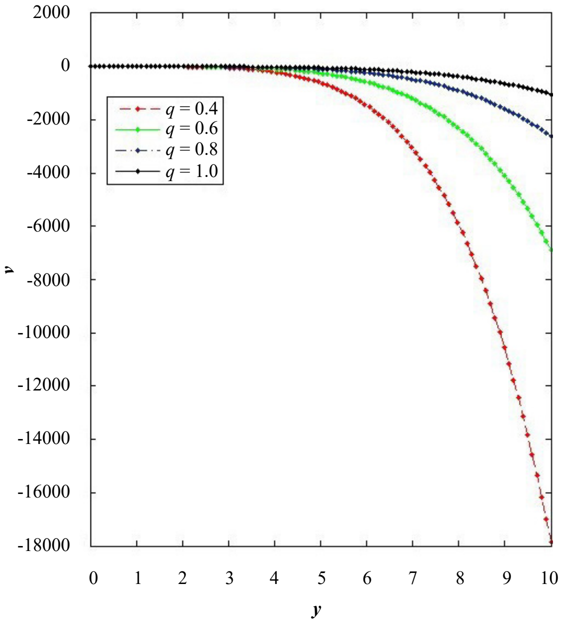

Figure 1. Exact solution of Example 1, for different values of q and y

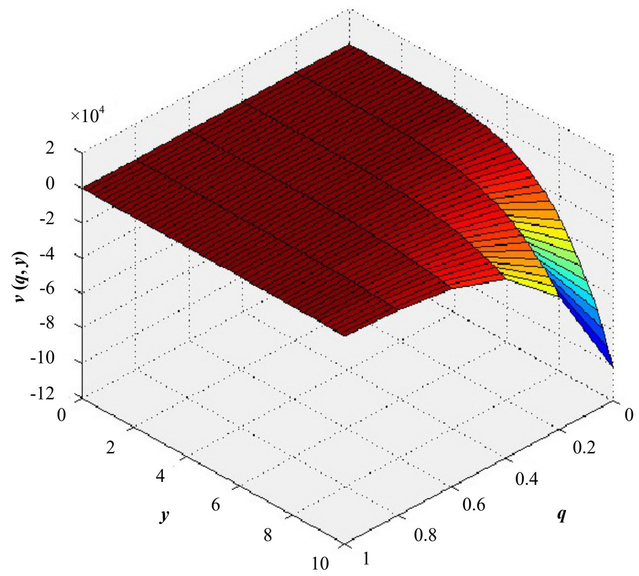

Figure 2. Exact solution of Example 1, for fractional order $0 < q < 1$

Example 2. Consider the linear Volterra q-integral equation

$$v(y) = y + \int_0^y (t - y)v(t)d_q(t), \quad 0 < y < 1, \ |q| < 1.$$

Sol: From the Theorem 1 above and from Table 1, the relation is given by

$$V(k) = \delta(k-1) + \sum_{l=0}^{k-1} \delta(l-l)\frac{V(k-l-1)}{k} - \sum_{l=0}^{k-1} \delta(l-1)\frac{V(k-l-1)}{k-1}, k \geq 1, V(0) = 0.$$

Consequently, we have

$$V(1) = 1, \qquad V(2) = 0,$$

$$V(3) = \frac{-1}{[3]_q!}, \quad V(4) = 0,$$

$$V(5) = \frac{-1}{[5]_q!} \dots .$$

Therefore, from (6), the solution of the above integral equation is given by

$$v(y) = 1 - \frac{1}{[3]_q!}y^3 - \frac{1}{[5]_q!}y^5 - \cdots = \sin_q(y),$$

which is an exact solution. The exact solution for different values of q and y are given in Figure 3 and exact solution for fractional order $0 < q < 1$ is given in Figure 4.

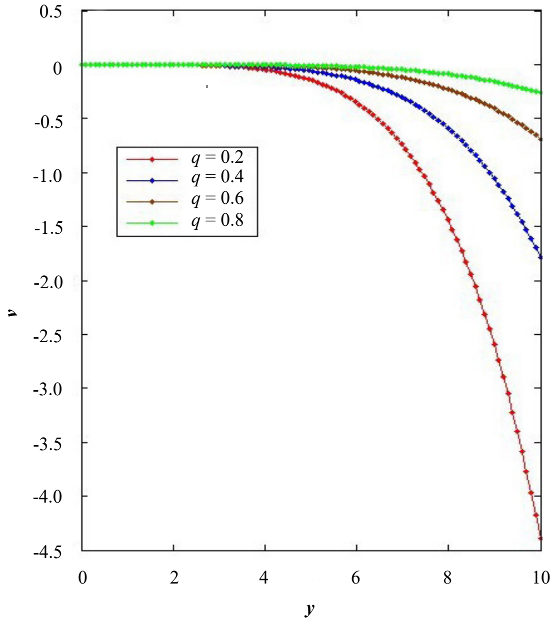

Figure 3. Exact solution of Example 2, for different values of q and y

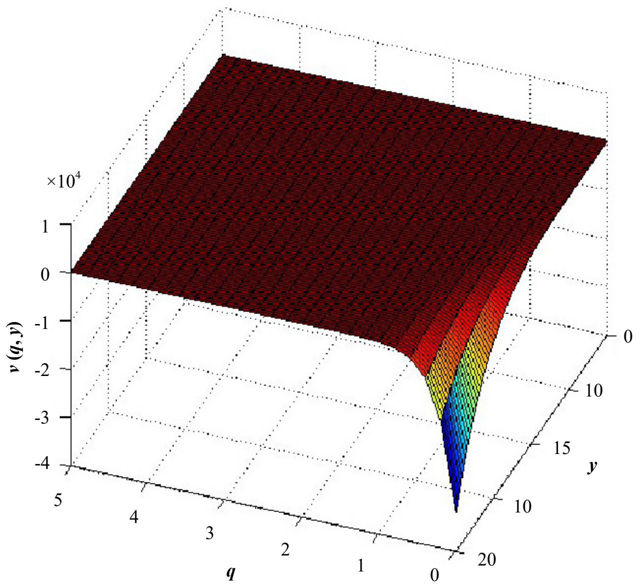

Figure 4. Exact solution of Example 2, for fractional order $0 < q < 1$

Example 3. Consider the nonlinear Volterra q-integral equation

$$v(y)+\int_0^y v^2(t)+v(t)d_q(t)=\frac{3}{2}-\frac{1}{2}\exp_q(-2y).,\quad 0<y<1.$$

Sol: We have following relation with the help of theorem

$$V(k)+\sum_{l=0}^{k-1}V(l)\frac{V(k-l-1)}{k}\frac{V(k-1)}{k}=\frac{3}{2}\delta(k)-\frac{(-2)^k}{[2k]_q!},\quad k\geq 1,\quad V(0)=1.$$

Consequently, we find

$$V(1)=-1,\qquad V(2)=\frac{1}{[2]_q!},$$

$$V(3)=\frac{-1}{[3]_q!},\quad V(4)=\frac{1}{[4]_q!},$$

$$V(5)=\frac{-1}{[5]_q!}\dots.$$

Therefore, from (6), we have

$$\begin{aligned}v(y)&=1-y+\frac{1}{[2]_q!}y^2-\frac{1}{[3]_q!}y^3-\cdots\\&=\exp_q(-y),\end{aligned}$$

which is an exact solution. The exact solution for different values of q and y are given in Figure 5 and exact solution for fractional order $0 < q < 1$ is given in Figure 6.

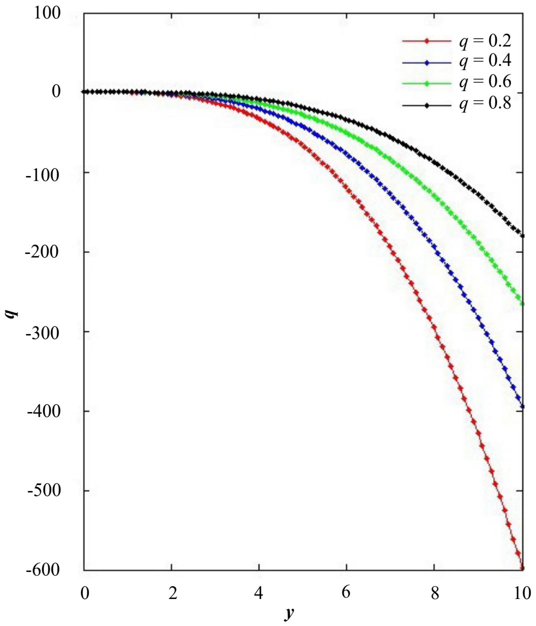

Figure 5. Exact solution of Example 3, for different values of q and y

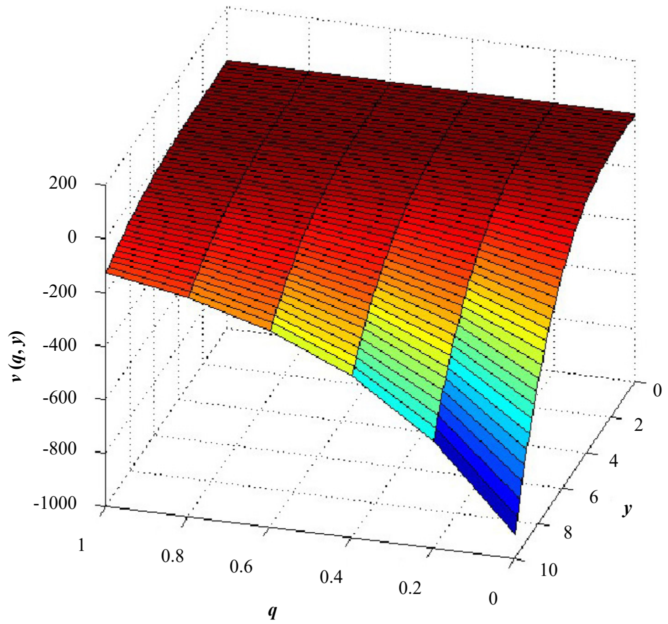

Figure 6. Exact solution of Example 3, for fractional order $0 < q < 1$

Example 4. Consider the nonlinear Volterra q-integral equation

$$v(y) = \cos_q(y) + \frac{1}{2}\sin_q(2y) + 3y - 2\int_0^y (1 + v^2(t))d_q(t).$$

Sol: Consider the following relation

$$V(k) = \frac{-1}{[k]_q!}\cos_q(\frac{\pi k}{2}) + \frac{2^k - 1}{[k]_q!}\sin_q(\frac{\pi k}{2}) + \delta(k-1) - \sum_{l=0}^{k-1} V(l)\frac{V(k-l-1)}{k}, \quad k \geq 1, \ V(0) = 1.$$

Consequently, we find

$$V(1) = 0, \quad V(2) = \frac{-1}{[2]_q!},$$

$$V(3) = 0, \quad V(4) = \frac{1}{[4]_q!},$$

$$V(5) = 0 \dots .$$

Therefore, from (6), the solution is given as

$$\begin{aligned} v(y) &= 1 - \frac{1}{[2]_q!}y^2 + \frac{1}{[4]_q!}y^4 - \cdots \\ &= \cos_q(y), \end{aligned}$$

which is an exact solution of the above integral equation. The exact solution for different values of q and y are given in

Figure 7 and exact solution for fractional order $0 < q < 1$ is given in Figure 8.

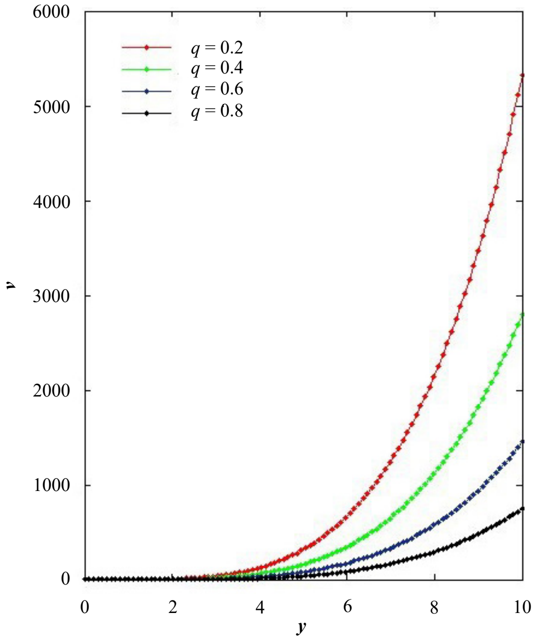

Figure 7. Exact solution of Example 4, for different values of q and y

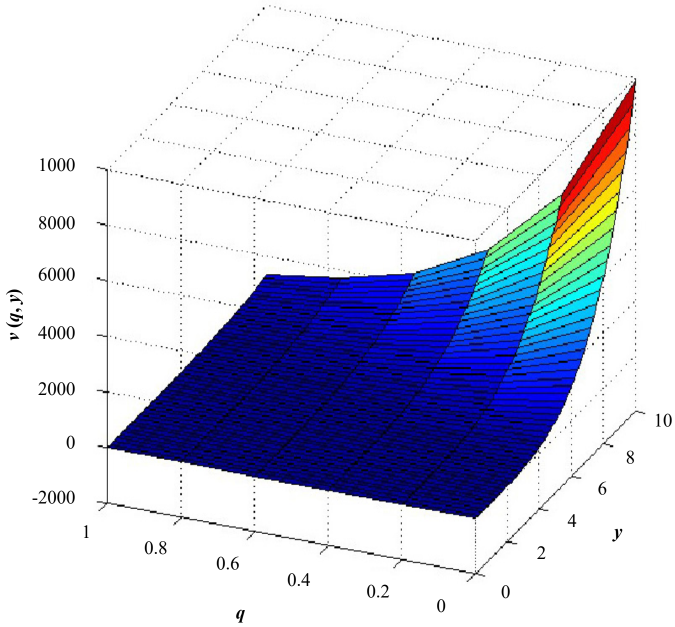

Figure 8. Exact solution of Example 4, for fractional order $0 < q < 1$

Table 2 gives the numerical estimation of exact solutions of the above mentioned examples.

Table 2. Numerical estimation of exact solution of Example 1, Example 2, Example 3, and Example 4 for different values of q and y

Example 1	q	y	$v(y)$	**Example 3**	q	y	$v(y)$
	0.1	1	-1.4830		0.1	1	0.0901
	0.2	2	-20.2259		0.2	2	-3.0430
	0.3	3	-84.2453		0.3	3	-10.0188
	0.4	4	-210.5381		0.4	4	-20.8755
	0.5	5	-387.8595		0.5	5	-34.9524
Example 2	q	y	$v(y)$	**Example 4**	q	y	$v(y)$
	0.1	1	-0.4830		0.1	1	0.8286
	0.2	2	-18.2259		0.2	2	6.3199
	0.3	3	-81.2453		0.3	3	26.0483
	0.4	4	-206.5381		0.4	4	63.0752
	0.5	5	-382.859		0.5	5	114.6265

6. Conclusion

It is well known that integral equations are used to model many physical problems given in the form of ordinary and partial differential equations. In this work, the q-differential transform method has been used to solve various Volterra q-integral equations. The solutions are given in exact form, and additionally, numerical results have been presented to show the efficiency of the given method. Moreover, numerical results have been presented graphically, which is well accepted by the obtained results. Thus, it can be concluded that the q-differential transform method is an easy and efficient method to solve various problems arising in different fields of applied mathematics.

Acknowledgment

The authors would like to thank the reviewers for their valuable comments and suggestions, which led to a great improvement in the original manuscript. The authors extend their appreciation to the Dean and Head of Research at the University of Technology and Applied Sciences, Salalah, Oman, for funding this work under the Internal Funded Project (01-IRFP-2022-23).

Conflict of interest

The authors declare no conflict of interest.

References

[1] Jing S-C, Fan H-Y. q-Taylor's formula with its q-remainder. *Communications in Theoretical Physics*. 1995; 23(1): 117-120. Available from: https://doi.org/10.1088/0253-6102/23/1/117.

[2] Kac V, Cheung P. *Quantum calculus*. New York: Springer-Verlag; 2002. Available from: https://doi.org/10.1007/978-1-4613-0071-7.

[3] Jackson FH. q-Difference equation. *American Journal of Mathematics*. 1910; 32(4): 305-314. Available from: https://doi.org/10.2307/2370183.

[4] Jackson FH. On q-definite integrals. *The Quarterly Journal of Pure and Applied Mathematics*. 1910; 41; 193-203.

[5] Euler L. *Introductio in analysin infinitorum*. Lausanne: Marc-Michael Bousque; 1748.

[6] Ernst T. *A comprehensive treatment of q-calculus*. Basel: Birkhäuser Basel; 2012. Available from: https://doi.org/10.1007/978-3-0348-0431-8.

[7] Bangerezako G. *An introduction to q-difference equations*. [Preprint]. 2007. Available from: https://perso.uclouvain.be/alphonse.magnus/gbang/qbook712.pdf.

[8] Zhou JK. *Differential transformation and its applications for electrical circuits*. Wuhan, China: Huazhong University Press; 1986.

[9] Odibat ZM. Differential transform method for solving Volterra integral equation with separable kernels. *Mathematical and Computer Modelling*. 2008; 48(7-8): 1144-1149. Available from: https://doi.org/10.1016/j.mcm.2007.12.022.

[10] Biazar J, Eslami M, Islam MR. Differential transform method for special systems of integral equations. *Journal of King Saud University - Science*. 2012; 24(3): 211-214. Available from: https://doi.org/10.1016/j.jksus.2010.08.015.

[11] Jafari H, Haghbin A, Hesam S, Baleanu D. Solving partial q-differential equations within reduced q-differential transformation method. *Mathematical and Theoretical Physics*. 2014; 59: 399-407.

[12] Wazwaz A-M. Introductory concepts of integral equations. In: *Linear and non-linear integral equations*. Berlin, Heidelberg: Springer; 2011. p.33-63. Available from: https://doi.org/10.1007/978-3-642-21449-3_2.

[13] Mansour ZSI. On a class of nonlinear Volterra-Fredholm q-integral equations. *Fractional Calculus and Applied Analysis*. 2014; 17(1): 61-78. Available from: https://doi.org/10.2478/s13540-014-0155-7.

[14] Jackson FH. A q-form of Taylor's theorem. *Messenger of Mathematics*. 1909; 38: 62-64.

[15] Exton H. *q-Hypergeometric functions and applications*. Halsted Press; 1983.

[16] Gasper G, Rahman M. *Basic hypergeometric series*. 2nd ed. Cambridge: Cambridge University Press; 2004. Available from: https://doi.org/10.1017/CBO9780511526251.

[17] Rajković PM, Stanković MS, Sladjana D. On q-iterative methods for solving equations and systems. *Novi Sad Journal of Mathematics*. 2003; 33(2): 127-137.

[18] Carmichael RD. The general theory of linear q-difference equation. *American Journal of Mathematics*. 1912; 34(2): 147-168. Available from: https://doi.org/10.2307/2369887.

[19] Ayaz F. Applications of differential transform method to differential-algebraic equations. *Applied Mathematics and Computation*. 2004; 152(3): 649-657. Available from: https://doi.org/10.1016/S0096-3003(03)00581-2.
[20] Abdel-Halim Hassan IH. Comparison differential transformation technique with Adomian decomposition method for linear and nonlinear initial value problems. *Chaos, Solitons & Fractals*. 2008; 36(1): 53-65. Available from: https://doi.org/10.1016/j.chaos.2006.06.040.
[21] Ayaz F. Solutions of the system of differential equations by differential transform method. *Applied Mathematics and Computation*. 2004; 147(2): 547-567. Available from: https://doi.org/10.1016/S0096-3003(02)00794-4.
[22] Arikoglu A, Ozkol I. Solution of boundary value problems for integro-differential equations by using differential transform method. *Applied Mathematics and Computation*. 2005; 168(2): 1145-1158. Available from: https://doi.org/10.1016/j.amc.2004.10.009.
[23] Bildik N, Konuralp A, Bek FO, Küçükarslan S. Solution of different type of the partial differential equation by differential transform method and Adomian's decomposition method. *Applied Mathematics and Computation*. 2006; 172(1): 551-567. Available from: https://doi.org/10.1016/j.amc.2005.02.037.
[24] Arikoglu A, Ozkol I. Solution of difference equations by using differential transform method. *Applied Mathematics and Computation*. 2006; 174(2): 1216-1228. Available from: https://doi.org/10.1016/j.amc.2005.06.013.
[25] Arikoglu A, Ozkol I. Solution of differential-difference equations by using differential transform method. *Applied Mathematics and Computation*. 2006; 181(1): 153-162. Available from: https://doi.org/10.1016/j.amc.2006.01.022.
[26] Liu H, Song Y. Differential transform method applied to high index differential-algebraic equations. *Applied Mathematics and Computation*. 2007; 184(2): 748-753. Available from: https://doi.org/10.1016/j.amc.2006.05.173.