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Research Article

# Data Dependence and Existence and Uniqueness for Hilfer Nabla Fractional Difference Equations

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**Abstract:** In the present article, we establish sufficient conditions for the existence of a unique bounded solution using a prominent fixed-point theorem for the non-linear initial value problem involving the recently introduced Hilfer nabla fractional difference operator.

$$\begin{cases} \left(\nabla_x^{\aleph,\beta}\xi\right)(\alpha) = j(\alpha,\xi(\alpha)), \quad \alpha \in \mathbb{N}_{x+1}, \\ \left[\left(\nabla_x^{-(1-\iota)}\xi\right)(\alpha)\right]_{\alpha=x} = \xi(x) = \xi_0, \end{cases}$$

where $0 < \aleph < 1, 0 \le \text{ß} \le 1, \iota = \aleph + \text{ß} - \aleph\text{ß}$ and $j : \mathbb{N}_x \times \mathbb{R}^n \to \mathbb{R}^n$. We also analyze the Ulam-Hyers stability of the considered problem and make some interesting observations on the dependence of its solutions on initial conditions and parameters. Finally, we conclude this article by constructing suitable examples to illustrate the applicability of established results.

***Keywords*:** Hilfer nabla fractional difference, initial value problem, existence, uniqueness, Ulam-Hyers stability

**MSC:** 26A33, 34A08, 39A12, 39A70

## 1. Introduction

The theory of nabla fractional calculus is a relatively new branch of mathematics that deals with arbitrary order differences and sums in the backward sense, with works drawing a lot of attention in the past decade [1-10]. The notions of nabla fractional difference and sum were put forward by Gray and Zhang [11] and Miller and Ross [12]. Following their works, several mathematicians' contributions have made the theory of nabla fractional calculus a worthy area of research in emerging science. For related theories of fractional difference equations and real-world applications that demonstrate the importance of discrete fractional calculus, we refer to [13-24].

Stability analysis of functionals has a vital role in various branches of mathematics. This analysis can be traced back to the question raised by Ulam [25], looking for conditions in order for a linear mapping close to an approximately


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linear mapping to exist. Hyers first approached this question, and later his approach was strengthened by Rassias [26], which we have today as Hyers-Ulam-Rassias stability theory.

In 2000, Hilfer [27] gave the generalized Riemann-Liouville fractional derivative, which contains a parameter that lets us interpolate between the Riemann-Liouville and the Caputo fractional derivatives as particular cases. Since then, the Hilfer fractional derivative has drawn the attention of many scientists. Motivated by Hilfer's definition, the authors in [28, 29] introduced the nabla Hilfer fractional derivative and obtained some of its important preliminary properties for initial value problems of order $0 < \aleph \le 1$ and $1 < \aleph \le 2$.

In the present article, we take the following initial value problem involving the Hilfer nabla fractional difference operator [28]:

$$\begin{cases} \left(\nabla_x^{\aleph,ß}\xi\right)(\alpha) = j(\alpha,\xi(\alpha)), \quad \alpha \in \mathbb{N}_{x+1}, \\ \left[\left(\nabla_x^{-(1-\iota)}\xi\right)(\alpha)\right]_{\alpha=x} = \xi(x) = \xi_0, \end{cases} \tag{1}$$

where $0 < \aleph < 1, 0 \le ß \le 1, \iota = \aleph + ß - \aleph ß$ and $j : \mathbb{N}_x \times \mathbb{R}^n \to \mathbb{R}^n$. The present article is organized as follows: We begin with some preliminary definitions and established results of nabla fractional calculus in Section 2. In Section 3, under suitable assumptions on the non-linear function $j$, we obtain sufficient conditions for the existence of a unique bounded solution of (1) on a well-defined space using the Banach fixed point theorem with respect to the weighted supremum norm. In Section 4, we analyze the dependence of solutions of (1) on initial conditions and parameters. Finally, in Section 5, we discuss the Ulam-Hyers stability of (1) and conclude the article with problems to illustrate the application of established results.

## 2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{Z}^-$ be the sets of all real numbers and non-positive integers, respectively. Define $\mathbb{N}_x = \{x, x+1, x+2, x+3, ...\}$ for any $x \in \mathbb{R}$. Assume empty sums and products are 0 and 1, respectively. Let $\xi : \mathbb{N}_x \to \mathbb{R}$ and $R \in \mathbb{N}_1$. The nabla difference of $s$ of order 1 is defined by $(\nabla s)(\alpha) = s(\alpha) - s(\alpha - 1)$ for $\alpha \in \mathbb{N}_{x+1}$, and the $R$th-order nabla difference of $\xi$ can be defined recursively by $\left(\nabla^R \xi\right)(\alpha) = \left(\nabla\left(\nabla^{R-1}\xi\right)\right)(\alpha)$ for $\alpha \in \mathbb{N}_{x+R}$.

**Definition 2.1.** (See [18]). The generalized rising function for $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ and $\rho \in \mathbb{R}$ such that $(\alpha + \rho) \in \mathbb{R} \setminus \mathbb{Z}^-$, is defined by

$$\alpha^{\overline{\rho}} = \frac{\Gamma(\alpha + \rho)}{\Gamma(\alpha)},$$

where $\Gamma(\cdot)$ represents the Euler gamma function $\alpha \in \mathbb{Z}^-$ and $s \in \mathbb{R}$ such that, $(\alpha + \rho) \in \mathbb{R} \setminus \mathbb{Z}^-$, then we use the convention that $\alpha^{\overline{\beta}} = 0$.

**Lemma 2.2.** (See [10]). Assume the following are well defined:

i. $\alpha^{\overline{\aleph}}(\alpha + \aleph)^{\overline{ß}} = \alpha^{\overline{\aleph + ß}}$;
ii. If $\alpha \le s$, then $\alpha^{\overline{\aleph}} \le s^{\overline{\aleph}}$;
iii. If $\aleph < \alpha \le s$, then $s^{\overline{-\aleph}} \le \alpha^{\overline{-\aleph}}$.

**Definition 2.3.** (See [18]). Let $\alpha, x \in \mathbb{R}$ and $\zeta \in \mathbb{R} \setminus -\mathbb{N}$. The nabla fractional Taylor monomial of $\zeta$th-order is given by

$$H_\zeta(\alpha, x) = \frac{(\alpha - x)^{\overline{\zeta}}}{\Gamma(\zeta + 1)},$$

provided the right-hand side exists.

**Lemma 2.4.** (See [3]). Let $\zeta > -1$ and $\rho \in \mathbb{N}_x$. Then, the following results of Taylor monomials hold:

1. If $\alpha \in \mathbb{N}_{\varrho(\rho)}$, then $H_{\zeta}(\alpha, \varrho(\rho)) \geq 0$.
2. If $\alpha \in \mathbb{N}_{\rho}$, then $H_{\zeta}(\alpha, \varrho(\rho)) \geq 0$.
3. If $\alpha \in \mathbb{N}_{\rho+1}$ and $-1 < \zeta < 0$, then $H_{\zeta}(\alpha, \varrho(\rho)) \geq 0$ is a decreasing function of $\alpha$.
4. If $\alpha \in \mathbb{N}_{\rho}$ and $-1 < \zeta < 0$, then $H_{\zeta}(\alpha, \varrho(\rho))$ is an increasing function of $\rho$.
5. If $0 < v \leq \zeta$, then $H_{v}(\alpha, x) \leq H_{\zeta}(\alpha, x)$ for each fixed $\alpha \in \mathbb{N}_{x}$.

**Definition 2.5.** (See [5]). Let $\xi : \mathbb{N}_x \to \mathbb{R}$ and $\zeta > 0$. The nabla sum of ζth-order of $\xi$ is given by

$$\left(\nabla_x^{-\zeta} \xi\right)(\alpha) = \sum_{\rho=x}^{\alpha} H_{\zeta-1}(\alpha, \varrho(\rho))\xi(\rho), \quad \alpha \in \mathbb{N}_x,$$

where $\varrho(\rho) = \rho - 1$.

**Theorem 2.6.** (See [5]). (Power Rule) Let $\aleph > 0$ and $\zeta > -1$. Then,

$$\nabla_x^{-\aleph}(\alpha - x + 1)^{\overline{\zeta}} = \frac{\Gamma(\zeta+1)}{\Gamma(\zeta+\aleph+1)}(\alpha - x + 1)^{\overline{\zeta+\aleph}} \quad \text{for } \alpha \in \mathbb{N}_x.$$

**Definition 2.7.** (See [5]). Let $\xi : \mathbb{N}_x \to \mathbb{R}$, $\zeta > 0$ and choose $W \in \mathbb{N}_1$ such that $W - 1 < \zeta \leq W$. The Riemann-Liouville nabla difference of ζth-order of $\xi$ is given by

$$\left(\nabla_x^{\zeta} \xi\right)(\alpha) = \left(\nabla^{W}\left(\nabla_x^{-(W-\zeta)} \xi\right)\right)(\alpha), \quad \alpha \in \mathbb{N}_{x+W}.$$

*Remark* 2.8. It can be observed from Definition 2.5 and Definition 2.7 that if $\xi : \mathbb{N}_x \to \mathbb{R}$, then $\left(\nabla_x^{-\zeta} \xi\right) : \mathbb{N}_x \to \mathbb{R}$ and $\left(\nabla_x^{\zeta} \xi\right) : \mathbb{N}_{x+W} \to \mathbb{R}$.

**Definition 2.9.** (See [4]). Let $\zeta > 0$, choose $W \in \mathbb{N}_1$ such that $W - 1 < \zeta \leq W$ and $\xi : \mathbb{N}_{x-W} \to \mathbb{R}$. The ζth-order Caputo nabla difference of $\xi$ is given by

$$\left(\nabla_{*x}^{\zeta} \xi\right)(\alpha) = \left(\nabla_x^{-(W-\zeta)}(\nabla^{W} \xi)\right)(\alpha), \quad \alpha \in \mathbb{N}_x.$$

**Definition 2.10.** (See [28, 29]). Let $\xi : \mathbb{N}_x \to \mathbb{R}, 0 \leq ß \leq 1$, and choose $W \in \mathbb{N}_1$ such that $W - 1 < \aleph \leq W$. The ℵth-order and ßth-type Hilfer nabla difference of $\xi$ is defined by

$$\left(\nabla_x^{\aleph, ß} \xi\right)(\alpha) = \left(\nabla_{x+W}^{-ß(W-\aleph)} \nabla^{W} \nabla_x^{-(1-ß)(W-\aleph)} \xi\right)(\alpha), \quad \alpha \in \mathbb{N}_{x+W}.$$

The type ß allows to interpolate continuously from the Riemann-Liouville case $\nabla_x^{\aleph,0} \equiv \nabla_x^{\aleph}$ to the Caputo case $\nabla_x^{\aleph,1} \equiv \nabla_{*x}^{\aleph}$.

**Definition 2.11.** [30] The one-parameter discrete Mittag-Leffler function is defined by

$$F_{\aleph}(\Lambda, \alpha - x) = \sum_{k=0}^{\infty} \Lambda^k \frac{(\alpha - x)^{\overline{\aleph k}}}{\Gamma(\aleph k + 1)},$$

where $0 < \aleph < 1, |\Lambda| < 1$ and $\alpha \in \mathbb{N}_x$.

**Lemma 2.12.** [10] Consider $\Lambda \in (0,1)$. Then,

*a.* $F_{\aleph}(\Lambda, 0) = 1$;

*b.* $F_{\aleph}(\Lambda, \alpha - x) \to \infty$ monotonically with respect to $\alpha$;

*c.* $F_{\aleph}(\Lambda, \alpha - x) : \mathbb{N}_x \to [1, \infty)$;

*d.* $\nabla_{x+1}^{-\aleph} F_{\aleph}(\Lambda, \alpha - x) = \frac{1}{\Lambda}\left(F_{\aleph}(\Lambda, \alpha - x) - 1\right)$.

Throughout Section 3, we take

$$w(\alpha) = F_{\aleph}(\Lambda, \alpha - x) \text{ for } 0 < \Lambda < 1 \text{ and } \alpha \in \mathbb{N}_x.$$

## 3. Unique bounded solution

In this section, under suitable assumptions on the non-linear function $j$, we obtain sufficient conditions for the existence of a unique bounded solution of (1) on a well-defined space using the Banach fixed point theorem with respect to the weighted supremum norm. First, we state some important definitions and theorems of functional analysis [31], which are vital in establishing the main results.

**Definition 3.1.** We define the set

$$l^\infty = l^\infty(\mathbb{R}^n) = \{\xi : \mathbb{N}_x \to \mathbb{R}^n \text{ with } \|\xi\|_\infty < \infty\}.$$

Then, $l^\infty = \left(l^\infty, \|.\|\right)$ is a Banach space consisting of bounded sequences of real vectors with $\|.\|$ been the supremum norm defined by

$$\|\xi\|_\infty = \sup_{\alpha \in \mathbb{N}_x} \|\xi(\alpha)\|.$$

**Definition 3.2.** $l_w^\infty = \left(l_w^\infty, \|.\|\right)$ denotes the Banach space consisting of bounded sequences of real vectors with $\|.\|$ been weighted supremum norm defined by

$$\|\xi\|_w = \sup_{\alpha \in \mathbb{N}_x} \frac{\|\xi(\alpha)\|}{w(\alpha)},$$

where $w : \mathbb{N}_x \to [1, \infty), w(x) = 1, w(\alpha) \to \infty$ monotonically with respect to $\alpha$.

**Theorem 3.3.** [28] $\xi$ is a solution of the initial value problem (1) if and only if $\xi$ is a solution of the Volterra summation equation

$$\xi(\alpha) = \xi_0 H_{\iota-1}(\alpha, \varrho(x)) + \sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha, \varrho(\rho)) j(\rho, \xi(\rho)), \quad \alpha \in \mathbb{N}_x. \tag{2}$$

Define the following operator

$$T\xi(\alpha) = \xi_0 H_{\iota-1}(\alpha, \varrho(x)) + \sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha, \varrho(\rho)) j(\rho, \xi(\rho)), \quad \alpha \in \mathbb{N}_x.$$

*Remark* 3.4. It is clear from Theorem 3.3 that $\xi$ is a fixed point of $T$ if and only if $\xi$ is a solution of (1).

**Theorem 3.5.** (See [10]). Consider $D > 0$ a random constant. Then, consider the function $j$ defined on $\mathbb{N}_x \times \mathbb{R}^n$ or on a sub-region of the type

$$M = \left\{(\alpha, \xi) : \|\xi(\alpha)\| \le D \text{ for all } \alpha \in \mathbb{N}_x\right\}.$$

Let $j$ be a continuously differentiable function with respect to the second argument on $\mathbb{N}_x \times \mathbb{R}^n$ (or $M$), and also assume there exists a constant $Q > 0$ such that for all $(\alpha, \xi) \in \mathbb{N}_x \times \mathbb{R}^n$ (or $M$),

$$\left\| \frac{\partial j(\alpha, \xi)}{\partial \xi_p} \right\| \le Q, \quad (p = 1, 2, \ldots, n),$$

then $j$ is Lipschitz continuous with Lipschitz constant $Q$ with respect to its second argument on $\mathbb{N}_x \times \mathbb{R}^n$ (or $M$),

**Theorem 3.6.** [Contraction Mapping Theorem] [31] Let $H$ be a closed subset of a Banach space $X$. Assume $Y: H \to H$ is a contraction mapping, that is, there exists a $\mu$, $0 < \mu < 1$ such that

$$\|Y\xi - Yh\| \le \mu \|\xi - h\|,$$

for all $\xi, h \in H$. Then, the operator $Y$ has a unique fixed-point $t$ in $H$.

We make the following assumptions on $j$:

(H1) $j$ is Lipschitz continuous with respect to its second variable on $\mathbb{N}_x \times \mathbb{R}^n$, i.e., there exists a constant $Q \in [0, \Lambda)$ such that for all $(\alpha, \xi), (\alpha, h) \in \mathbb{N}_x \times \mathbb{R}^n$,

$$\|j(\alpha, \xi) - j(\alpha, h)\| \le Q \|\xi - h\|.$$

(H1)' $j$ is Lipschitz continuous with respect to its second variable on $M$, i.e., there exists $Q \in [0, \Lambda)$ a constant such that for all $(\alpha, \xi), (\alpha, h) \in \mathbb{N}_x \times M$,

$$\|j(\alpha, \xi) - j(\alpha, h)\| \le Q \|\xi - h\|.$$

(H2)

$$\|j\|_w = \sup_{\alpha \in \mathbb{N}_x} \frac{\|j(\alpha, 0)\|}{w(\alpha)} < \infty$$

(H2)'

$$\|j\|_w = \sup_{\alpha \in \mathbb{N}_x} \frac{\|j(\alpha, 0)\|}{w(\alpha)} < \frac{D}{\Lambda - Q} < \infty$$

(H3) For any pair of elements $\xi$ and $h$ in

$$\|j(\alpha, \xi) - j(\alpha, h)\| \le M_1 (\alpha - x)^{\overline{-t_2}} \|\xi - h\|, \quad \alpha \in \mathbb{N}_{x+1},$$

where $M_1 \ge 0$ and $\aleph < t_2 < 1$.

(H4) $x_1 = M_1 \Gamma(1 - t_2) < 1$.

(H5) $j$ is continuous concerning the second argument.

The following result can be easily verified with Definition 2.3 and Lemma 2.4.

**Lemma 3.7.** Consider $\iota = \aleph + ß - \aleph ß$ where $0 < \aleph < 1$ and $0 \le ß \le 1$. Then,

$$H_{\iota-1}(\alpha, \varrho(x)) \le 1, \quad t \in \mathbb{N}_x.$$

**Theorem 3.8.** Suppose that (H1) and (H2) hold. Then, there exists a unique solution for the initial value problem (1) defined on $\mathbb{N}_x$.

*Proof.* We first show that $Y: l_w^\infty \to l_w^\infty$. We have seen that $l_w^\infty$ is a complete metric space with the norm-weighted sup-metric defined by

$$\varrho(\xi, h) = \sup_{\alpha \in \mathbb{N}_x} \frac{\|\xi(\alpha) - h(\alpha)\|}{w(\alpha)},$$

for each pair $\xi, h \in l_w^\infty$. Using Lemmas 2.12 and 3.7, we have

$$
\begin{aligned}
\|Y\xi\|_w &= \sup_{\alpha\in\mathbb{N}_x}\frac{\|Y\xi(\alpha)\|}{w(\alpha)} \\
&\le \sup_{\alpha\in\mathbb{N}_x}\frac{1}{w(\alpha)}\left[\|H_{\iota-1}(\alpha,\varrho(x))\xi_0\| + \sum_{\rho=x+1}^{t} H_{\aleph-1}(\alpha,\varrho(\rho))\|j(\rho,\xi(\rho))\|\right] \\
&= \sup_{\alpha\in\mathbb{N}_x}\frac{1}{w(\alpha)}\left[H_{\iota-1}(\alpha,\varrho(a))\|\xi_0\| + \sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))\|j(\rho,\xi(\rho)) - j(\rho,0) + j(\rho,0)\|\right] \\
&\le \sup_{\alpha\in\mathbb{N}_x}\frac{1}{w(\alpha)}\left[\|\xi_0\| + \sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))\|j(\rho,\xi(\rho)) - j(\rho,0)\| + \sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))\|j(\rho,0)\|\right] \\
&\le \|\xi_0\|_w + \sup_{\alpha\in\mathbb{N}_x}\frac{Q}{w(\alpha)}\sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))w(\rho)\frac{\|\xi(\rho)\|}{w(\rho)} + \sup_{\alpha\in\mathbb{N}_x}\frac{1}{w(\alpha)}\sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))w(\rho)\frac{\|j(\rho,0)\|}{w(\rho)} \\
&\le \|\xi_0\|_w + \left[Q\|\xi\|_w + \|j\|_w\right]\sup_{\alpha\in\mathbb{N}_x}\left[\frac{1}{w(\alpha)}\sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))w(\rho)\right] \\
&= \|\xi_0\|_w + \left[Q\|\xi\|_w + \|j\|_w\right]\sup_{\alpha\in\mathbb{N}_x}\frac{1}{w(\alpha)}(\nabla_{x+1}^{-\aleph} w)(\alpha) \\
&= \|\xi_0\|_w + \left[Q\|\xi\|_w + \|j\|_w\right]\sup_{\alpha\in\mathbb{N}_x}\frac{1}{\Lambda w(\alpha)}[w(\alpha)-1] \\
&= \|\xi_0\|_w + \frac{1}{\Lambda}\left[Q\|\xi\|_w + \|j\|_w\right]\sup_{\alpha\in\mathbb{N}_x}\left[1-\frac{1}{w(\alpha)}\right] \\
&= \|\xi_0\|_w + \frac{1}{\Lambda}\left[Q\|\xi\|_w + \|j\|_w\right] < \infty.
\end{aligned}
$$

Thus, we have $Y\xi \in l_w^\infty$. Let $\xi, h \in l_w^\infty$ and consider

$$
\begin{aligned}
\|Y\xi - Yh\|_w &= \sup_{\alpha\in\mathbb{N}_x}\left[\frac{\|Y\xi(\alpha) - Yh(\alpha)\|}{w(\alpha)}\right] \\
&\le \sup_{\alpha\in\mathbb{N}_x}\frac{1}{w(\alpha)}\left[\sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))\|j(\alpha,\xi(\alpha)) - j(\alpha,h(\alpha))\|\right] \\
&\le \sup_{\alpha\in\mathbb{N}_x}\frac{Q}{w(\alpha)}\left[\sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))w(\rho)\frac{\|\xi(\alpha)-h(\alpha)\|}{w(\rho)}\right] \\
&\le Q\|\xi-h\|_w \sup_{\alpha\in\mathbb{N}_x}\frac{1}{w(\alpha)}\left[\sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))w(\rho)\right] \\
&= Q\|\xi-h\|_w \sup_{\alpha\in\mathbb{N}_x}\frac{1}{w(\alpha)}\left(\nabla_{x+1}^{-\aleph} w\right)(\alpha) \\
&= Q\|\xi-h\|_w \sup_{\alpha\in\mathbb{N}_x}\frac{1}{\Lambda w(\alpha)}[w(\alpha)-1] \\
&= \frac{Q}{\Lambda}\|\xi-h\|_w \sup_{\alpha\in\mathbb{N}_x}\left[1-\frac{1}{w(\alpha)}\right] \\
&= \frac{Q}{\Lambda}\|\xi-h\|_w.
\end{aligned}
$$

Since $Q < \Lambda, Y$ is a contraction. Hence, by Theorem 3.6, $Y$ has a unique solution in $l_w^\infty$.

We now state here the local existence result using the Banach fixed point theorem.

**Theorem 3.9.** (See [25]). Let $X = (X, \varrho)$ be a complete metric space containing an open ball $\dot{C}_{r'}(t_0)$ having centre $t_0$

and radius $r'$. Let $E:\dot{C}_{r'}(t_0)\to X$ be a contractive map with a positive number $1>q$ as the contraction constant. If

$$\varrho(E\alpha_0,\alpha_0)<(1-q)r',$$

then there exists a unique fixed point in $\dot{C}_{r'}(\alpha_0)$ for the operator $E$.

**Theorem 3.10.** Suppose that the conditions (H1)' and (H2)' hold. Let $l>0$ and define a set

$$B_l^w(0)=\left\{\xi:\|\xi\|_w<l\right\}\subset l_w^\infty,$$

where

$$l=\frac{D}{(\Lambda-Q)^2}.$$

Then, there exists a unique bounded solution of (1) in $B_l^w(0)$.

*Proof.* Clearly, $Y$ maps $B_l^w(0)$ into $l_w^\infty$. We have seen that $Y$ is a contraction, with $\frac{Q}{\Lambda}<1$. Now, we use Theorem 3.9 and Lemma 2.12 to establish a unique bounded solution in $B_l^w(0)$. Consider

$$\begin{aligned}
\|Y0-0\|_w &= \sup_{\alpha\in\mathbb{N}_a}\frac{\|Y0-0\|}{w(\alpha)}\\
&=\sup_{\alpha\in\mathbb{N}_x}\frac{1}{w(\alpha)}\left\|\sum_{s=x+1}^{\xi}H_{\aleph-1}(\alpha,\varrho(s))j(s,0)\right\|\\
&\le\sup_{\alpha\in\mathbb{N}_x}\frac{1}{w(\alpha)}\left[\sum_{s=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(s))w(s)\frac{\|j(s,0)\|}{w(s)}\right]\\
&\le\|j\|_w\sup_{\alpha\in\mathbb{N}_x}\frac{1}{w(\alpha)}\left[\sum_{s=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(s))w(s)\right]\\
&=\|j\|_w\sup_{\alpha\in\mathbb{N}_x}\frac{1}{w(\alpha)}\left(\nabla_{x+1}^{-\aleph}w\right)(\alpha)\\
&<\frac{D}{\Lambda-Q}\sup_{\alpha\in\mathbb{N}_x}\frac{1}{\Lambda w(\alpha)}[w(\alpha)-1]\\
&=\frac{D}{\Lambda(\Lambda-Q)}\sup_{\alpha\in\mathbb{N}_x}\left[1-\frac{1}{w(\alpha)}\right]\\
&=\left(1-\frac{Q}{\Lambda}\right)l.
\end{aligned}$$

Hence, from Theorem 3.9 the result follows and there exists a unique bounded solution in $B_l^w(0)$.

Now, we show the boundedness of the solution of (1) under suitable assumptions on the non-linear function $j$.

**Theorem 3.11.** Assume $j$ satisfies (H5) and there exists constants $\iota_1\in[\aleph,1)$ and $C_1\ge 0$ such that

$$\|j(\alpha,\xi(\alpha))\|\le C_1(\alpha-x)^{-\iota_1},\quad \alpha\in\mathbb{N}_{x+1}. \tag{2}$$

Then, there exists a positive constant

$$C_2 = \max\left\{\xi_0, \frac{C_1\Gamma(1-\iota_1)}{\Gamma(\aleph-\iota_1+1)}\right\},$$

such that the solution of (1) satisfies

$$\|\xi(\alpha)\| \le C_2\left(1+(\alpha-x)^{\overline{\aleph-\iota_1}}\right), \quad \alpha \in \mathbb{N}_x.$$

*Proof.* Using Lemma 3.7 in (2), for $\alpha \in \mathbb{N}_x$, we have

$$\begin{aligned}
\|\xi(\alpha)\| &\le \|\xi_0 H_{\iota-1}(\alpha,\varrho(x))\| + \sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))\|j(\rho,\xi(\rho))\| \\
&\le H_{\iota-1}(\alpha,\varrho(x))\|\xi_0\| + C_1\sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))(\rho-x)^{\overline{-\iota_1}} \\
&\le \|\xi_0\| + C_1\nabla_{x+1}^{-\aleph}(\alpha-x)^{\overline{-\iota_1}} = \|\xi_0\| + \frac{C_1\Gamma(1-\iota_1)}{\Gamma(\aleph-\iota_1+1)}(\alpha-x)^{\overline{\aleph-\iota_1}} \\
&\le C_2\left(1+(\alpha-x)^{\overline{\aleph-\iota_1}}\right).
\end{aligned}$$

As $\alpha \to \infty, (\alpha-x)^{\overline{\aleph-\iota_1}} \to 0$, implying that

$$\|\xi(\alpha)\| \le 2C_2, \quad t \in \mathbb{N}_x.$$

## 4. Data dependence

In this section, we analyze the dependence of solutions of (1) on initial conditions and parameters. It can be easily shown that $\iota = \aleph + ß - \aleph ß \le 1$, the result of which helps us to say $H_{\iota-1}(\alpha,\varrho(x))$ is a decreasing function of $\alpha$ by Lemma 2.4.

**Theorem 4.1**. Assume that $j$ satisfies conditions (H3) and (H5). Suppose $\xi$ and $h$ satisfy the initial value problems

$$\left(\nabla_x^{\aleph+\epsilon,ß}\xi\right)(\alpha) = j(\alpha,\xi(\alpha)), \quad \left[\left(\nabla_x^{-(1-\iota)}\xi\right)(\alpha)\right]_{\alpha=x} = \xi(x) = \xi_0, \quad \alpha \in \mathbb{N}_{x+1}, \tag{3}$$

$$\left(\nabla_x^{\aleph,ß}h\right)(\alpha) = j(\alpha,h(\alpha)), \quad \left[\left(\nabla_x^{-(1-\iota)}h\right)(\alpha)\right]_{\alpha=x} = h(x) = \xi_0, \quad \alpha \in \mathbb{N}_{x+1}, \tag{4}$$

respectively, where $\epsilon > 0$ and $0 < \aleph < \aleph + \epsilon < 1$. Then,

$$\|\xi(\alpha) - h(\alpha)\| = O(\epsilon), \tag{5}$$

provided (H4) holds.

*Proof.* The initial value problems (3) and (4) are equivalent to

$$\begin{aligned}
\xi(\alpha) &= \xi_0 H_{\aleph+ß+\epsilon-\aleph ß-ß\epsilon-1}(\alpha,\varrho(x)) + \sum_{\rho=x+1}^{\alpha} H_{\aleph+\epsilon-1}(\alpha,\varrho(\rho))j(\rho,\xi(\rho)), \quad \alpha \in \mathbb{N}_x, \\
h(\alpha) &= \xi_0 H_{\iota-1}(\alpha,\varrho(x)) + \sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))j(\rho,h(\rho)), \quad \alpha \in \mathbb{N}_x,
\end{aligned}$$

respectively. Using Lemma 2.2 and Lemma 2.4, for $\alpha \in \mathbb{N}_{x+1}$, we have

$$\left\|\xi(\alpha)-h(\alpha)\right\| \le \left|\frac{(\alpha-\varrho(x))^{\overline{\aleph+ß+\epsilon-\aleph ß-ß\epsilon-1}}}{\Gamma(\aleph+ß+\epsilon-\aleph ß-ß\epsilon)}-\frac{(\alpha-\varrho(x))^{\overline{\iota-1}}}{\Gamma(\iota)}\right|\left\|\xi_0\right\|$$

$$+\left\|\sum_{\rho=x+1}^{\alpha} H_{\aleph+\epsilon-1}(\alpha,\varrho(\rho))\, j(\rho,\xi(\rho))-\sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))\, j(\rho,h(\rho))\right\|$$

$$\le \left|\frac{(\alpha-\varrho(x))^{\overline{\iota+\epsilon(1-ß)-1}}}{\Gamma(\iota+\epsilon(1-ß))}-\frac{(\alpha-\varrho(x))^{\overline{\iota-1}}}{\Gamma(\iota)}\right|\left\|\xi_0\right\|$$

$$+\left\|\sum_{\rho=x+1}^{\alpha}\frac{(\alpha-\varrho(\rho))^{\overline{\aleph+\epsilon-1}}}{\Gamma(\aleph+\epsilon)} j(\rho,\xi(\rho))-\sum_{\rho=x+1}^{\alpha}\frac{(\alpha-\varrho(\rho))^{\overline{\aleph-1}}}{\Gamma(\aleph)} j(\rho,h(\rho))\right\|$$

$$\le \frac{(\alpha-x+1)^{\overline{\iota-1}}}{\Gamma(\iota)}\left|\frac{\Gamma(\iota)}{\Gamma(\iota+\epsilon(1-ß))}(\alpha-x+\iota)^{\overline{\epsilon(1-ß)}}-1\right|\left\|\xi_0\right\|$$

$$+\sum_{\rho=x+1}^{\alpha}\frac{(\alpha-\varrho(\rho))^{\overline{\aleph+\epsilon-1}}}{\Gamma(\aleph+\epsilon)}\left\|j(\rho,\xi(\rho))-j(\rho,h(\rho))\right\|$$

$$+\left\|\sum_{\rho=x+1}^{\alpha}\frac{(\alpha-\varrho(\rho))^{\overline{\aleph-1}}}{\Gamma(\aleph)} j(\rho,h(\rho))\left[\frac{\Gamma(\aleph)}{\Gamma(\aleph+\epsilon)}(\alpha-\rho+\aleph)^{\overline{\epsilon}}-1\right]\right\|$$

$$\le \frac{(2)^{\iota-1}}{\Gamma(\iota)}\left|\frac{\Gamma(\iota)\Gamma(\alpha-x+\iota+\epsilon(1-ß))}{\Gamma(\iota+\epsilon(1-ß))\Gamma(\alpha-x+\iota)}-1\right|\left\|\xi_0\right\|$$

$$+\sum_{\rho=x+1}^{\alpha}\frac{(\alpha-\varrho(\rho))^{\overline{\aleph+\epsilon-1}}}{\Gamma(\aleph+\epsilon)}\left\|j(\rho,\xi(\rho))-j(\rho,h(\rho))\right\|$$

$$+\sum_{\rho=x+1}^{\alpha}\frac{(\alpha-\varrho(\rho))^{\overline{\aleph-1}}}{\Gamma(\aleph)}\left\|j(\rho,h(\rho))\right\|\left|1-\frac{\Gamma(\aleph)}{\Gamma(\aleph+\epsilon)}\frac{\Gamma(\alpha-\rho+\aleph+\epsilon)}{\Gamma(\alpha-\rho+\aleph)}\right|. \tag{6}$$

Since

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left|\frac{\Gamma(\iota)\Gamma(\alpha-x+\iota+\epsilon(1-ß))}{\Gamma(\iota+\epsilon(1-ß))\Gamma(\alpha-x+\iota)}-1\right|=A_1$$

and

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left|1-\frac{\Gamma(\aleph)}{\Gamma(\aleph+\epsilon)}\frac{\Gamma(\alpha-\rho+\aleph+\epsilon)}{\Gamma(\alpha-\rho+\aleph)}\right|=A_2,$$

we have

$$\left|\frac{\Gamma(\iota)\Gamma(\alpha-x+\iota+\epsilon(1-ß))}{\Gamma(\iota+\epsilon(1-ß))\Gamma(\alpha-x+\iota)}-1\right|=O(\epsilon), \tag{7}$$

$$\left|1-\frac{\Gamma(\aleph)}{\Gamma(\aleph+\epsilon)}\frac{\Gamma(\alpha-\rho+\aleph+\epsilon)}{\Gamma(\alpha-\rho+\aleph)}\right|=O(\epsilon). \tag{8}$$

Substituting (7) and (8) in (6), for $\alpha\in\mathbb{N}_{x+1}$, we have

$$\|\xi(\alpha)-h(\alpha)\| \le O(\epsilon)\iota\|\xi_0\| + M_1\|\xi(\alpha)-h(\alpha)\| \sum_{\rho=x+1}^{\alpha} \frac{(\alpha-\varrho(\rho))^{\overline{\aleph+\epsilon-1}}}{\Gamma(\aleph+\epsilon)}(\rho-x)^{\overline{-\iota_2}} + O(\epsilon)\|j\|_\infty \sum_{\rho=x+1}^{\alpha} \frac{(\alpha-\varrho(\rho))^{\overline{\aleph-1}}}{\Gamma(\aleph)}(\rho-x)^{\overline{-\iota_2}}$$

$$= O(\epsilon)\iota\|\xi_0\| + M_1\|\xi(\alpha)-h(\alpha)\|\nabla_{x+1}^{-(\aleph+\epsilon)}(\alpha-x)^{\overline{-\iota_2}} + O(\epsilon)\|j\|_\infty \nabla_{x+1}^{-\aleph}(\alpha-x)^{\overline{-\iota_2}}$$

$$= O(\epsilon)\iota\|\xi_0\| + M_1\|\xi(\alpha)-h(\alpha)\|\frac{\Gamma(1-\iota_2)}{\Gamma(1-\iota_2+\aleph+\epsilon)}(\alpha-x)^{\overline{\aleph-\iota_2+\epsilon}} + O(\epsilon)\|j\|_\infty \frac{\Gamma(1-\iota_2)}{\Gamma(1-\iota_2+\aleph)}(\alpha-x)^{\overline{\aleph-\iota_2}}$$

$$\le O(\epsilon)\iota\|\xi_0\| + M_1\|\xi(\alpha)-h(\alpha)\|\frac{\Gamma(1-\iota_2)}{\Gamma(1-\iota_2+\aleph+\epsilon)}(1)^{\overline{\aleph-\iota_2+\epsilon}} + O(\epsilon)\|j\|_\infty \frac{\Gamma(1-\iota_2)}{\Gamma(1-\iota_2+\aleph)}(1)^{\overline{\aleph-\iota_2}}$$

$$= O(\epsilon)\iota\|\xi_0\| + M_1\|\xi(\alpha)-h(\alpha)\|\Gamma(1-\iota_2) + O(\epsilon)\|j\|_\infty \Gamma(1-\iota_2).$$

Then, we have the relation

$$\|\xi(\alpha)-h(\alpha)\| \le \left[\frac{\iota\|\xi_0\| + \|j\|_\infty \Gamma(1-\iota_2)}{1-M_1\Gamma(1-\iota_2)}\right]O(\epsilon),$$

implies that

$$\|\xi(\alpha)-h(\alpha)\| = O(\epsilon).$$

**Theorem 4.2.** Assume that $j$ satisfies conditions (H3) and (H5). Suppose $\xi$ and $h$ satisfy the initial value problems

$$\left(\nabla_x^{\aleph,\beta}\xi\right)(\alpha) = j(\alpha,\xi(\alpha)), \quad \left[\left(\nabla_x^{-(1-\iota)}\xi\right)(\alpha)\right]_{\alpha=x} = \xi(x) = \xi_0, \quad \alpha \in \mathbb{N}_{x+1}, \tag{9}$$

$$\left(\nabla_x^{\aleph,\beta}v\right)(\alpha) = j(\alpha,h(\alpha)), \quad \left[\left(\nabla_x^{-(1-\iota)}h\right)(\alpha)\right]_{\alpha=x} = h(x) = h_0, \quad \alpha \in \mathbb{N}_{x+1}, \tag{10}$$

respectively, where $0 < \aleph < 1$. Then,

$$\|\xi(\alpha)-h(\alpha)\| = O\left(\|\xi_0-h_0\|\right),$$

provided (H4) holds.

*Proof.* The initial value problems (9) and (10) are equivalent to

$$\xi(\alpha) = \xi_0 H_{\iota-1}(\alpha,\varrho(x)) + \sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))j(\rho,\xi(\rho)), \quad \alpha \in \mathbb{N}_x,$$

$$h(\alpha) = h_0 H_{\iota-1}(\alpha,\varrho(x)) + \sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha,\varrho(\rho))j(\rho,h(\rho)), \quad \alpha \in \mathbb{N}_x,$$

respectively. Using Lemma 2.4, we have for $\alpha \in \mathbb{N}_{x+1}$,

$$\|\xi(\alpha)-h(\alpha)\| \le \|\xi_0-h_0\|\frac{(\alpha-x+1)^{\overline{\iota-1}}}{\Gamma(\iota)}+\sum_{\rho=x+1}^{\alpha}\frac{(\alpha-\varrho(\rho))^{\overline{\aleph-1}}}{\Gamma(\aleph)}\|j(\rho,\xi(\rho))-j(\rho,h(\rho))\|$$

$$\le \|\xi_0-h_0\|\frac{(2)^{\overline{\iota-1}}}{\Gamma(\iota)}+M_1\|\xi(\alpha)-h(\alpha)\|\sum_{\rho=x+1}^{\alpha}\frac{(\alpha-\varrho(\rho))^{\overline{\aleph-1}}}{\Gamma(\aleph)}(\rho-x)^{\overline{-\iota_2}}$$

$$=\iota\|\xi_0-h_0\|+M_1\|\xi(\alpha)-h(\alpha)\|\nabla_{x+1}^{-\aleph}(\alpha-x)^{\overline{-\iota_2}}$$

$$=\iota\|\xi_0-h_0\|+M_1\|\xi(\alpha)-h(\alpha)\|\frac{\Gamma(1-\iota_2)}{\Gamma(1-\iota_2+\aleph)}(\alpha-x)^{\overline{\aleph-\iota_2}}$$

$$\le \iota\|\xi_0-h_0\|+M_1\|\xi(\alpha)-h(\alpha)\|\frac{\Gamma(1-\iota_2)}{\Gamma(1-\iota_2+\aleph)}(1)^{\overline{\aleph-\iota_2}}$$

$$=\iota\|\xi_0-h_0\|+M_1\|\xi(\alpha)-h(\alpha)\|\Gamma(1-\iota_2).$$

Thus, we have

$$\|\xi(\alpha)-h(\alpha)\| \le \frac{\iota\|\xi_0-h_0\|}{M_1\Gamma(1-\iota_2)},$$

implies that

$$\|\xi(\alpha)-h(\alpha)\| = O\left(\|\xi_0-h_0\|\right).$$

**Theorem 4.3.** Assume that $j$ and $j_1$ both satisfies (H3) and (H5). Suppose $\xi$ and $h$ are the solutions to the initial value problems

$$\left(\nabla_x^{\aleph,\beta}\xi\right)(\alpha)=j(\alpha,\xi(\alpha)),\quad \left[\left(\nabla_x^{-(1-\iota)}\xi\right)(\alpha)\right]_{\alpha=x}=\xi(x)=\xi_0,\quad \alpha\in\mathbb{N}_{x+1}, \tag{11}$$

$$\left(\nabla_x^{\aleph,\beta}h\right)(\alpha)=j_1(\alpha,h(\alpha)),\quad \left[\left(\nabla_x^{-(1-\iota)}h\right)(\alpha)\right]_{\alpha=x}=h(x)=\xi_0,\quad \alpha\in\mathbb{N}_{x+1}, \tag{12}$$

respectively, where $0<\aleph<1$. Then,

$$\|\xi(\alpha)-h(\alpha)\| = O\left(\|j-j_1\|_\infty\right),$$

provided (H4) holds.

*Proof*. The initial value problems (11) and (12) are equivalent to

$$\xi(\alpha)=g_0H_{\iota-1}(\alpha,\varrho(x))+\sum_{\rho=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(\rho))j(\rho,\xi(\rho)),\quad \alpha\in\mathbb{N}_x,$$

$$h(\alpha)=\xi_0H_{\iota-1}(\alpha,\varrho(x))+\sum_{\rho=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(\rho))j_1(\rho,h(\rho)),\quad \alpha\in\mathbb{N}_x,$$

respectively. Using Lemma 2.4, we have for $\alpha\in\mathbb{N}_{x+1}$,

$$\|\xi(\alpha)-h(\alpha)\| \leq \sum_{\rho=x+1}^{\alpha} \frac{(\alpha-\varrho(\rho))^{\overline{\aleph-1}}}{\Gamma(\aleph)} \|j(\rho,\xi(\rho))-j_1(\rho,h(\rho))\|$$

$$= \sum_{\rho=x+1}^{\alpha} \frac{(\alpha-\varrho(\rho))^{\overline{\aleph-1}}}{\Gamma(\aleph)} \|j(\rho,\xi(\rho))-j(\rho,h(\rho))+j(\rho,h(\rho))-j_1(\rho,h(\rho))\|$$

$$\leq \sum_{\rho=x+1}^{\alpha} \frac{(\alpha-\varrho(\rho))^{\overline{\aleph-1}}}{\Gamma(\aleph)} \|j(\rho,\xi(\rho))-j(\rho,h(\rho))\|$$

$$+ \sum_{\rho=x+1}^{\alpha} \frac{(\alpha-\varrho(\rho))^{\overline{\aleph-1}}}{\Gamma(\aleph)} \|j(\rho,h(\rho))-j_1(\rho,h(\rho))\|$$

$$\leq \left[M_1\|\xi(\alpha)-h(\alpha)\|+\|j-j_1\|_\infty\right] \sum_{\rho=x+1}^{\alpha} \frac{(\alpha-\varrho(\rho))^{\overline{\aleph-1}}}{\Gamma(\aleph)} (\rho-x)^{\overline{-\iota_2}}$$

$$= \left[M_1\|\xi(\alpha)-h(\alpha)\|+\|j-j_1\|_\infty\right] \nabla_{x+1}^{-\aleph}(\alpha-x)^{\overline{-\iota_2}}$$

$$= \left[M_1\|\xi(\alpha)-h(\alpha)\|+\|j-j_1\|_\infty\right] \frac{\Gamma(1-\iota_2)}{\Gamma(1-\iota_2+\aleph)} (\alpha-x)^{\overline{\aleph-\iota_2}}$$

$$\leq \left[M_1\|\xi(\alpha)-h(\alpha)\|+\|j-j_1\|_\infty\right] \frac{\Gamma(1-\gamma_2)}{\Gamma(1-\iota_2+\aleph)} (1)^{\overline{\aleph-\iota_2}}$$

$$= \left[M_1\|\xi(\alpha)-h(\alpha)\|+\|j-j_1\|_\infty\right] \Gamma(1-\iota_2).$$

Thus, we have

$$\|\xi(\alpha)-h(\alpha)\| \leq \frac{\Gamma(1-\iota_2)\|j-j_1\|_\infty}{\left[1-M_1\Gamma(1-\iota_2)\right]},$$

implies that

$$\|\xi(\alpha)-h(\alpha)\| = O\left(\|j-j_1\|_\infty\right).$$

## 5. Ulam-Hyers stability

In this section, we discuss the Ulam-Hyers stability of (1) and conclude the article with two examples to demonstrate the applicability of the established results. We consider the following nabla fractional difference equation:

$$\left(\nabla_x^{\aleph,\text{ß}} \xi\right)(\alpha) = j(\alpha,\xi(\alpha)), \quad \alpha \in \mathbb{N}_{x+1}^d, \tag{13}$$

where $0 < \aleph < 1, 0 \leq \text{ß} \leq 1, j: \mathbb{N}_{x+1}^d \times \mathbb{R}^n \to \mathbb{R}^n$. Let for $\epsilon > 0$ and $\psi: \mathbb{N}_x^d \to \mathbb{R}^{n+}$, we consider the following inequalities:

$$\left\|\nabla_x^{\aleph,\text{ß}} \tau(\alpha) - j(\alpha,\tau(\alpha))\right\| \leq \epsilon, \quad \alpha \in \mathbb{N}_{x+1}^d, \tag{14}$$

$$\left\|\nabla_x^{\aleph,\text{ß}} \tau(\alpha) - j(\alpha,\tau(\alpha))\right\| \leq \psi(\alpha), \quad \alpha \in \mathbb{N}_{x+1}^d, \tag{15}$$

$$\left\|\nabla_x^{\aleph,\text{ß}} \tau(\alpha) - j(\alpha,\tau(\alpha))\right\| \leq \epsilon\psi(\alpha), \quad \alpha \in \mathbb{N}_{x+1}^d. \tag{16}$$

**Definition 5.1.** The equation (13) is Ulam-Hyers stable if there exists a real number $k_j > 0$ such that for each $\epsilon > 0$, and for each solution $\tau : \mathbb{N}_x^d \to \mathbb{R}^n$ of the inequality (14), there exists a solution $\xi : \mathbb{N}_x^d \to \mathbb{R}^n$ of the equation (13) with

$$\|\tau(\alpha) - \xi(\alpha)\| \le k_j \epsilon, \quad \alpha \in \mathbb{N}_x^d.$$

**Definition 5.2.** The equation (13) is Ulam-Hyers-Rassias stable with respect to $\psi$ if there exists $k_{j,\psi} > 0$ such that for each $\epsilon > 0$, and for each solution $\tau : \mathbb{N}_x^d \to \mathbb{R}^n$ of the inequality (15), there exists a solution $\xi : \mathbb{N}_x^d \to \mathbb{R}^n$ of the equation (13) with

$$\|\tau(\alpha) - \xi(\alpha)\| \le k_{j,\psi} \epsilon \psi(\alpha), \quad \alpha \in \mathbb{N}_x^d.$$

**Definition 5.3.** The equation (13) is generalized Ulam-Hyers-Rassias stable with respect to $\psi$ if there exists $k_{j,\psi} >$ 0 such that for each solution $\tau : \mathbb{N}_x^d \to \mathbb{R}^n$ of the inequality (15) there exists a solution $u : \mathbb{N}_x^d \to \mathbb{R}^n$ of the equation (13) with

$$\|\tau(\alpha) - \xi(\alpha)\| \le k_{j,\psi} \psi(\alpha), \quad \alpha \in \mathbb{N}_x^d.$$

*Remark* 5.4. It can be observed that Definition 5.2 ⇒ Definition 5.1 and Definition 5.2 ⇒ Definition 5.3.

We now state here a discrete analogue of Gronwall's inequality.

**Definition 5.5.** (See [13]). The nabla Mittag-Leffler function for $|f| < 1, \aleph > 0$, and $ß \in \mathbb{R}$, is defined by

$$E_{f,\aleph,ß}(\alpha, x) = \sum_{k=0}^{\infty} f^k H_{\aleph k + ß}(\alpha, x), \quad \alpha \in \mathbb{N}_x.$$

**Theorem 5.6.** (See [2]). (Generalized Gronwall Inequality) Let $\aleph > 0$, $j$ be a non-negative function and $q$, $s$ be non-negative and non-decreasing functions defined on $\mathbb{N}_0$ such that $s(\alpha) \le M$ for all $\alpha \in \mathbb{N}_0$, where $M$ is a constant. If

$$j(\alpha) \le q(\alpha) + s(\alpha)\Gamma(\aleph)\left(\nabla_0^{-\aleph} j\right)(\alpha), \quad \alpha \in \mathbb{N}_0,$$

then

$$j(\alpha) \le q(\alpha) E_{s(\alpha)\Gamma(\aleph),\aleph,0}(\alpha, 0), \quad \alpha \in \mathbb{N}_0.$$

*Remark* 5.7. The function $\tau : \mathbb{N}_x \to \mathbb{R}^n$ is a solution of the inequality (14) if and only if there exists a function $j_1 : \mathbb{N}_x \to \mathbb{R}^n$ such that

(i) $\|\xi(\alpha)\| \le \epsilon, \alpha \in \mathbb{N}_{x+1}^d$,

(ii) $\left(\nabla_x^{\aleph,ß} \tau\right)(\alpha) = j(\alpha, \tau(\alpha)) + j_1(\alpha), \quad \alpha \in \mathbb{N}_{x+1}^d$.

*Remark* 5.8. If $\tau : \mathbb{N}_x^d \to \mathbb{R}^n$ is a solution of (13) then $\tau$ is a solution of the following inequality

$$\left\| \tau(\alpha) - \tau(x) H_{\iota-1}(\alpha, \varrho(x)) - \sum_{\rho=x+1}^{\alpha} H_{\aleph-1}(\alpha, \varrho(\rho)) j(\rho, \tau(\rho)) \right\| \le \epsilon H_{\aleph}(d, x+1).$$

By Remark 5.7, we have

$$\left(\nabla_x^{\aleph,ß} \tau\right)(\alpha) = j(\alpha, \tau(\alpha)) + j_1(\alpha), \quad \alpha \in \mathbb{N}_{x+1}^d. \tag{17}$$

Then, the solution of (17) is given by

$$\tau(\alpha)=\tau(x)H_{\iota-1}(\alpha,\varrho(x))+\sum_{\rho=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(\rho))\left[j(\rho,\tau(\rho))+j_1(\rho)\right],$$

then we can write

$$\begin{aligned}\left\|\tau(\alpha)-\tau(x)H_{\iota-1}(\alpha,\varrho(x))-\sum_{\rho=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(\rho))j(\rho,\tau(\rho))\right\|&=\left\|\sum_{\rho=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(\rho))j_1(\rho)\right\|\\&\leq\epsilon\sum_{\rho=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(\rho))\\&=\epsilon H_{\aleph}(\alpha,x+1)\\&\leq\epsilon H_{\aleph}(d,x+1).\end{aligned}$$

We assume

(L1) $j$ is continuous function with respect to second variable;

(L2) There exists $x_j > 0$ such that

$$\|j(\alpha,\xi)-j(\alpha,h)\|\leq x_j\|\xi-h\| \text{ for each } \alpha\in\mathbb{N}_x^d \text{ and } \xi,h\in\mathbb{R}^n.$$

**Theorem 5.9.** (13) with $\xi(x) = \xi(0)$ is Ulam-Hyers stable if the conditions (L1) and (L2) hold.

*Proof.* Let $\tau$ satisfy the inequality (16). From Theorem 3.3, the unique solution $\xi$ of (13) with initial condition $\left[\left(\nabla_x^{-(1-\iota)}\xi\right)(\alpha)\right]_{\alpha=x}=\xi(x)=\xi_0$ is given by

$$\xi(\alpha)=\xi_0H_{\iota-1}(\alpha,\varrho(x))+\sum_{\rho=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(\rho))j(\rho,\xi(\rho)),\quad \alpha\in\mathbb{N}_x^d.$$

It follows with the help of Remark 5.8 that

$$\begin{aligned}\|\tau(\alpha)-\xi(\alpha)\|&\leq\left\|\tau(\alpha)-\xi_0H_{\iota-1}(\alpha,\varrho(x))-\sum_{\rho=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(\rho))j(\rho,\xi(\rho))\right\|\\&\leq\Bigg\|\tau(\alpha)-\xi_0H_{\iota-1}(\alpha,\varrho(x))-\sum_{\rho=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(\rho))j(\rho,\tau(\rho))\\&\quad+\sum_{\rho=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(\rho))j(\rho,\tau(\rho))-\sum_{\rho=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(\rho))j(\rho,\xi(\rho))\Bigg\|\\&\leq\left\|\tau(\alpha)-\xi_0H_{\iota-1}(\alpha,\varrho(x))-\sum_{\rho=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(\rho))j(\rho,\tau(\rho))\right\|+\sum_{\rho=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(\rho))\|j(\rho,\tau(\rho))-j(\rho,\xi(\rho))\|\\&\leq\epsilon H_{\aleph}(d,x+1)+x_j\sum_{\rho=x+1}^{\alpha}H_{\aleph-1}(\alpha,\varrho(\rho))\,\|\,\tau(\rho)-\xi(\rho)\,\|.\end{aligned}$$

Then, from Theorem 5.6, we have

$$\|\tau(\alpha)-\xi(\alpha)\|\leq\epsilon H_{\aleph}(d,x+1)E_{x_j,\aleph,0}(\alpha,0)\leq\epsilon H_{\aleph}(d,x+1)E_{x_j,\aleph,0}(d,0)=\epsilon k_{j,\psi},\text{ for }\alpha\in\mathbb{N}_x^d.$$

Thus, (13) is Ulam-Hyers stable.

### 5.1 *Example*

We construct here two examples to demonstrate the applicability of the established results in the preceding sections.

**Example 5.10.** Consider the initial value problem

$$\begin{cases} \left(\nabla_0^{0.5,ß} u\right)(\alpha) = \alpha + (0.1)\sin\xi(\alpha), \quad \alpha \in \mathbb{N}_1, \\ \xi(0) = 1. \end{cases} \tag{18}$$

Take $\Lambda = 0.4$. Then, we see that $j$ satisfies (H1) with $Q = 0.1$. Thus, we have $Q < \Lambda$ and

$$\sup_{\alpha\in\mathbb{N}_x} \frac{|j(\alpha,0)|}{w(\alpha)} = \sup_{\alpha\in\mathbb{N}_0} \frac{\alpha}{F_{0.5}(0.4.(\alpha-0))} < \infty.$$

Thus, by Theorem 3.8, (18) has a unique bounded solution defined on $\mathbb{N}_0$.

**Example 5.11.** Consider the initial value problem

$$\begin{cases} \left(\nabla_0^{0.5,ß} \xi\right)(\alpha) = \alpha - 1 + (0.05)\left[g^2(\alpha)\right], \quad \alpha \in \mathbb{N}_1, \\ \xi(0) = 1. \end{cases} \tag{19}$$

Take $\Lambda = 0.4$ and $D = 1$. Then, we see that $j$ satisfies (H1)' with $Q = 0.1$. Thus, we have $Q < \Lambda$ and

$$\sup_{\alpha\in\mathbb{N}_x} \frac{|j(\alpha,0)|}{w(\alpha)} = \sup_{\alpha\in\mathbb{N}_0} \frac{\alpha-1}{F_{0.5}(0.4.(\alpha-0))} < 1 < \frac{D}{\Lambda - Q} = 3.333.$$

Thus, in Theorem 3.10, (19) has the unique bounded solution $B_q^w(0)$, where

$$q = \frac{D}{(\Lambda - Q)^2} = 11.111.$$

## 6. Conclusion

In this work, we, under suitable assumptions on the non-linear function, have established sufficient conditions for the existence of a unique bounded solution. For this purpose, we have used the Banach fixed point theorem on a well-defined space for a non-linear initial value problem involving the Hilfer nabla fractional difference operator of order $0 < \aleph < 1$. We have also analyzed the Ulam-Hyers stability of the considered problem and made some interesting observations on the dependence of its solutions on initial conditions and parameters. Finally, we have concluded this article by constructing suitable problems to illustrate the application of established results. The development of the Hilfer nabla fractional operator and its properties are scarce in the literature. Conditions on existence, uniqueness, and stability analysis such as Ulam-Hyers stability for the considered initial value problem involving the Hilfer nabla fractional operator would play a vital role in the analysis of possible models involving the nabla fractional Riemann-Liouville and Caputo operators.

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## Conflict of interest

There is no conflict of interest in this study.

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