Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

Information Asymmetric Cooperative Game with Agreements Implemented by a Third Party

Jeanpantz Chen

Department of Economics and Finance, South China University of Technology, Guangzhou, China
E-mail: jeanpantz@126.com

Received: 24 February 2023; **Revised:** 22 May 2023; **Accepted:** 6 July 2023

Abstract: This paper introduces asymmetric information into the analysis of cooperative games with agreements implemented by a third party and establishes theoretical models of a one-time information asymmetric cooperative game with agreements implemented by a third party for the first time, using the methodology proposed in previously done research studying coalition formation and cooperative payoff distribution. In an information asymmetric cooperative game with agreements implemented by a third party, players may retain their own private information, and make decisions through their virtual games on the basis of their own information sets. This paper defines the virtual cooperative games of the players and demonstrates the equilibrium of the virtual cooperative game of a player; proposes the condition for the existence of the coalition equilibrium in an information asymmetric cooperative game with agreements implemented by a third party, defines and provides the existence proof of this coalition equilibrium when it does exist; defines the public choice game of a coalition on the strategic combination choice in a certain coalition situation, defines and provides the existence proof of the equilibrium of this game; examines the condition for the existence of the bargaining game on the distribution of the cooperative payoff of a coalition, defines and provides existence proof of the bargaining game, when the coalition members are allied or unallied in the bargaining game.

***Keywords*:** information asymmetry, cooperative game with agreements implemented by a third party, virtual game, coalition equilibrium, bargaining game

MSC: 91A12, 91A27

1. Introduction

An important hypothesis in early game theory is that, there is no asymmetric information among the players and that all the players have complete information in common. Afterward, one important kind of model is used to examine the games of incomplete information, in which the incompleteness of information means that, some players don't know the "types" of others Harsanyi [1-3]. When the types of other players are unknown, the decision of a player depends on his belief, that is, his pre-estimation of the types of those players.

However, even a person who knows game theory just a little will feel doubtful about the complete information hypothesis. The game players may lack understanding of their opponent's strategy sets, payoff functions, and the completeness of their information sets, not just their "types". It's unsurprising that Kadane et al. [4] asked whether game

DOI: https://doi.org/10.37256/cm.5220242557

models make sense according to the symmetric information hypothesis. This hypothesis neglects the brass tacks that things the players believe rely on the situations they know.

This paper attempts to introduce asymmetric information into the analysis of cooperative games, assuming that: (1) any player may retain private information, therefore, information in the cooperative game is asymmetric; (2) any player decides through his virtual game based on his information set; (3) any player is rational and always pursues the maximization of his expected cooperative payoff distribution in the game; (4) the agreements are implemented by a third party cost-free; in the game, no player will take opportunism behaviour.

This paper is based on Chen's theoretic models of symmetric information cooperative games with agreements implemented by a third party [5]. The basic methodology proposed by Chen in [5] will be applied to our analysis of an asymmetric information cooperative game with agreements implemented by a third party: (1) the formation of the coalition equilibrium is the result of the choices of the players who pursue the maximization of their welfares, and (2) the cooperative payoff of a coalition can always be decomposed into the common payoffs of different member sets, and the common payoff of a member set will be evenly distributed to each member. Therefore, the equilibrium of the bargaining game of a coalition on the cooperative payoff distribution can be obtained by applying the distribution rule of common payoffs.

Based on [5], this paper tries to examine an information asymmetric cooperative game with agreements implemented by a third party. The main components include:

(1) The virtual games of the players.

(2) The coalition formation in the game and the condition for the existence of the coalition equilibrium.

(3) The public choice game on the strategic combination choice of a coalition.

(4) The Nash equilibrium of the bargaining game on the distribution of the cooperative payoff of a coalition in the coalition equilibrium (if it does exist) when the coalition members are allied or unallied in the game.

This paper establishes the theoretical models of information asymmetric cooperative games with agreements implemented by a third party for the first time using the methodology proposed by Chen [5] for studying coalition formation and cooperative payoff distribution. In the theoretical literature on cooperative games, asymmetric information is often introduced into theoretical models only in repeated games rather than one-time games, and the theoretical models of one-time information asymmetry in cooperative games have not been established. This paper defines the virtual cooperative games of the players and demonstrates the equilibrium of the virtual cooperative game of a player, proposes the condition for the existence of the coalition equilibrium in an information asymmetric cooperative game with agreements implemented by a third party, defines and provides the existence proof of this coalition equilibrium when it does exist, defines the public choice game of a coalition on the strategic combination choice in a certain coalition situation, defines and provides the existence proof of the equilibrium of this game; examines the condition for the existence of the bargaining game on the distribution of the cooperative payoff of a coalition, defines and provides existence proof of the bargaining game, when the coalition members are allied or unallied in the game.

2. Literature review

Researchers have been committed to introducing asymmetric information into non-cooperative game theory since the 1960s. Aumann et al. [6] studied a Bayesian persuasion problem in a repeated non-cooperative game. Mertens et al. [7] proposed the model of repeated non-cooperative games of asymmetric information, provided a rich framework to model situations in which one player lacks complete knowledge about the "state of nature". Novika et al. [8] substantiated the feasibility and reasonability of employing the framework of reflexive games for describing decision-making and provided reflexive non-cooperative game models with players making decisions on the hierarchy of knowledge about essential parameters and knowledge about knowledge.

On the other hand, researchers have also attempted to introduce asymmetric information into cooperative game theory models. Based on the equilibrium definition introduced by Konishi et al. [9], the extended definition by Hyndman et al. [10], the solution concept used in Gomes et al. [11], Gomes [12], and Aumann [13] examination and definition of the equilibrium process of coalition formation (EPCF), it is believed that in a repeated cooperative game, coalition formation is a process with the property that at every stage, every active coalition, will be given a set of potential partners, to make a profitable and maximal move. However, in a cooperative game of information symmetry, if the players are completely rational, the coalition equilibrium must exist, and it does not need to be formed in a process while the game is infinitely repeated.

In addition to examining the formation process of coalitions, a lot of literature investigate the notions of solutions to a cooperative game with asymmetric information. Noguchi [14] studied the conditions of a game with asymmetric information on a continuum of states that will admits a non-empty core α-core; Kamishiro [15] investigated the core with asymmetric information and the Shapley value with externalities and investigated to what extent the core convergence results hold for core notions with asymmetric information. Myerson [16] proposed a bargaining solution concept that generalizes the Nash bargaining solution and the Shapley non-transferable utility (NTU) value, which is defined for cooperative games with incomplete information, and showed that these bargaining solutions are efficient and equitable when interpersonal comparisons are made in terms of certain virtual utility scales. Masuya [17] and Masuya et al. [18] studied cooperative transferable utility (TU) games in which the worth of some coalitions are unknown and investigated the super additive games and the Shapley values on a class of cooperative games under incomplete information. Masuya et al. [19] took the first step toward the theory of cooperative games under incomplete information on coalitional values. They defined the concepts related to such incomplete games and investigated the solution concepts in a special case when only the values of the grand coalition and singleton coalitions are known and showed that there exists a focal point solution which is commonly suggested from many points of view. Lin et al. [20] introduced a concerned definition of incomplete cooperative games and presented an effective model to examine the super additivity of the games. Based on the criterion for minimizing the deviation of excess and expected excess value, they proposed the L-nucleolus, constructed the deviation of imputation and ideal vectors, and discussed the existence and rationality of the L-nucleolus and the I-Shapley value. Petrosjan [21] proposed the definition of a cooperative game in characteristic function form with incomplete information on a game tree, the notions of optimality principle, and introduced the solution concept based on it. He also defined the “regularized” core, proved the strong time-consistency, and investigated the special case of stochastic games in detail.

However, the literature on asymmetric information cooperative games mentioned above has not found the correct direction in the study of coalition formation and the distribution schemes of coalition cooperative payoffs (which is, the solution to a cooperative game). The most important reason is that, as its basis, symmetric information cooperative game theory has not provided satisfactory answers in both aspects. Moreover, the discussions on asymmetric information cooperative games are based on repeated games rather than one-time games.

The problem in the literature of information symmetric cooperative game theory lies not only in the failure to define the coalition equilibrium in a cooperative game but also in the ever-proposed inappropriate distribution schemes of the cooperative payoff that a coalition gets in the game.

In symmetric information cooperative game theory, researchers often assume that there is only one coalition in a one-time cooperative game. If there is no dummy in the game, all the players will join the only coalition; and if there are dummies in the game, all players except the dummies will join this coalition. Such assumption can only accord with a special situation as in a general cooperative game, the situation may be different from the assumption above. In many cases, there may be not only one coalition but a series of cooperative coalitions. If there are “dummies” for a coalition in a cooperative game, there may be synergies among the “dummies”. Therefore, they may form one or more coalitions to benefit from cooperation.

After eliminating some obvious unreasonable feasible allocations, some researchers regard the set of the remaining feasible allocations as the “solution” to the cooperative game. This kind of literature includes notions such as the stable set, the core, and the bargaining set (Aumann et al. [22]; Gillies [23]; Shapley [24]; Von Neumann et al. [25]), and also the extended idea of Harsanyi [26] introducing the notions of farsighted behavior, and the following of Aumann et al. [27], Chwe [28], Ray et al. [29], Diamantoudi et al. [30], and others. These are the only sets of feasible allocations to which the equilibrium distribution scheme belongs, but not the equilibrium distribution scheme itself.

Some researchers tried to find the “real” solution to the game on the cooperative payoff distribution among the coalition members, that is, the single-point solution, such as the Shapley value in [31], and the nucleolus in [32]. In their models, it is assumed that the coalition members have a common goal or follow some collectivistic behavioural logic. However, the distribution game for the cooperative payoff of a coalition is a typical non-cooperative game. Individually rational coalition members would not sacrifice their interests for the common goal of the coalition, nor would they follow some collectivist behavior mode. Therefore, Nash [33] believed that the distribution process of the cooperative payoffs in a coalition is the bargaining process among the coalition members. After establishing some axioms, Nash proved that there exists only one bargaining process that can satisfy the axioms that should be satisfied; this only bargaining process is called the Nash negotiation solution. Unfortunately, because of Nash's assumptions are not all axiomatic, the Nash negotiation solution is not reasonable.

Chen [5] examined the coalition formation in a cooperative game with agreements implemented by a third party, providing the existence proof and the algorithm of the coalition equilibrium, which is formed when each player tries to maximize their cooperative payoff distribution plays the coalition-choosing strategy that is the best response to the collective actions of other players. Chen inherited Nash's idea that the distribution of the cooperative payoff of a coalition is the bargaining process among the coalition members. He defined the common payoff of a member set of the coalition and showed that the equilibrium of the bargaining game on the distribution of the common payoff is that each member in the set will get the same distribution share. Chen proved that the cooperative payoff of a coalition can be decomposed into the sum of the common payoffs of all its member sets, and the cooperative payoff distribution that each member gets is equal to the sum of the distribution shares of the common payoffs he can get from the different member sets to which he belongs.

In recent years, asymmetric information coordination has been studied in networked system theory. Shang [34] studied a simple three-body consensus model that favourably incorporates higher-order network interactions, higher-order dimensional states, the group reinforcement effect, and the homophily principle. He proposed a system model of three-body interactions in complex networks. Shang [35] introduced a novel multiplex network presentation for directed graphs and their associated connectivity concepts, including pseudo-strong connectivity and graph robustness, which provide a resilience characterization in the presence of malicious nodes. Qi et al. [36] investigated the linear quadratic (LQ) control problem for a stochastic system with different intermittent observations. However, the goal of such literature on networked system theory is not to establish a general information asymmetry cooperative game theory.

3. Unallied bargaining games and information asymmetric cooperative games with agreements implemented by a third party

In this section, assuming that members of each coalition are unallied in the bargaining game on the distribution of the cooperative payoff of the coalition, we will analyze the information asymmetric cooperative game with agreements implemented by a third party. At the same time, it is assumed that in each coalition, the supervision is perfect. An information asymmetric cooperative game with agreements implemented by a third party when coalition members are under imperfect supervision is similar to an information asymmetric cooperative game with self-implemented agreements since the third party does not have enough information to implement the agreements and punish opportunistic behaviors.

When coalition members are under imperfect supervision in an information asymmetric cooperative game with agreements implemented by a third party, the following three questions are the most important and must be addressed:

(1) The coalition formation in the information asymmetric cooperative game with agreements implemented by a third party. Does there exist a (mixed strategic) coalition equilibrium in the information asymmetric cooperative game with agreements implemented by a third party? If the coalition equilibrium does exist, what is the coalition equilibrium?

(2) The public choices of the coalitions' strategic combinations. Under information asymmetry, how does a cooperative coalition make its public choice of strategic combination?

(3) The cooperative payoff distribution scheme. After the completion of the information asymmetric cooperative game with agreements implemented by a third party, what kind of cooperative payoff distribution scheme will a coalition adopt?

3.1 *Information asymmetry and the information sets of players*

In an information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$, (where N denotes the player set, $N = \{1, 2, ..., n\}$; S_i denotes the strategy set of any player i, $S_i = \{s_{i1}, s_{i2}, ..., s_{im_i}\}$, $i = 1, 2, ..., n$; and u_i denotes the payoff function of any player i, when each player i selects a strategy $s_i \in S_i$ ($i = 1, 2, \cdots, n$), strategic situation $s = (s_1, s_2, \cdots, s_n) \in \Pi_{i\in N} S_i$ will come into being. For some situation s, the payoff that player i gets is $u_i(s) = u_i(s_1, s_2, \cdots, s_n)$, $i = 1, 2, \cdots, n$ every player may retain his private information. The information set of player i about the strategy set, the payoff function and so on of another player j does not only depend on the signal set $I_{j\to i}^{(j,\, j)}$ released by player j to player i, but also depends on player i's information set about player j that player i already has, and the signal set $I_{k\to i}^{(k,\, j)}$ about player j's strategy set, payoff function, and so on, released by another player $k\,(k \neq i, j)$ to player i. In the information asymmetric cooperative game, information sets $I_{j\to i}^{(j,\, j)}$, $I_{i\to i}^{(i,\, j)}$ and $I_{k\to i}^{(k,\, j)}\left(k \neq i, j\right)$ ultimately form the information set $I_i^{(i,\, j)} = I_{1\to i}^{(1,\, j)} \cup \cdots \cup I_{k\to i}^{(k,\, j)} \cup \cdots \cup I_{n\to i}^{(n,\, j)}$ of player i about player j. This information set forms the basis for player i's estimation of player j's strategy set, payoff function and his virtual game.

After obtaining the information about other players, estimating other players' strategy sets, payoff functions and virtual games, using his own strategy set and payoff function, player i can make up his virtual cooperative game $\Gamma^{(i)}\left(N, \left\{S_j^{(i)}\right\}, \left\{u_j^{(i)}\right\}\right)$, where N denotes the player set, $N = \{1, 2, ..., n\}$; $S_j^{(i)}$ denotes player i's estimation of the strategy set of any player j, $S_j^{(i)} = \left\{s_{j1}^{(i)}, s_{j2}^{(i)}, \cdots, s_{jm_j}^{(i)}\right\}$, $j = 1, 2, \cdots, n$; and $u_j^{(i)}$ denotes player i's estimation of the payoff function of any player j, $j = 1, 2, ..., n$.

In fact in the process of information, that is, until the cooperation agreements of the coalitions are signed, there are always exchanges of information among the players: persuading and being persuaded, tempting and being tempted, deceiving and being deceived. The signal set released by any player i, and also his information set about other players have been in a dynamic adjustment process. In the discussion in this paper, we do not consider the adjustment process of the information sets of the players (this is actually the process in which the players develop and implement their signal strategies); the information set of any player we refer to is the final information set of this player at the end of the cooperation negotiation.

3.2 *Public choice of strategic combination of a coalition in coalition situation c*

Our discussion begins with the public choice of strategic combination of a coalition in some coalition situation c. In information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ we assume that the players are independent in choosing coalitions before they are formed. They are free to choose their coalition-choosing strategies. Through free assemble, the players in cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ can form n coalitions (if a coalition is empty, we also define it as a coalition with no member). The players can form different coalition situations through their different coalition-choosing strategies.

A coalition situation $c = (c_1, c_2, ..., c_n)$, in cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ is the situation when each player i chooses to join coalition $C_{c_i}\,(1 \leq c_i \leq n)$.

Apparently, according to the definition of a coalition situation above, when the n players in the game form n 1-member coalitions, a maximum of n non-empty coalitions can be formed in the game. Denote these n coalitions as $C_1, C_2, ..., C_n$, respectively. Through free combination, the players can form different coalition situations, and in each coalition situation there are n coalitions (including empty coalitions). But some coalition situations may be duplicated because the coalitions in these situations are just different in order. So, we set special arrangement rules that can guarantee that the elements in the coalition situation are all unique.

Rules on coalition-choosing. In cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$, assume that that all the players follow the rules below when they choose their coalition-choosing strategies:

$$c_1 = 1;$$

$$c_2 = 1,\ 2;$$

$$c_3 = 1,\ 2,\ 3\ (\text{if } c_2 \neq 2,\ c_3 \neq 2);$$

……

$$c_i = 1,\ 2,\ 3,\ \cdots,\ i\ \ (\text{if } c_j \neq j,\ c_i \neq j,\ j < i);$$

……

$$c_n = 1,\ 2,\ 3,\ \cdots,\ n\ \ (\text{if } c_j \neq j,\ c_n \neq j,\ j < n);$$

we get unrepeated coalition situations in cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$.

However, the rules above don't mean that the players are deprived of their choice options.

Given some coalition situation $c = (c_1, c_2, ..., c_n)$, how does coalition c_m $(m = 1, 2, \cdots, n)$ choose its strategic combination?

Theorem 1 In coalition situation $c = (c_1, c_2, ..., c_n)$ of information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$, with agreements implemented by a third party, in the virtual game $\Gamma^{(i)}\left(N, \left\{S_j^{(i)}\right\}, \left\{u_j^{(i)}\right\}\right)$ of some member i of coalition c_m $(m = 1, 2, \cdots, n)$, the strategic combination of the coalition which maximizes his expected cooperative payoff distribution is just the one which maximizes the expected cooperative payoff of the coalition.

Proof. Give some coalition situation $c = (c_1, c_2, ..., c_n)$, in the virtual game $\Gamma^{(i)}\left(N, \left\{S_j^{(i)}\right\}, \left\{u_j^{(i)}\right\}\right)$ of some member i, of coalition c_m $(m = 1, 2, \cdots, n)$. Assume that the strategic combination of the coalition which maximizes his expected cooperative payoff distribution is $s_{C_m}^{(i)} = \left(s_1^{(i)}, s_2^{(i)}, \cdots, s_i^{(i)}, \cdots, s_K^{(i)}\right)$, where K is the number of members of the coalition; and that the Nash equilibrium of the non-cooperative game among coalitions is $\left(s_{C_m}^{*(i)}, s_{-C_m}^{*(i)}\right)$, where $s_{C_m}^{*(i)} = \left(s_1^{*(i)}, s_2^{*(i)}, \ldots, s_i^{*(i)}, \ldots, s_K^{*(i)}\right)$, and $s_{-C_m}^{*(i)}$ is player i's estimation of the strategic combinations of other coalitions.

Since $s_{C_m}^{*(i)} = \left(s_1^{*(i)}, s_2^{*(i)}, \ldots, s_i^{*(i)}, \ldots, s_K^{*(i)}\right)$ is the equilibrium strategic combination of coalition C_m that player i belongs to when the equilibrium strategic combinations of other coalitions are $s_{-C_m}^{*(i)}$ in the virtual game of player i, we have:

$$V_{C_m}^{(i)}(s_{C_m}^{*(i)}, s_{-C_m}^{*(i)}) \geq V_{C_m}^{(i)}(s_{C_m}^{(i)}, s_{-C_m}^{*(i)}),$$

where $V_{C_m}^{(i)}\left(s_{C_m}^{*(i)}, s_{-C_m}^{*(i)}\right)$ and $V_{C_m}^{(i)}\left(s_{C_m}^{(i)}, s_{-C_m}^{*(i)}\right)$ are the cooperative payoff (the sum of the payoffs the members get in the game) of coalition C_m in strategic situations $\left(s_{C_m}^{*(i)}, s_{-C_m}^{*(i)}\right)$ and $\left(s_{C_m}^{(i)}, s_{-C_m}^{*(i)}\right)$, respectively.

If $V_{C_m}^{(i)}\left(s_{C_m}^{*(i)}, s_{-C_m}^{*(i)}\right) > V_{C_m}^{(i)}\left(s_{C_m}^{(i)}, s_{-C_m}^{*(i)}\right)$, in the virtual game of player i, compared with strategic combination $s_{C_m}^{(i)}$, when equilibrium strategic combination $s_{C_m}^{*(i)}$, is adopted by coalition C_m, the distribution scheme of coalition C_m will inevitably lead to an improvement of the cooperative payoff distribution of player i while the cooperative payoff distributions of other coalition members at least remain unchanged. Comparing to the strategic combination $s_{C_m}^{(i)}$, when equilibrium strategic combination $s_{C_m}^{*(i)}$ is adopted, the cooperative payoff distributions of all the coalition members can get a Pareto improvement. Therefore, for player i, the equilibrium strategic combination $s_{C_m}^{*(i)}$ in his virtual game is the optimal choice, which maximizes his expected cooperative payoff distribution.

Theorem 1 shows that in the virtual game of any player i, the optimal strategic combination choice of the coalition he belongs to under the criterion of maximum expected cooperative payoff distribution of his own (individual rationality) and the optimal strategic combination choice under the criterion of maximum expected cooperative payoff of the coalition (collective rationality) are just the same. However, even if player i follows the criterion of maximum expected cooperative payoff of the coalition and able to find the optimal strategic combination choice of the coalition in his virtual game, this does not mean that the optimal strategic combination choices of the coalition that all the coalition members find in their respective virtual games are the same. Because the information sets of the coalition members are different, the equilibrium strategic combination choices under the criterion of maximum expected cooperative payoff of the coalition, which they find, are necessarily different.

If the optimal strategic combination choices that the coalition C_m, "should" adopt in the virtual games of the coalition members are different, that is, the members' estimations of the optimal strategic combination of the coalition are different, what strategic combination should coalition C_m adopt? Obviously, the coalition cannot only adopt the "optimal" strategic combination in some members' virtual games; at the same time, no mixed strategic combination can maximize the expected utilities of all members.

What is the public choice of the strategic combination of some coalition C_m, if the optimal mixed strategic combinations of the coalition that members get in their virtual games are different? There is no doubt that if a coalition has its own private owner, the right to choose the strategic combination of the coalition must belong to its owner. However, if the private ownership of a coalition is not introduced, or if the ownership of a coalition is public and its decision-making rule is unanimous, in the public choice game of the coalition's strategic combination, all the coalition members must agree on the same public choice.

Denote the strategic combination set of coalition C_m as S_{C_m}. Assuming that in the virtual game of any member i, the Nash equilibrium of the non-cooperative game among coalitions is $\left(s_{C_m}^{*(i)}, s_{-C_m}^{*(i)}\right)$, at this point, when coalition C_m adopts some strategic combination S_{C_m}, the expected cooperative payoff of coalition C_m is $V_{C_m}^{(i)}(s_{C_m}) = V_{C_m}^{(i)}\left(s_{C_m}, s_{-C_m}^{*(i)}\right)$. Therefore, we can get a public choice game in which the player set is comprised of all the members of coalition C_m, and the strategy set of each member is S_{C_m}. Before we analyze the equilibrium public choice $s_{C_m}^*$ in the public choice game on the strategic combination choice of coalition C_m, first we examine any coalition member i's, estimation of the cooperative payoff distribution scheme of coalition C_m, in the equilibrium public choice $s_{C_m}^*$ in his virtual game.

In the public choice game on the strategic combination choice of coalition C_m, each coalition member has the same strategy set, the coalition's strategic combination set S_{C_m}. Since the target of the expected cooperative payoff distribution of each coalition member is consistent with the goal of the expected cooperative payoff of the coalition, in the public choice game on the strategic combination choice of coalition C_m, the goal of any member i is to maximize the expected cooperative payoff of coalition C_m, $V_{C_m}^{(i)}(s_{C_m}) = V_{C_m}^{(i)}\left(s_{C_m}, s_{-C_m}^{*(i)}\right)$, that is to say, any member i regards the cooperative payoff function as his own payoff function in the public choice game on the strategic combination choice of coalition C_m. Since in the virtual game of any coalition member, when the expected cooperative payoff of the coalition he belongs to is maximized, his cooperative payoff distribution is maximized too. Therefore, in the public choice game Γ_{C_m} under unanimity rule on the strategic combination choice of coalition C_m, the payoff function of each player, that is, his expected cooperative payoff distribution, can be replaced by the expected cooperative payoff of the coalition.

Theorem 2 In coalition situation c of an information asymmetric cooperative game with agreements implemented by a third party, assume that the equilibrium public choice in public choice game Γ_{C_m} under unanimity rule on the strategic combination choice of coalition C_m is $s_{C_m}^*$, and in the virtual game of any member i, in the Nash equilibrium $\left(s_{C_m}^{*(i)}, s_{-C_m}^{*(i)}\right)$ of the non-cooperative game among coalitions, member i's estimation of the expected cooperative payoff of coalition C_m is $V_{C_m}^{(i)}\left(s_{C_m}^{*(i)}, s_{-C_m}^{*(i)}\right)$, then member i's estimation of his expected cooperative payoff distribution in public choice game Γ_{C_m} is:

$$x_i = \frac{1}{K}\sum_{j=1}^{K}\left[V_{C_m}^{(j)}\left(s_{C_m}^*, s_{-C_m}^{*(j)}\right) - V_{C_m}^{(j)}\left(s_{C_m}^{\circ(j)}, s_{-C_m}^{\circ(j)}\right)\right] + V_{C_m}^{(i)}\left(s_{C_m}^{\circ(i)}, s_{-C_m}^{\circ(i)}\right), i = 1, 2, \cdots, K,$$

where K is the number of the members of coalition C_m, $\left(s_{C_m}^{\circ(j)}, s_{-C_m}^{\circ(j)}\right)$ is the situation when the coalition members fail to reach any agreement on the public choice of the strategic combination of coalition C_m and the coalition dissolves, $V_{C_m}^{(j)}\left(s_{C_m}^{\circ(j)}, s_{-C_m}^{\circ(j)}\right)$ is the "cooperative" payoff of coalition C_m in situation $\left(s_{C_m}^{\circ(j)}, s_{-C_m}^{\circ(j)}\right)$ in the virtual game of player j.

Proof. In public choice game Γ_{C_m} under unanimity rule on the strategic combination choice of coalition C_m, all the players (that is, all the members of coalition C_m) form the only cooperative coalition. All the coalition members must reach an agreement in order to obtain the cooperative payoff surplus $\sum_{j=1}^{K}\left[V_{C_m}^{(j)}\left(s_{C_m}^*, s_{-C_m}^{*(j)}\right) - V_{C_m}^{(j)}\left(s_{C_m}^{\circ(j)}, s_{-C_m}^{\circ(j)}\right)\right]$. If any member withdraws from the coalition, the entire cooperative payoff surplus of the coalition disappears. Therefore, we define the coalition's cooperative payoff surplus $\sum_{j=1}^{K}\left[V_{C_m}^{(j)}\left(s_{C_m}^*, s_{-C_m}^{*(j)}\right) - V_{C_m}^{(j)}\left(s_{C_m}^{\circ(j)}, s_{-C_m}^{\circ(j)}\right)\right]$ as the common payoff of all coalition members, according to the distribution rule of common payoff Chen [5], the expected cooperative payoff distribution of each member is:

$$y_i = \frac{1}{K}\sum_{j=1}^{K}\left[V_{C_m}^{(j)}\left(s_{C_m}^*, s_{-C_m}^{*(j)}\right) - V_{C_m}^{(j)}\left(s_{C_m}^{\circ(j)}, s_{-C_m}^{\circ(j)}\right)\right], i = 1,\ 2,\ \ldots,\ K.$$

When member i, withdraws from the coalition, the escape payoff (reservation distribution) he gets, is:

$$w_i = V_{C_m}^{(i)}\left(s_{C_m}^{\circ(i)}, s_{-C_m}^{\circ(i)}\right), i = 1, 2, \ldots, K.$$

Therefore, the total expected cooperative payoff distribution of member i is:

$$x_i = \frac{1}{K}\sum_{j=1}^{K}\left[V_{C_m}^{(j)}\left(s_{C_m}^{*}, s_{-C_m}^{*(j)}\right) - V_{C_m}^{(j)}\left(s_{C_m}^{\circ(j)}, s_{-C_m}^{\circ(j)}\right)\right] + V_{C_m}^{(i)}\left(s_{C_m}^{\circ(i)}, s_{-C_m}^{\circ(i)}\right), i = 1, 2, \ldots, K.$$

After examining the cooperative payoff distribution scheme in the equilibrium situation of the public choice game Γ_{C_m} under unanimity rule, we will analyze the Nash equilibrium of the public choice game.

Theorem 3 In coalition situation c of information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, in the public choice game under unanimity rule on the strategic combination choice of coalition C_m, the Nash equilibrium of the public choice game is:

$$s_{C_m}^{*} = \underset{s_{C_m}}{\operatorname{argmax}} TV_{C_m}(s_{C_m}) = \underset{s_{C_m}}{\operatorname{argmax}} \sum_{i=1}^{K} V_{C_m}\left(s_{C_m}, s_{-C_m}^{*(i)}\right),$$

where K is the number of members of coalition C_m, $TV_{C_m}(s_{C_m})$ is the sum of the estimations of the cooperative payoff of the coalition in the virtual games of all the coalition members.

Proof. Under unanimity rule, in the Nash equilibrium $s_{C_m}^{*}$ of public choice game Γ_{C_m} on the strategic combination choice of coalition C_m (in which the player set is member set M of coalition C_m, the strategy set of each player is the coalition's strategic combination set S_{C_m}, and the payoff function of each player is his estimation of the cooperative payoff of the coalition in corresponding coalition situation c), the expected cooperative payoff distribution of member i is:

$$x_i = \frac{1}{K}\sum_{j=1}^{K}\left[V_{C_m}^{(j)}\left(s_{C_m}^{*}, s_{-C_m}^{*(j)}\right) - V_{C_m}^{(j)}\left(s_{C_m}^{\circ(j)}, s_{-C_m}^{\circ(j)}\right)\right] + V_{C_m}^{(i)}\left(s_{C_m}^{\circ(i)}, s_{-C_m}^{\circ(i)}\right), i = 1, 2, \ldots, K.$$

In the virtual game of any player j, $V_{C_m}^{(j)}\left(s_{C_m}^{\circ(j)}, s_{-C_m}^{\circ(j)}\right)$ has nothing to do with the public choice of the coalition and in the virtual game of player i, $V_{C_m}^{(i)}\left(s_{C_m}^{\circ(i)}, s_{-C_m}^{\circ(i)}\right)$ has nothing to do with the public choice of the coalition.

In public choice game Γ_{C_m} under unanimity rule on the strategic combination choice of coalition C_m, if $s_{C_m}^{*} = \underset{s_{C_m}}{\operatorname{argmax}}\sum_{i=1}^{K} V_{C_m}^{(i)}\left(s_{C_m}, s_{-C_m}^{*(i)}\right)$ is adopted by the coalition as the equilibrium public choice, the expected cooperative payoff distribution of member i,

$$x_i = \frac{1}{K}\sum_{j=1}^{K}\left[V_{C_m}^{(j)}\left(s_{C_m}^{*}, s_{-C_m}^{*(j)}\right) - V_{C_m}^{(j)}\left(s_{C_m}^{\circ(j)}, s_{-C_m}^{\circ(j)}\right)\right] + V_{C_m}^{(i)}\left(s_{C_m}^{\circ(i)}, s_{-C_m}^{\circ(i)}\right) (i = 1, 2, \ldots, K)$$

is maximized. Therefore, for each coalition member, $s_{C_m}^{*} = \underset{s_{C_m}}{\operatorname{argmax}}\sum_{i=1}^{K} V_{C_m}^{(i)}\left(s_{C_m}, s_{-C_m}^{*(i)}\right)$ is a Pareto improvement of all other public choices. For any member i,

$$x_i(s_{C_m}^{*}) = \max x_i(s_{C_m}) = \frac{1}{K}\sum_{j=1}^{K}\left[V_{C_m}^{(j)}\left(s_{C_m}^{*}, s_{-C_m}^{*(j)}\right) - V_{C_m}^{(j)}\left(s_{C_m}^{\circ(j)}, s_{-C_m}^{\circ(j)}\right)\right] + V_{C_m}^{(i)}\left(s_{C_m}^{\circ(i)}, s_{-C_m}^{\circ(i)}\right),$$

$$x_i(s_{C_m}^{*}) \geq x_i(s_{C_m}).$$

That is, all the coalition members have no motivation to break away from equilibrium $s_{C_m}^{*}$. Therefore, $s_{C_m}^{*}$ is the equilibrium of the public choice game Γ_{C_m} under unanimity rule on the strategic combination choice of coalition C_m.

The above rule for public choice is easy to understand, but the distribution scheme of the expected cooperative payoff is not realistic. In the case of public choice, if some member i makes decision according to his own information set, he certainly thinks that his estimation of the strategic combinations of other coalitions is correct, but in order to achieve the equilibrium of the public choice game, he must pay the price for other coalition members' misjudgement. However, in the distribution process of the actual cooperative payoff of the coalition, a member who accurately judges the Nash equilibrium of the non-cooperative game among coalitions is unlikely to be willing to take responsibility for other members' misjudgements. That is to say, even if he makes a concession in the public choice game, he will no longer compromise in the distribution process of the actual cooperative payoff.

In the cooperative payoff distribution process of a coalition, a reasonable assumption for the distribution rule of cooperative payoff should be: each member should take responsibility for his own misjudgement, but not other's. Therefore, we assume that this is a basic criterion for the distribution of the cooperative payoff of a coalition in an information asymmetric cooperative game, and this criterion is a common knowledge of all players. According to this assumption, even if the equilibrium of the public choice game Γ_{C_m} under unanimity rule on the strategic combination choice of coalition C_m is $s^*_{C_m} = \underset{s_{C_m}}{\operatorname{argmax}} \sum_{i=1}^{K} V^{(i)}_{C_m}\left(s_{C_m}, s^{*(i)}_{-C_m}\right)$, the expected cooperative payoff distribution of member i is still the expected cooperative payoff distribution, $x^{*(i)}_i\left(s^{*(i)}_{C_m}, s^{*(i)}_{-C_m}\right)$, corresponding to the expected cooperative payoff is $V^{(i)}_{C_m}\left(s^{*(i)}_{C_m}\right) = V^{(i)}_{C_m}\left(s_{C_m}, s^{*(i)}_{-C_m}\right)$ of coalition C_m in this virtual game. Since member i, believes in his own information set, he will believe that his judgment of the Nash equilibrium of the non-cooperative game among coalitions is correct.

3.3 *Equilibrium of the public choice game on strategic combination choice of a coalition in member i_1's virtual game in coalition situation c*

Now, we will examine the member i_1's ($i_1 \in C_{k_1}$) virtual non-cooperative game $\Gamma^{(i_1)}(c)\left(N, \left\{S^{(i_1)}_{C_{k_1}}, S^{(i_1)}_{-C_{k_1}}\right\}, \left\{TV^{(i_1)}_{C_{k_1}}, TV^{(i_1)}_{-C_{k_1}}\right\}\right)$ among coalitions in coalition situation c, where TV is the sum of the expected cooperative payoffs of all the members of a coalition under some strategic combination. Since the maximum number of expected cooperative payoff distribution of a member is consistent with the coalition's maximum expected cooperative payoff, the goal of the coalition's public choice is to maximize the sum of the expected cooperative payoffs of all the coalition members in the coalition's public choice game on the strategic combination choice. Therefore, in the virtual game of member i_1, each coalition aims at its maximum sum of the expected cooperative payoffs of all its members.

Given the coalition situation c, the coalitions in the cooperative game and their members, after player i_1 has obtained information about other players, estimated other players' strategy sets, payoff functions and virtual games, utilizing his own strategy set and payoff function, he can make up his virtual non-cooperative game $\Gamma^{(i_1)}(c)\left(N, \left\{S^{(i_1)}_{C_{k_1}}, S^{(i_1)}_{-C_{k_1}}\right\}, \left\{TV^{(i_1)}_{C_{k_1}}, TV^{(i_1)}_{-C_{k_1}}\right\}\right)$ among coalitions, which is called the virtual game of player i_1 in coalition situation c, or is denoted as virtual game $\Gamma^{(i_1)}(c)$.

Next, we will review the decision-making process of any player i_1.

Virtual game of player i_1 in coalition situation c. After player i_1 ($i_1 \in C_{k_1}$) obtained information about other players, estimated other players' strategy sets, payoff functions and virtual games, utilizing his own strategy set and payoff function, he can make up a virtual non-cooperative game $\Gamma^{(i_1)}(c)\left(N, \left\{S^{(i_1)}_{C_{k_1}}, S^{(i_1)}_{-C_{k_1}}\right\}, \left\{TV^{(i_1)}_{C_{k_1}}, TV^{(i_1)}_{-C_{k_1}}\right\}\right)$ among coalitions C_1, C_2, C_{k_1}, ⋯, C_n. Player i_1 clearly understands that the coalition's final public choices are not involved in the Nash equilibrium of this non-cooperative virtual game, because the information base of any other player, is not the same as player i_1's information set. That is to say, the Nash equilibrium of the virtual game of player i_1 is not the Nash equilibrium of non-cooperative game $\Gamma^{(i_1)}(c)\left(N, \left\{S^{(i_1)}_{C_{k_1}}, S^{(i_1)}_{-C_{k_1}}\right\}, \left\{TV^{(i_1)}_{C_{k_1}}, TV^{(i_1)}_{-C_{k_1}}\right\}\right)$.

In coalition situation c, the optimal strategic combination choice of coalition C_{k_1} which maximizes player i_1's expected cooperative payoff distribution is the optimal solution to his virtual non-cooperative game among coalitions $C_1, C_2, C_{k_1}, \cdots, C_n$ in coalition situation c. In the virtual game of player i_1 the optimal strategic combination that coalition C_{k_1} should adopt is the solution to the following optimization problems:

$$\max_{s^{(i_1)}_{C_{k_1}}} V^{(i_1)}_{C_{k_1}}\left(s^{(i_1)}_{C_{k_1}}, s^{*(i_1)}_{-C_{k_1}}\right).$$

Therefore,

$$s_{C_{k_1}}^{*(i_1)} = \underset{s_{C_{k_1}}^{(i_1)}}{\text{argmax}} V_{C_{k_1}}^{(i_1)}\left(s_{C_{k_1}}^{(i_1)}, s_{-C_{k_1}}^{*(i_1)}\right),$$

where $s_{C_{k_1}}^{*(i_1)}$ is the strategic combination that coalition C_{k_1} "should" be adopted in player i_1's virtual game, $s_{-C_{k_1}}^{*(i_1)}$ stands for the strategic combinations that other coalitions adopt in the Nash equilibrium of player i_1's virtual game.

According to the analysis above, coalition C_{k_1}'s strategic combination is not determined by player i_1, but by the coalition's public choice game. Therefore, in the virtual game of player i_1, the public choice of the coalition's strategic combination is the solution to the following optimization problems:

$$\max_{s_{C_{k_1}}^{(i_1, C_{k_1})}} TV_{C_{k_1}}^{(i_1)}\left(s_{C_{k_1}}^{(i_1, C_{k_1})}\right) = \max_{s_{C_{k_1}}^{(i_1, C_{k_1})}} \sum\nolimits_{q_1=1}^{K_1} V_{C_{k_1}}^{(i_1, q_1)}\left(s_{C_{k_1}}^{(i_1, C_{k_1})}, s_{-C_{k_1}}^{*(i_1, q_1)}\right), q_1 \in C_{k_1}, C_{k_1} = \{1, 2, \cdots, K_1\}$$

$$s_{C_{k_1}}^{*(i_1, C_{k_1})} = \underset{s_{C_{k_1}}^{(i_1, C_{k_1})}}{\text{argmax}} TV_{C_{k_1}}^{(i_1)}\left(s_{C_{k_1}}^{(i_1, C_{k_1})}\right) = \underset{s_{C_{k_1}}^{(i_1, C_{k_1})}}{\text{argmax}} \sum\nolimits_{q_1=1}^{K_1} V_{C_{k_1}}^{(i_1, q_1)}\left(s_{C_{k_1}}^{(i_1, C_{k_1})}, s_{-C_{k_1}}^{*(i_1, q_1)}\right)$$

where K_1 is the number of members of coalition C_{k_1}.

That is to say, player i_1's estimation of $TV_{C_{k_1}}^{(i_1)}\left(s_{C_{k_1}}^{(i_1, C_{k_1})}\right)$ the sum of the expected cooperative payoff of coalition $C_{k_1}\,(i_1 \in C_{k_1})$ of all the members in player i_1's virtual game depends on player i_1's estimation of any member q_1's, estimation of the equilibrium strategic combinations $s_{-C_{k_1}}^{*(i_1, q_1)}$ of other coalitions, that rely on player i_1's estimation of the virtual game of any member q_1, of coalition $-C_{k_1}$:

$$s_{C_{k_2}}^{*(i_1, q_1)} = s_{C_{k_2}}^{*(i_1, q_1, C_{k_2})} = \underset{s_{C_{k_2}}^{(i_1, q_1, C_{k_1})}}{\text{argmax}} \sum\nolimits_{C_{k_2}} V_{C_{k_2}}^{(i_1, q_1)}\left(s_{C_{k_2}}^{(i_1, q_1, C_{k_2})}, s_{-C_{k_2}}^{*(i_1, q_1, q_2)}\right),$$

$$s_{C_{k_3}}^{*(i_1, q_1, q_2)} = s_{C_{k_3}}^{*(i_1, q_1, q_2, C_{k_3})} = \underset{s_{C_{k_3}}^{(i_1, q_1, q_2, C_{k_3})}}{\text{argmax}} \sum\nolimits_{q_3 \in C_{k_3}} V_{C_{k_3}}^{(i_1, q_1, q_2)}\left(s_{C_{k_3}}^{(i_1, q_1, q_2, C_{k_3})}, s_{-C_{k_3}}^{*(i_1, q_1, q_2, q_3)}\right), q_1 \in C_{k_1}, q_2 \in C_{k_2}, q_3 \in C_{k_3},$$

.....

Player i_1's estimation of the equilibrium public choice of strategic combination of coalition C_{k_1} he belongs to, depends on player i_1's estimation of all the coalition members' estimations of the equilibrium public choice of strategic combination of any other coalition $C_{k_2}\,(C_{k_2} \neq C_{k_1})$; and any other member q_1's estimation of the equilibrium public choice of strategic combination of any other coalition C_{k_2} depends on member q_1's estimation of any member q_2's $(q_2 \in C_{k_2}, C_{k_2} \neq C_{k_1})$ estimation of the equilibrium public choice of strategic combination of any other coalition $C_{k_3}\,(C_{k_3} \neq C_{k_2})$.

At this point, the virtual game of player i_1 enters the second and third-level.

By analogy, we can get the first level, the second level and the r-th level virtual game of player i_1 in coalition situation c of the information asymmetric cooperative game with agreements implemented by a third party.

Player i_1's first level, second level, and r-th level virtual games in coalition situation c. Player i_1's first level virtual game in coalition situation c includes his estimation of the virtual games of all the members of coalition C_{k_1}. His first level virtual game in coalition situation c is shown as follows:

$$s_{C_{k_1}}^{*(i_1)} = \underset{s_{C_{k_1}}^{(i_1)}}{\text{argmax}} V_{C_{k_1}}^{(i_1)}\left(s_{C_{k_1}}^{(i_1)}, s_{-C_{k_1}}^{*(i_1)}\right),$$

$$s_{C_{k_1}}^{*(i_1, q_1)} = \underset{s_{C_{k_1}}^{(i_1, q_1)}}{\text{argmax}} V_{C_{k_1}}^{(i_1, q_1)}\left(s_{C_{k_1}}^{(i_1, q_1)}, s_{-C_{k_1}}^{*(i_1, q_1)}\right),$$

$$s_{C_{k_1}}^{*(i_1, C_{k_1})} = \underset{s_{C_{k_1}}^{(i_1, C_{k_1})}}{\text{argmax}} TV_{C_{k_1}}^{(i_1)}\left(s_{C_{k_1}}^{(i_1, C_{k_1})}\right) = \underset{s_{C_{k_1}}^{(i_1, C_{k_1})}}{\text{argmax}} \sum\nolimits_{q_1 \in C_{k_1}} V_{C_{k_1}}^{(i_1, q_1)}\left(s_{C_{k_1}}^{(i_1, C_{k_1})}, s_{-C_{k_1}}^{*(i_1, q_1)}\right), q_1 \in C_{k_1}.$$

Player i_1's second level virtual game in coalition situation c is shown as follows:

$$s_{C_{k_2}}^{*(i_1, q_1, q_2)} = \underset{s_{C_{k_2}}^{(i_1, q_1, q_2)}}{\operatorname{argmax}} V_{C_{k_2}}^{(i_1, q_1, q_2)} \left(s_{C_{k_2}}^{(i_1, q_1, q_2)}, s_{-C_{k_2}}^{*(i_1, q_1, q_2)} \right),$$

$$\begin{aligned} s_{C_{k_2}}^{*(i_1, q_1)} &= s_{C_{k_2}}^{*(i_1, q_1, C_{k_2})} \\ &= \underset{s_{C_{k_2}}^{(i_1, q_1, C_{k_2})}}{\operatorname{argmax}} TV_{C_{k_2}}^{(i_1, q_1)} \left(s_{C_{k_2}}^{(i_1, q_1, C_{k_2})} \right) \\ &= \underset{s_{C_{k_2}}^{(i_1, q_1, C_{k_2})}}{\operatorname{argmax}} \sum\nolimits_{q_2 \in C_{k_2}} V_{C_{k_2}}^{(i_1, q_1, q_2)} \left(s_{C_{k_2}}^{(i_1, q_1, C_{k_2})}, s_{-C_{k_2}}^{*(i_1, q_1, q_2)} \right), q_1 \in C_{k_1}, q_2 \in C_{k_2}. \end{aligned}$$

Player i_1's third level virtual game in coalition situation c is shown as follows:

$$s_{C_{k_3}}^{*(i_1, q_1, q_2, q_3)} = \underset{s_{C_{k_3}}^{(i_1, q_1, q_2, q_3)}}{\operatorname{argmax}} V_{C_{k_3}}^{(i_1, q_1, q_2, q_3)} \left(s_{C_{k_3}}^{(i_1, q_1, q_2, q_3)}, s_{-C_{k_3}}^{*(i_1, q_1, q_2, q_3)} \right),$$

$$\begin{aligned} s_{C_{k_3}}^{*(i_1, q_1, q_2)} &= s_{C_{k_3}}^{*(i_1, q_1, q_2, C_{k_3})} \\ &= \underset{s_{C_{k_3}}^{(i_1, q_1, q_2, C_{k_3})}}{\operatorname{argmax}} TV_{C_{k_3}}^{(i_1, q_1, q_2)} \left(s_{C_{k_3}}^{(i_1, q_1, q_2, C_{k_3})} \right) \\ &= \underset{s_{C_{k_3}}^{(i_1, q_1, q_2, C_{k_3})}}{\operatorname{argmax}} \sum\nolimits_{q_3 \in C_{k_3}} V_{C_{k_3}}^{(i_1, q_1, q_2, q_3)} \left(s_{C_{k_3}}^{(i_1, q_1, q_2, C_{k_3})}, s_{-C_{k_3}}^{*(i_1, q_1, q_2, q_3)} \right), q_1 \in C_{k_1}, q_2 \in C_{k_2}, q_3 \in C_{k_3}, \end{aligned}$$

$\dots$

Player i_1's r-th level virtual game in coalition situation c is shown as follows:

$$s_{C_{k_r}}^{*(i_1, q_1, \dots, q_{r-1}, q_r)} = \underset{s_{C_{k_r}}^{(i_1, q_1, \dots, q_{r-1}, q_r)}}{\operatorname{argmax}} V_{C_{k_r}}^{(i_1, q_1, \dots, q_{r-1}, q_r)} \left(s_{C_{k_r}}^{(i_1, q_1, \dots, q_{r-1}, q_r)}, s_{-C_{k_r}}^{*(i_1, q_1, \dots, q_{r-1}, q_r)} \right),$$

$$\begin{aligned} s_{C_{k_r}}^{*(i_1, q_1, \dots, q_{r-1})} &= s_{C_{k_r}}^{*(i_1, q_1, \dots, q_{r-1}, C_{k_r})} \\ &= \underset{s_{C_{k_r}}^{(i_1, q_1, \dots, q_{r-1}, C_{k_r})}}{\operatorname{argmax}} TV_{C_{k_r}}^{(i_1, q_1, \dots, q_{r-1})} \left(s_{C_{k_r}}^{(i_1, q_1, \dots, q_{r-1}, C_{k_r})} \right) \\ &= \underset{s_{C_{k_r}}^{(i_1, q_1, \dots, q_{r-1}, C_{k_r})}}{\operatorname{argmax}} \sum\nolimits_{q_r \in C_{k_r}} V_{C_{k_r}}^{(i_1, q_1, \dots, q_{r-1}, q_r)} \left(s_{C_{k_r}}^{(i_1, q_1, \dots, q_{r-1}, C_{k_r})}, s_{-C_{k_r}}^{*(i_1, q_1, \dots, q_{r-1}, q_r)} \right), q_1 \in C_{k_1}, q_2 \in C_{k_2}, \dots, q_r \in C_{k_r}, \end{aligned}$$

$\dots$

Equilibrium of player i_1's virtual game in coalition situation c. For the p-th level virtual game of player i_1 in coalition situation c,

$$s_{C_{k_p}}^{*(i_1, q_1, \cdots, q_p)} = \underset{s_{C_{k_p}}^{(i_1, q_1, \cdots, q_p)}}{\operatorname{argmax}} V_{C_{k_p}}^{(i_1, q_1, \cdots, q_p)} \left(s_{C_{k_p}}^{*(i_1, q_1, \cdots, q_p)}, s_{-C_{k_p}}^{*(i_1, q_1, \cdots, q_p)} \right),$$

$$\begin{aligned} s_{C_{k_p}}^{*(i_1, q_1, \cdots, q_{p-1})} &= s_{C_{k_p}}^{*(i_1, q_1, \cdots, q_{p-1}, C_{k_p})} \\ &= \underset{s_{C_{k_p}}^{(i_1, q_1, \cdots, q_{p-1}, C_{k_p})}}{\arg\max} TV_{C_{k_p}}^{(i_1, q_1, \cdots, q_{p-1})} \left(s_{C_{k_p}}^{(i_1, q_1, \cdots, q_{p-1}, C_{k_p})} \right) \\ &= \underset{s_{C_{k_p}}^{(i_1, q_1, \cdots, q_{p-1}, C_{k_p})}}{\arg\max} \sum_{q_p \in C_{k_p}} V_{C_{k_p}}^{(i_1, q_1, \cdots, q_p)} \left(s_{C_{k_p}}^{(i_1, q_1, \cdots, q_{p-1}, C_{k_p})}, s_{-C_{k_p}}^{*(i_1, q_1, \cdots, q_{p-1}, q_p)} \right), q_1 \in C_{k_1}, q_2 \in C_{k_2}, \cdots, q_p \in C_{k_p} \end{aligned}$$

if conditions $s_i^{(i_1, q_1, \ldots, q_{p-1})} = s_i^{(i_1, q_1, \ldots, q_p)}, s_i^{(i_1, q_1, \ldots, q_{p-1}, C_{k_{p-1}})} = s_i^{(i_1, q_1, \ldots, q_p, C_{k_p})}, u_i^{(i_1, q_1, \ldots, q_{p-1})} = u_i^{(i_1, q_1, \ldots, q_p)}$ always hold, player i_1's virtual game in coalition situation c is called a p-order game. At this time, equilibrium solution $s_{C_{k_p}}^{*(i_1, q_1, \ldots, q_{p-1})}$ to the p-level virtual game of player i_1 is the Nash equilibrium of the non-cooperative information symmetric cooperative game $\Gamma^{(i_1)}\left(N, \left\{S_i^{(i_1, q_1, \cdots, q_p)}\right\}, \left\{u_i^{(i_1, q_1, \cdots, q_p)}\right\}\right)$ with agreements implemented by a third party among coalitions in coalition situation c. By backward induction, from the (p-1)-th level virtual game to the first level virtual game, we can finally get player i_1's estimation of the public choice of strategic combination of the coalition he belongs to in his virtual game, and also his estimation of the public choices of strategic combination of all other coalitions.

3.4 *Distribution of the expected cooperative payoff of a coalition in its member i_1's virtual game in coalition situation c*

The optimal strategic combination $s_{C_{k_1}}^{*(i_1)}$ of coalition C_{k_1} in player i_1's virtual game is not necessarily equal to the equilibrium public choice of strategic combination $s_{C_{k_1}}^{*(i_1, C_{k_1})}$ of the coalition, except for special situations. That is to say, according to player i_1's information set, the optimal strategic combination of the coalition is not necessarily equal to the public choice of strategic combination of the coalition.

Next, we will analyze the distribution scheme of the expected cooperative payoff of coalition C_{k_1} in player i_1's virtual game in coalition situation c of the information asymmetric cooperative game with agreements implemented by a third party. According to player i_1's information set, the optimal strategic combination that coalition C_{k_1} should adopt is $s_{C_{k_1}}^{*(i_1)}$, but the equilibrium public choice of strategic combination of the coalition is $s_{C_{k_1}}^{*(i_1, C_{k_1})}$. According to player i_1's information set, by adopting strategic combination $s_{C_{k_1}}^{*(i_1, C_{k_1})}$ the coalition will obtain a cooperative payoff $V_{C_{k_1}}^{(i_1)}\left(s_{C_{k_1}}^{*(i_1, C_{k_1})}, s_{-C_{k_1}}^{*(i_1, -C_{k_1})}\right)$. How should this cooperative payoff be distributed?

(1) Assume that all coalition members are not responsible for their own misjudgements. Assume that the members of some coalition are not responsible for their misjudgements of the equilibrium strategic combinations of other cooperative coalitions, the distribution of the coalition's cooperative payoff $V_{C_{k_1}}^{(i_1)}\left(s_{C_{k_1}}^{*(i_1, C_{k_1})}, s_{-C_{k_1}}^{*(i_1, -C_{k_1})}\right)$ should be based on the cooperative payoff distribution rule in the unallied bargaining game of the coalition, as shown in Chen [5].

According to player i_1's information set, in coalition situation c of the information symmetric cooperative game with agreements implemented by a third party, when the coalition-choosing strategies of other players remain unchanged with player $q_1(q_1 \in C_{k_1})$ withdraws from coalition C_{k_1} and joins another cooperative coalition $C_a(C_a \neq C_{k_1})$, new coalition situation $c' = \left(c_1, c_2, \cdots, c'_{q_1} = a, \cdots, c_n\right)$ will be formed. In two different coalition situations c and c', denote the corresponding cooperative payoffs of coalition C_{k_1} as $\left(V_{C_{k_1}}^{(i_1)}(c), V_{C_{k_1}}^{(i_1)}(c')\right)$ and the corresponding cooperative payoffs of coalition C_a as $\left(V_{C_a}^{(i_1)}(c), V_{C_a}^{(i_1)}(c')\right)$ respectively.

Thus, defining the maximum value of the marginal contributions of player q_1 to all possible target coalitions when he withdraws from coalition C_{k_1} which player q_1 belongs to and join any other coalition in coalition situation c as the escape payoff of player q_1 in coalition situation c:

$$w_{q_1}^{(i_1)}(C_{k_1}, c) = \max_{\substack{C_h = C_1, \cdots, C_n \\ C_h \neq C_{k_1}}} Mv_{q_1}^{(i_1)}(C_{k_1}, C_h).$$

In coalition situation c, when the coalition-choosing strategies of other players remain unchanged, if player q_1 decides to withdraws from coalition C_{k_1} which he belongs to, the target coalition must be the one to which player q_1's marginal contribution is the largest, that is, $C_a = \underset{\substack{C_h = C_1, \cdots, C_n \\ C_h \neq C_{k_1}}}{\operatorname{argmax}} Mv_{q_1}^{(i_1)}(C_{k_1}, C_h)$, where $Mv_{q_1}^{(i_1)}\left(C_{k_1}\right) = V_{C_{k_1}}^{(i_1)}\left(c'\right) - V_{C_{k_1}}^{(i_1)}\left(c\right)$ is the marginal contribution of player q_1 to coalition C_{k_1} in coalition situation c.

Assume that member set $M_{q_1, q_2, \cdots, q_k}$ composed of members $q_1, q_2, \cdots, q_k$ is a subset of coalition member set M of coalition C_{k_1} in coalition situation c, $M_{q_1, q_2, \cdots, q_k} \subseteq M \, (k \leq K_1)$, $\delta^{(i_1)}(M_{q_1, q_2, \cdots, q_k})$ is called the common payoff function of member set $M_{q_1, q_2, \cdots, q_k}$ in the virtual game of player i_1:

$$\delta^{(i_1)}(M_{q_1, q_2, \cdots, q_k}) = V_{M_{q_1, q_2, \cdots, q_k}}^{(i_1)} - \sum_{i=1}^{k} w_{q_i}^{(i_1)} - \sum \delta_{(2)}^{(i_1)}(M_{q_1, q_2, \cdots, q_k}) - \cdots - \sum \delta_{(k-1)}^{(i_1)}(M_{q_1, q_2, \cdots, q_k}),$$

where $V_{M_{q_1, q_2, \cdots, q_k}}^{(i_1)}$ is player i_1's estimation of the cooperative payoff of coalition C_{k_1} when the rest of the members (other than those in member set $M_{q_1, q_2, \cdots, q_k}$) withdraw from the coalition and join the same coalition as a whole to maximize their escape payoff, while members of other coalitions keep their coalition-choosing strategies unchanged, $\sum \delta_{(j)}^{(i_1)}(M_{q_1, q_2, \cdots, q_k})$ is the sum of the common payoffs of all the j-member subsets of member set $M_{q_1, q_2, \cdots, q_k}$, $\sum_{i=1}^{k} w_{q_i}^{(i_1)}$ is the sum of the escape payoffs of all the members in member set $M_{q_1, q_2, \cdots, q_k}$.

According to the distribution rule in an unallied bargaining game Chen [5], if coalition C_{k_1} adopts the strategic combination $s_{C_{k_1}}^{*(i_1)}$, player i_1's estimation of the cooperative payoff distribution of another member q_1 of coalition C_{k_1} is:

$$x_{q_1}^{(i_1)}\left(s_{C_{k_1}}^{*(i_1)}, s_{-C_{k_1}}^{*(i_1)}\right) = w_{q_1}^{(i_1)} + \frac{1}{2}\sum_{\substack{j=1 \\ j \neq q_1}}^{K_1} \delta^{(i_1)}(M_{q_1, j}) + \frac{1}{3}\sum_{\substack{j=1 \\ j \neq q_1}}^{K_1} \sum_{\substack{k=1 \\ k \neq q_1}}^{j-1} \delta^{(i_1)}(M_{q_1, j, k}) + \cdots + \frac{1}{K_1}\delta^{(i_1)}(M_{1, 2, \cdots, K_1}).$$

According to player i_1's information set, in coalition situation c of the information asymmetric cooperative game with agreements implemented by a third party, if coalition C_{k_1} adopts the strategic combination $s_{C_{k_1}}^{*(i_1)}$, when the cooperative payoff $V_{C_{k_1}}^{(i_1)}\left(s_{C_{k_1}}^{*(i_1)}, s_{-C_{k_1}}^{*(i_1)}\right)$ of coalition C_{k_1} is distributed according to the above distribution rule, the expected cooperative payoff distribution of player i_1 is:

$$x_{i_1}^{(i_1)}\left(s_{C_{k_1}}^{*(i_1)}, s_{-C_{k_1}}^{*(i_1)}\right) = w_{i_1}^{(i_1)} + \frac{1}{2}\sum_{\substack{j=1 \\ j \neq i_1}}^{K_1} \delta^{(i_1)}(M_{i_1, j}) + \frac{1}{3}\sum_{\substack{j=1 \\ j \neq i_1}}^{K_1} \sum_{\substack{k=1 \\ k \neq i_1}}^{j-1} \delta^{(i_1)}(M_{i_1, j, k}) + \cdots + \frac{1}{K_1}\delta^{(i_1)}(M_{1, 2, \cdots, K_1}).$$

After coalition C_{k_1} decided its strategic combination in its public choice game, that is, when the coalition adopts strategic combination $s_{C_{k_1}}^{*(i_1, C_{k_1})}$ player i_1's expected cooperative payoff distribution in coalition situation c is:

$$x_{i_1}^{(i_1)}(c) = \frac{1}{K_1}\sum_{j=1}^{K_1}\left[u_j^{(i_1)}\left(s_{C_{k_1}}^{*(i_1, C_{k_1})}, s_{-C_{k_1}}^{*(i_1)}\right) - u_j^{(i_1)}\left(s_{C_{k_1}}^{*(i_1, j)}, s_{-C_{k_1}}^{*(i_1, j)}\right)\right] + x_{i_1}^{(i_1)}\left(s_{C_{k_1}}^{*(i_1)}, s_{-C_{k_1}}^{*(i_1)}\right).$$

where $\frac{1}{K_1}\sum_{j=1}^{K_1}\left[u_j^{(i_1)}\left(s_{C_{k_1}}^{*(i_1, C_{k_1})}, s_{-C_{k_1}}^{*(i_1)}\right) - u_j^{(i_1)}\left(s_{C_{k_1}}^{*(i_1, j)}, s_{-C_{k_1}}^{*(i_1, j)}\right)\right]$ is the expected loss of his cooperative payoff distribution due to the coalition's misjudgement in the public choice game.

(2) Assume that all the members of a coalition are responsible for their own misjudgements. Assume that all the members of a coalition are responsible for their own misjudgements of the equilibrium strategic combination choices of other cooperative coalitions, a coalition member who believes in his judgment will require the coalition distributes its cooperative payoff in a "correct" way, and he should not be responsible for other coalition members' misjudgements. Herein, the so-called "correct" distribution way of the coalition according to his point of view is that he should get the "correct" cooperative payoff distribution when the coalition adopts the "correct" strategic combination according to his

"correct" judgment.

According to the assumption above, considering that a coalition member i_1 believes in his judgment of the equilibrium strategic combination choices of other cooperative coalitions, the expected cooperative payoff distribution of member i_1 of coalition C_{k_1} "should" be:

$$x_{i_1}^{(i_1)}(c) = x_{i_1}^{(i_1)}\left(s_{C_{k_1}}^{*(i_1)}, s_{-C_{k_1}}^{*(i_1)}\right).$$

Obviously, it is a more reasonable assumption that all the members of any coalition are responsible for their own misjudgements.

3.5 *Coalition equilibrium of the virtual game of player i_1*

Assume that all the members of a coalition must be responsible for their own misjudgements, according to the analysis above, in the virtual game of player i_1, in coalition situation c, player i_1's expected cooperative payoff distribution is $x_{i_1}^{(i_1)}(c) = x_{i_1}^{(i_1)}\left(s_{C_{k_1}}^{*(i_1)}, s_{-C_{k_1}}^{*(i_1)}\right)$.

In the virtual game of player i_1, the player estimates his own expected cooperative payoff distribution and the expected cooperative payoff distributions of other players. According to the first level of the virtual game of player i_1,

$$s_{C_{k_1}}^{*(i_1, q_1)} = \underset{s_{C_{k_1}}^{(i_1, q_1)}}{\operatorname{argmax}} V_{C_{k_1}}^{(i_1, q_1)}\left(s_{C_{k_1}}^{(i_1, q_1)}, s_{-C_{k_1}}^{*(i_1, q_1)}\right), \; q_1 \in C_{k_1}.$$

On the basis of his information set, player i_1's estimation of the strategic combination choice of coalition C_{k_1} that its member, q_1 thinks the coalition "should" adopt is $s_{C_{k_1}}^{*(i_1, q_1)}$, because it is the only "correct" choice in member q_1's virtual game.

According to the second level virtual game of player i_1,

$$\begin{aligned}
s_{C_{k_2}}^{*(i_1, q_1)} &= s_{C_{k_2}}^{*(i_1, q_1, C_{k_2})} \\
&= \underset{s_{C_{k_2}}^{(i_1, q_1, C_{k_2})}}{\operatorname{argmax}} TV_{C_{k_2}}^{(i_1, q_1)}\left(s_{C_{k_2}}^{(i_1, q_1, C_{k_2})}\right) \\
&= \underset{s_{C_{k_2}}^{(i_1, q_1, C_{k_2})}}{\operatorname{argmax}} \sum_{q_2 \in C_{k_2}} V_{C_{k_2}}^{(i_1, q_1, q_2)}\left(s_{C_{k_2}}^{(i_1, q_1, C_{k_2})}, s_{-C_{k_2}}^{*(i_1, q_1, q_2)}\right), \; q_1 \in C_{k_1}, \; q_2 \in C_{k_2}.
\end{aligned}$$

On the basis of his information set, player i_1's estimation of member q_1's estimation of the strategic combination choice of any other coalition $C_{k_2}(C_{k_2} \neq C_{k_1})$ is $s_{C_{k_2}}^{*(i_1, q_1, C_{k_2})}$. Obviously, if in the cooperative payoff distribution process of coalition C_{k_1}, member q_1 should not be responsible for other's misjudgement. According to the distribution rule proposed by Chen [5], player i_1's estimation of coalition member q_1's expected cooperative payoff distribution is:

$$x_{i_1}^{(i_1, q_1)}\left(s_{C_{k_1}}^{*(i_1, q_1)}, s_{-C_{k_1}}^{*(i_1, q_1)}\right) = w_{q_1}^{(i_1, q_1)} + \frac{1}{2}\sum_{\substack{j=1 \\ j \neq q_1}}^{K_1} \delta^{(i_1, q_1)}(M_{q_1, j}) + \frac{1}{3}\sum_{\substack{j=1 \\ j \neq q_1}}^{K_1} \sum_{\substack{k=1 \\ k \neq q_1}}^{j-1} \delta^{(i_1, q_1)}(M_{q_1, j, k}) + \cdots + \frac{1}{K_1}\delta^{(i_1)}(M_{1, 2, \cdots, K_1}),$$

where $w_{q_1}^{(i_1, q_1)}$ is the escape payoff of coalition member q_1 in player i_1's estimation of member q_1's virtual game; $\delta^{(i_1, q_1)}(M_{q_1, j, k})$ is the common payoff of member set $M_{q_1, j, k}$ in player i_1's estimation of member q_1's virtual game.

According to the second level virtual game of player i_1,

$$s_{C_{k_2}}^{*(i_1, q_1, q_2)} = \underset{s_{C_{k_2}}^{(i_1, q_1, q_2)}}{\operatorname{argmax}} V_{C_{k_2}}^{(i_1, q_1, q_2)}\left(s_{C_{k_2}}^{(i_1, q_1, q_2)}, s_{-C_{k_2}}^{*(i_1, q_1, q_2)}\right).$$

$s_{C_{k_2}}^{*(i_1, q_1, q_2)}$ is player i_1's estimation of the strategic combination of coalition C_{k_2}, which player q_1 regards as the only "correct" strategic combination choice of coalition C_{k_2} according to player q_1's virtual game. If $i_1 = q_1$, $s_{C_{k_2}}^{*(i_1,\, q_1,\, q_2)} = s_{C_{k_2}}^{*(i_1,\, q_2)}$ that is to say, player i_1's estimation of player q_1's estimation of the "correct" strategic combination choice of coalition C_{k_2} is $s_{C_{k_2}}^{*(i_1,\, q_2)}$.

According to the third level virtual game of player i_1,

$$s_{C_{k_3}}^{*(i_1,\, q_1,\, q_2)} = s_{C_{k_3}}^{*(i_1,\, q_1,\, q_2,\, C_{k_3})}$$

$$= \underset{s_{C_{k_3}}^{(i_1,\, q_1,\, q_2,\, C_{k_3})}}{\operatorname{argmax}} TV_{C_{k_3}}^{(i_1,\, q_1,\, q_2)}\left(s_{C_{k_3}}^{(i_1,\, q_1,\, q_2,\, C_{k_3})}\right)$$

$$= \underset{s_{C_{k_3}}^{(i_1,\, q_1,\, q_2,\, C_{k_3})}}{\operatorname{argmax}} \sum_{q_3 \in C_{k_3}} V_{C_{k_3}}^{(i_1,\, q_1,\, q_2,\, q_3)}\left(s_{C_{k_3}}^{(i_1,\, q_1,\, q_2,\, C_{k_3})}, s_{-C_{k_3}}^{*(i_1,\, q_1,\, q_2,\, q_3)}\right), q_1 \in C_{k_1}, q_2 \in C_{k_2}, q_3 \in C_{k_3}.$$

On the basis of his information set, player i_1's estimation of member q_1's estimation of the equilibrium public choice of strategic combination of another coalition $C_{k_3}\,(C_{k_3} \neq C_{k_2})$ is $s_{C_{k_3}}^{*(i_1,\, q_1,\, q_2)}$. If $i_1 = q_1$, according to player q_1's virtual game, $s_{C_{k_3}}^{*(i_1,\, q_1,\, q_2)} = s_{C_{k_3}}^{*(i_1,\, q_2)}$, player i_1's estimation of player q_1's estimation of the equilibrium public choice of strategic combination of coalition $C_{k_3}\,(C_{k_3} \neq C_{k_2})$ is $s_{C_{k_3}}^{*(i_1,\, q_1,\, q_2)}$.

According to player i_1's estimation of player q_2's virtual game, player q_2's estimation of the equilibrium public choice of strategic combination of coalition C_{k_2} is $s_{C_{k_2}}^{*(i_1,\, q_2)}$, and his estimation of the equilibrium public choice of strategic combination of coalition C_{k_3} is $s_{C_{k_3}}^{*(i_1,\, q_2)}$. If a coalition member should not be responsible for other members' judgments, according to the distribution rule proposed by Chen [5], player q_2's will consider that the cooperative payoff distribution that he "should" get is:

$$x_{q_2}^{(i_1,\, q_2)}\left(s_{C_{k_2}}^{*(i_1,\, q_2)}, s_{-C_{k_2}}^{*(i_1,\, q_2)}\right) = w_{q_2}^{(i_1,\, q_2)} + \frac{1}{2}\sum_{\substack{j=1 \\ j \neq q_2}}^{K_2} \delta^{(i_1,\, q_2)}(M_{q_2,\, j}) + \frac{1}{3}\sum_{\substack{j=1 \\ j \neq q_2}}^{K_1} \sum_{\substack{k=1 \\ k \neq q_2}}^{j-1} \delta^{(i_1,\, q_2)}(M_{q_2,\, j,\, k}) + \cdots + \frac{1}{K_2}\delta^{(i_1)}(M_{1,\, 2,\, \cdots,\, K_2}).$$

Therefore, in player i_1's virtual game, in coalition situation c of the information asymmetric cooperative game with agreements implemented by a third party, player i_1's estimation of his expected cooperative payoff distribution and those of other players are respectively shown as follows:

$$x_{i_1}^{(i_1)}(c) = x_{i_1}^{(i_1)}\left(s_{C_{k_1}}^{*(i_1)}, s_{-C_{k_1}}^{*(i_1)}\right), i_1 \in C_{k_1};$$

$$x_{q_1}^{(i_1,\, q_1)}(c) = x_{q_1}^{(i_1,\, q_1)}\left(s_{C_{k_1}}^{*(i_1,\, q_1)}, s_{-C_{k_1}}^{*(i_1,\, q_1)}\right), q_1 \in C_{k_1};$$

$$x_{q_2}^{(i_1, q_2)}(c) = x_{q_2}^{(i_1, q_2)}\left(s_{C_{k_2}}^{*(i_1, q_2)}, s_{-C_{k_2}}^{*(i_1, q_2)}\right), q_2 \in C_{k_2}, C_{k_2} \neq C_{k_1}.$$

Thus, player i_1 can get the estimations of the payoff functions of all the players in his virtual coalition-choosing game $\Gamma_\Gamma^{(i_1)}\left(N, \left\{C_{i_1}, C_{-i_1}\right\}, \left\{x_{i_1}^{(i_1)}, x_{-i_1}^{(i_1, -i_1)}\right\}\right)$, the equilibrium situation of non-cooperative game $\Gamma_\Gamma^{(i_1)}\left(N, \left\{C_{i_1}, C_{-i_1}\right\}, \left\{x_{i_1}^{(i_1)}, x_{-i_1}^{(i_1, -i_1)}\right\}\right)$ is the coalition equilibrium of the information asymmetric cooperative game with agreements implemented by a third party, which is called the coalition equilibrium of the virtual cooperative game with agreements implemented by a third party of player i_1. Herein, in the virtual coalition-choosing game $\Gamma_\Gamma^{(i_1)}\left(N, \left\{C_{i_1}, C_{-i_1}\right\}, \left\{x_{i_1}^{(i_1)}, x_{-i_1}^{(i_1, -i_1)}\right\}\right)$ of player i_1, if the infeasible coalition situations are not removed, the Nash equilibrium of the virtual coalition-choosing game $\Gamma_\Gamma^{(i_1)}\left(N, \left\{C_{i_1}, C_{-i_1}\right\}, \left\{x_{i_1}^{(i_1)}, x_{-i_1}^{(i_1, -i_1)}\right\}\right)$ of player i_1 is not the coalition equilibrium of his virtual information asymmetry cooperative game with agreements implemented by a third party. In the coalition-choosing game, all the players must agree on their choices of the coalition situations, that is to say, in the coalition-choosing game, all the players play the same mixed coalition-choosing strategy at the same time. Therefore, the Nash equilibrium of player i_1's virtual coalition-choosing game $\Gamma_\Gamma^{(i_1)}\left(N, \left\{C_{i_1}, C_{-i_1}\right\}, \left\{x_{i_1}^{(i_1)}, x_{-i_1}^{(i_1, -i_1)}\right\}\right)$, can be obtained through the way in which all the players play the same mixed coalition situation-choosing strategies (with the same probability vector for all feasible coalition situations), and all the players will achieve their own maximum expected cooperative payoff distributions under the same mixed coalition-choosing strategy.

The Nash equilibrium $c^{*(i_1)}$, of the virtual coalition-choosing game $\Gamma_{\Gamma}^{(i_1)}\left(N,\left\{C_{i_1},C_{-i_1}\right\},\left\{x_{i_1}^{(i_1)},x_{-i_1}^{(i_1,-i_1)}\right\}\right)$, of player i_1 is not the real Nash equilibrium of the coalition-choosing game of the information asymmetric cooperation game with agreements implemented by a third party, or it is not the real coalition equilibrium of the information asymmetric cooperation game with agreements implemented by a third party, because player i_1 does not have the complete information.

When all the players form their virtual coalition-choosing games of the information asymmetric cooperative game with agreements implemented by a third party respectively on the basis of their own information sets, they often get different estimations of the coalition equilibrium, that is, usually,

$$c^{*(i_1)} \neq c^{*(-i_1)}.$$

At this point, the information asymmetric cooperative game with agreements implemented by third party finally reaches its coalition equilibrium. If the information sets of the players refer to their final information sets before the actual coalition equilibrium of the information asymmetric cooperative game with agreements implemented by third party is reached, it is obvious that:

$$c^{*(i_1)} = c^{*(-i_1)} = c^{*}.$$

3.6 *Unallied bargaining game of a coalition and the distribution of its cooperative payoff*

If information is still asymmetric after the information asymmetric cooperative game with agreements implemented by a third party is completed, there is no equilibrium in the bargaining game of each coalition about the distribution scheme of its cooperative payoff, thus there is no coalition equilibrium in the information asymmetric cooperative game with agreements implemented by a third party. Therefore, in this section, we only analyze the cooperative payoff distribution process of a coalition in the case of information symmetry among all players after the completion of the information asymmetric cooperative game with agreements implemented by a third party. Herein, we assume that coalition members are not allied in the bargaining game of the coalition.

After the information asymmetric cooperative game reaches its coalition equilibrium c^*, the expected cooperative payoff distribution that any player k_1 gets from coalition C_k (in which the number of the members is m) which he belongs to is $x_{k_1}^{(k_1)}(c^*)(c_{k_1}=k, k_1 \in C_k)$. Assume that each member should be responsible for their own misjudgement in the actual cooperative payoff distribution process, the expected cooperative payoff distribution that player k_1 get from coalition C_k is $x_{k_1}^{(k_1)}(c^*) = x_{k_1}^{(k_1)}\left(s_{C_k}^{*(k_1)}, s_{-C_k}^{*(k_1)}\right)$.

But, as a member of coalition C_k, the expected cooperative payoff distribution $x_{k_1}^{(k_1)}(c^*)$ that player k_1 assumed to receive from coalition C_k is not the final actual cooperative payoff distribution. Member k_1 is not necessarily correct in estimating the strategic combination choices of other coalitions in the non-cooperative game among coalitions. The equilibrium public choice of coalition C_k's strategic combination as the optimal response to other coalitions' strategic combination choices is not necessarily correct too. The final equilibrium strategic combinations of other coalitions do not necessarily appear in the strategic situation corresponding to the equilibrium public choice of strategic combination of coalition C_k.

Assume that in coalition equilibrium c^* of the information asymmetric cooperative game with agreements implemented by a third party, the equilibrium public choice of strategic combination of coalition C_k is $s_{C_k}^*$, while

$$s_{C_k}^* = \underset{s_{C_k}}{\operatorname{argmax}}\, TV_{C_k}(s_{C_k}) = \underset{s_{C_k}}{\operatorname{argmax}} \sum_{q_1=1}^{m} V_{C_k}^{(q_1)}\left(s_{C_k}, s_{-C_k}^{*(q_1)}\right),$$

denote the actual strategic combinations that other coalitions adopt as $s_{-C_k}^{*T}$. After the game is completed, the actual cooperative payoff of coalition C_k is:

$$V_{C_k}\left(s^*_{C_k}, s^{*T}_{-C_k}\right) = \sum\nolimits_{i \in C_k} u_i \left(s^*_{C_k}, s^{*T}_{-C_k}\right).$$

This cooperative payoff is the one that coalition C_k can ultimately use for distribution. It is not necessarily equal to the sum of the expected cooperative payoff distributions of all the coalition members. Due to information asymmetry, the equilibrium public choice of strategic combination of coalition C_k is not the optimal response to the equilibrium public choices of strategic combination of other coalitions, $s^{*T}_{-C_k}$. How does the actual cooperative payoff of coalition C_k, or, its actual cooperative payoff surplus (which is the difference between the actual cooperative payoff of coalition C_k and its cooperative payoff when the coalition members' judgments of the strategic choices of other coalitions are not correct), be distributed among all its members?

If all the coalition members' judgments of the strategic combinations ultimately adopted by other coalitions are correct, the maximum cooperative payoff that coalition C_k can be obtained is:

$$\max V_{C_k}\left(s_{C_k}, s^{*T}_{-C_k}\right) = \max \sum\nolimits_{i \in C_k} u_i \left(s_{C_k}, s^{*T}_{-C_k}\right),$$

$$s^{**}_{C_k} = \underset{s_{C_k}}{\text{argmax}}\, V_{C_k}\left(s_{C_k}, s^{*T}_{-C_k}\right) = \underset{s_{C_k}}{\text{argmax}} \sum\nolimits_{i \in C_k} u_i \left(s_{C_k}, s^{*T}_{-C_k}\right).$$

Therefore, if $s^{**}_{C_k} \neq s^*_{C_k}$, the actual cooperative payoff of coalition C_k is less than the above maximum cooperative payoff that coalition C_k can obtain when all the coalition members' judgments of the strategic combinations ultimately adopted by other coalitions are correct. The cooperative payoff surplus

$$Gap_{C_k} = V_{C_k}\left(s^*_{C_k}, s^{*T}_{-C_k}\right) - V_{C_k}\left(s^{**}_{C_k}, s^{*T}_{-C_k}\right) = \sum\nolimits_{i \in C_k} u_i \left(s^*_{C_k}, s^{*T}_{-C_k}\right) - \sum\nolimits_{i \in C_k} u_i \left(s^{**}_{C_k}, s^{*T}_{-C_k}\right) \leq 0,$$

is caused by coalition C_k's misjudgement of its equilibrium public choice of strategic combination. The misjudgement is caused by the coalition members' misjudgements of the equilibrium public choices of strategic combination of other coalitions. Therefore, in the cooperative payoff distribution process of surplus Gap_{C_k}, of coalition C_k, each member should be responsible for their own misjudgement of the equilibrium public choices of strategic combination of other coalitions.

In coalition equilibrium c^*, if all judgments of the equilibrium public choices of strategic combination of other coalitions by the members are correct, the equilibrium public choice of strategic combination of coalition C_k, should be $s^{**}_{C_k} \neq s^*_{C_k}$. At this time, the coalition C_k's cooperative payoff is $V_{C_k}\left(s^*_{C_k}, s^{*T}_{-C_k}\right) = \sum\nolimits_{i \in C_k} u_i \left(s^*_{C_k}, s^{*T}_{-C_k}\right)$. How is this cooperative payoff of coalition C_k distributed among its members? If after the game is completed and there are no information asymmetry, when coalition members are not allied in the bargaining game, according to Chen [5], the cooperative payoff of coalition C_k will be distributed among its members according to the distribution rule in the corresponding information symmetric unallied bargaining game:

$$x^{**}_{k_1} = w^{**}_{k_1} + \frac{1}{2}\sum\nolimits_{\substack{j=1 \\ j \neq k_1}}^{m} \delta^{**}(M_{k_1, j}) + \frac{1}{3}\sum\nolimits_{\substack{j=1 \\ j \neq k_1}}^{m} \sum\nolimits_{\substack{q=1 \\ q \neq k_1}}^{j-1} \delta^{**}(M_{k_1, j, q}) + \cdots + \frac{1}{m}\delta^{**}(M_{1, 2, \cdots, m}),$$

where $w^{**}_{k_1}$ is the escape payoff of player k_1, and $\delta^{**}(M_{k_1, j}), \delta^{**}(M_{k_1, j, q}), \cdots, \delta^{**}(M_{1, 2, \cdots, m})$ are respectively the common payoffs of member sets $M_{k_1, j}, M_{k_1, j, q}, \ldots, M_{1, 2, \cdots, m}$ of coalition C_k.

If the judgments of all the members of coalition C_k of the equilibrium public choices of strategic combination of other coalitions are not correct, that is, $s^{*(i)}_{C_k} \neq s^{**}_{C_k}$, $i \in C_k$, generally speaking,

$$s^*_{C_k} \neq s^{**}_{C_k}.$$

Correspondingly, $V_{C_k}\left(s^*_{C_k}, s^{*T}_{-C_k}\right) \neq V_{C_k}\left(s^{**}_{C_k}, s^{*T}_{-C_k}\right)$. Then, coalition C_k's cooperative payoff $V_{C_k}\left(s^*_{C_k}, s^{*T}_{-C_k}\right)$ can be regarded as the "cooperation" result of the members of coalition C_k starting from strategic combination $s^{**}_{C_k}$, and through the misjudgements of the equilibrium public choice of strategic combination of coalition C_k, and the equilibrium public choices of strategic combination of other coalitions. The distribution of the cooperative payoff $V_{C_k}\left(s^*_{C_k}, s^{*T}_{-C_k}\right)$ should be based on the distribution of the cooperative payoff $V_{C_k}\left(s_{C_k}^{**}, s_{-C_k}^{*T}\right)$ with an additional distribution of the "cooperative" payoff $V_{C_k}\left(s_{C_k}^{*}, s_{-C_k}^{*T}\right)-V_{C_k}\left(s_{C_k}^{**}, s_{-C_k}^{*T}\right)$ which is caused by the misjudgements of the members of coalition C_k.

For any member k_1 of coalition C_k, when he withdraws from "cooperative" state $s_{C_k}^{**}$, to state $s_{C_k}^{*}\left(s_{C_k}^{*}=\underset{s_{C_k}}{\operatorname{argmax}}\sum_{q_1=1}^{m}u_{q_1}\left(s_{C_k}, s_{-C_k}^{*(q_1)}\right)\right)$, if member k_1 misjudges but other members still keep their judgments correct, the loss of the cooperative payoff of coalition C_k shall obviously be the responsibility of member k_1, which is member k_1's escape payoff in the "cooperation" of misjudgement. When member k_1 escapes, coalition C_k's strategic combination choice is:

$$s_{C_k}^{*}(EM_{k_1})=\operatorname{argmax}\left\{\sum_{\substack{q_1=1\\q_1\neq k_1}}^{m}V_{C_k}\left(s_{C_k}, s_{-C_k}^{*T}\right)+V_{C_k}\left(s_{C_k}, s_{-C_k}^{*(k_1)}\right)\right\}$$

$$=\operatorname{argmax}\left\{(m-1)V_{C_k}\left(s_{C_k}, s_{-C_k}^{*T}\right)+V_{C_k}\left(s_{C_k}, s_{-C_k}^{*(k_1)}\right)\right\},$$

where $s_{C_k}^{*}(EM_{k_1})$ stands for coalition C_k's strategic combination choice after member k_1 escapes. The escape payoff of member k_1 is:

$$w_{k_1}^{*}=V_{C_k}\left(s_{C_k}^{*}(EM_{k_1}), s_{-C_k}^{*T}\right)-V_{C_k}\left(s_{C_k}^{**}, s_{-C_k}^{*T}\right).$$

If members k_1 and k_2 escape at the same time, coalition C_k's strategic combination choice is:

$$s_{C_k}^{*}(EM_{k_1,k_2})=\operatorname{argmax}\left\{\sum_{\substack{q_1=1\\q_1\neq k_1,k_2}}^{m}V_{C_k}\left(s_{C_k}, s_{-C_k}^{*T}\right)+V_{C_k}\left(s_{C_k}, s_{-C_k}^{*(k_1)}\right)+V_{C_k}\left(s_{C_k}, s_{-C_k}^{*(k_2)}\right)\right\}$$

$$=\operatorname{argmax}\left\{(m-2)V_{C_k}\left(s_{C_k}, s_{-C_k}^{*T}\right)+V_{C_k}\left(s_{C_k}, s_{-C_k}^{*(k_1)}\right)+V_{C_k}\left(s_{C_k}, s_{-C_k}^{*(k_2)}\right)\right\}.$$

The common payoff of members k_1 and k_2 is:

$$\delta^{*}(EM_{k_1,k_2})=V_{C_k}\left(s_{C_k}^{*}(EM_{k_1,k_2}), s_{-C_k}^{*T}\right)-V_{C_k}\left(s_{C_k}^{**}, s_{-C_k}^{*T}\right)-w_{k_1}^{*}-w_{k_2}^{*},$$

$$\cdots$$

When all the members in member set $M_{k_1,k_2,\ldots,k_i}$ escape, coalition C_k's strategic combination choice is:

$$s_{C_k}^{*}(EM_{k_1,k_2,\ldots,k_i})=\operatorname{argmax}\left\{\sum_{\substack{q_1=1\\q_1\neq k_1,k_2,\ldots,k_i}}^{m}V_{C_k}\left(s_{C_k}, s_{-C_k}^{*T}\right)+\sum_{r_1=k_1}^{k_i}V_{C_k}\left(s_{C_k}, s_{-C_k}^{*(r_1)}\right)\right\}$$

$$=\operatorname{argmax}\left\{(m-i)V_{C_k}\left(s_{C_k}, s_{-C_k}^{*T}\right)+\sum_{r_1=k_1}^{k_i}V_{C_k}\left(s_{C_k}, s_{-C_k}^{*(r_1)}\right)\right\}.$$

The common payoff of member set $M_{k_1,k_2,\ldots,k_i}$ is:

$$\delta^{*}(M_{k_1,k_2,\ldots,k_i})=V_{C_k}\left(s_{C_k}^{*}(EM_{k_1,k_2,\ldots,k_i}), s_{-C_k}^{*T}\right)-V_{C_k}\left(s_{C_k}^{**}, s_{-C_k}^{*T}\right)-\sum\nolimits_{j=1}^{i}w_{k_j}^{*}-\sum\delta_{(2)}(M_{k_1,k_2,\ldots,k_i})-\cdots-\sum\delta_{(k-1)}(M_{k_1,k_2,\ldots,k_i}),$$

where $\sum_{j=1}^{i} w_{k_j}^{*}$ is the sum of the escape payoffs of all the members in member set $M_{k_1, k_2, \ldots, k_i}$, $\sum \delta_{(j)}(M_{k_1, k_2, \ldots, k_i})$ is the sum of the common payoffs of all the j-member subsets of member set $M_{k_1, k_2, \ldots, k_i}$.

The cooperative payoff distribution that member k_1 gets from the misjudgement "cooperation" is:

$$x_{k_1}^{*} = w_{k_1}^{*} + \frac{1}{2}\sum_{\substack{q_j=1 \\ q_j \neq k_1}}^{m} \delta^{*}(M_{k_1, q_j}) + \frac{1}{3}\sum_{\substack{q_j=1 \\ q_j \neq k_1}}^{m} \sum_{\substack{q_k=1 \\ q_k \neq k_1}}^{q_j-1} \delta^{*}(M_{k_1, q_j, q_k}) + \cdots + \frac{1}{m}\delta^{*}\left(M_{1, 2, \cdots, m}\right).$$

And the actual total cooperative payoff distribution that member k_1 gets is:

$$\begin{aligned} x_{k_1} &= x_{k_1}^{*} + x_{k_1}^{**} \\ &= w_{k_1}^{*} + w_{k_1}^{**} + \frac{1}{2}\left[\sum_{\substack{q_j=1 \\ q_j \neq k_1}}^{m} \delta^{*}(M_{k_1, q_j}) + \sum_{\substack{q_j=1 \\ q_j \neq k_1}}^{m} \delta^{**}(M_{k_1, q_j})\right] + \frac{1}{3}\left[\sum_{\substack{q_j=1 \\ q_j \neq k_1}}^{m} \sum_{\substack{q_k=1 \\ q_k \neq k_1}}^{q_j-1} \delta^{*}(M_{k_1, q_j, q_k}) + \right. \\ &\quad \left. \sum_{\substack{q_j=1 \\ q_j \neq k_1}}^{m} \sum_{\substack{q_k=1 \\ q_k \neq k_1}}^{q_j-1} \delta^{**}(M_{k_1, q_j, q_k}) \right] \cdots + \frac{1}{m}\left[\delta^{*}(M_{1, 2, \cdots, m}) + \delta^{**}(M_{1, 2, \cdots, m})\right], \end{aligned}$$

where $x_{k_1}^{**}$ is the cooperative payoff distribution got by player k_1 when the judgments of the equilibrium public choices of strategic combination of other coalitions by all members are correct. $x_{k_1}^{*}$ is the cooperative payoff distribution that member k_1 gets from the misjudgement "cooperation".

When the game is completed, there may still be information asymmetry among players, and information asymmetry may still exist among the members of coalition C_k. At this point, after the game is completed, the members of coalition C_k will modify their information set and re-send signals, the purpose of such a signal is obviously a larger cooperative payoff distribution from coalition C_k.

If there is still information asymmetry among the members of coalition C_k in the unallied bargaining game which determines the cooperative payoff distribution scheme $V_{C_k}\left(s_{C_k}^{*}, s_{-C_k}^{*T}\right) = \sum_{i=1}^{m} u_i\left(s_{C_k}^{*}, s_{-C_k}^{*T}\right)$ of the coalition, there exists no Nash equilibrium in the unallied bargaining game. Assume that for any member k_1, according to his information set, the Nash equilibrium of the unallied bargaining game is $(x_1^{*(k_1)}, \cdots, x_i^{*(k_1)}, \cdots, x_K^{*(k_1)})$, where $x_i^{*(k_1)}$ is the threat-point of any member i in the virtual bargaining game of member k_1. Thus, for any two members k_1 and k_2 usually,

$$x_i^{*(k_1)} \neq x_i^{*(k_2)}.$$

Therefore, if information is still asymmetric after the game is completed, the members of coalition C_k cannot agree on the distribution of the cooperative payoff of coalition C_k.

Similarly, in the bargaining game on the distribution of "cooperative" payoff $V_{C_k}\left(s_{C_k}^{*}, s_{-C_k}^{*T}\right) - V_{C_k}\left(s_{C_k}^{**}, s_{-C_k}^{*T}\right)$ brought by the misjudgement of "cooperation", since the information set of each member is different, for any two members k_1 and k_2 usually,

$$x_i^{**(k_1)} \neq x_i^{**(k_2)}.$$

Coalition members may also be unable to reach the Nash equilibrium of the unallied bargaining game on the distribution of "cooperative" payoff brought about by the misjudgement "cooperation".

In general, for any two members k_1 and k_2 usually,

$$x_i^{*(k_1)} + x_i^{**(k_1)} \neq x_i^{*(k_2)} + x_i^{**(k_2)}.$$

If there is still information asymmetry among the coalition members in the unallied bargaining game on the distribution of the cooperative payoff of the coalition, there is no Nash equilibrium in the unallied bargaining game on the distribution of the cooperative payoff.

In real cooperative games, there are numerous examples of the difficulty in achieving the equilibrium of an unallied bargaining game on the distribution of the cooperative payoff under information asymmetry. The simplest example is the bargaining game between a tourist and a hawker on commodity prices. Even if both parties understand that cooperation is beneficial to both of them, the distribution of the cooperative payoff, that is, the price negotiation, often cannot reach the equilibrium because of the information asymmetry.

In an information asymmetric cooperative game with agreements implemented by a third party, when any player k_1 decides his own coalition-choosing strategy, he follows the coalition-choosing criterion of maximum expected cooperative payoff distribution. Player k_1's expected cooperative payoff distribution that he gets from coalition C_k, according to his own virtual game, is $x_{k_1}^{*(k_1)}$. If player k_1 knows that in the distribution process of actual cooperative payoff $V_{C_k}\left(s_{C_k}^{*}, s_{-C_k}^{*T}\right)$ his expected distribution is not guaranteed, the coalition-choosing criterion of maximum expected cooperative payoff distribution is not equivalent to the coalition-choosing criterion of maximum actual cooperative payoff distribution. That is to say, if among members of coalition C_k, information is still asymmetric after the game is completed, the virtual games of coalition members cannot lead to the coalition equilibrium under the criterion of maximum expected cooperative payoff distribution.

According to the above discussion on the information asymmetric cooperative games with agreements implemented by a third party, we can reach the conclusions shown in Theorem 4 and Theorem 5.

Theorem 4 In information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, assume that the information is symmetric after the game is completed and that the assumption is a common knowledge of all the players; there exists the mixed strategic Nash equilibrium in the unallied bargaining game of each coalition. At the same time, there exists a mixed strategic coalition equilibrium under the criterion of maximum expected cooperative payoff distribution in information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party:

$$\forall i = 1, 2, \cdots, n, c_i^* = \begin{cases} i, & \text{if for any } c_i,\ x_i^{(i)}(i, c_{-i}^*) \geq x_i^{(i)}(c_i, c_{-i}^*), \\ \operatorname{argmax} x_i^{(i)}(c_i, c_{-i}^*), & \text{if at least for a certain } c_i \neq i, x_i^{(i)}(c_i, c_{-i}^*) > x_i^{(i)}(i, c_{-i}^*). \end{cases}$$

Proof. Assume that information is symmetric after the game is completed, and that the assumption is common knowledge of all the players; in the virtual game of any player i, the unallied bargaining game of coalition C which he belongs to, will be an information of symmetric game. Therefore, the mixed strategic Nash equilibrium of this game exist Chen [5]. Moreover, since the game is a non-cooperative game in which the payoff functions of the players are continuous and quasi-convex, its Nash equilibrium is a pure strategic equilibrium.

According to the information set of player i, the equilibrium of the threat-point choices of the coalition members in the above-mentioned unallied bargaining game is the cooperative payoff distribution scheme of coalition C. Since the pure strategic equilibrium of the unallied bargaining game is one and only, the cooperative payoff distribution that each member can get is unique.

According to the information set of player i, in any situation c, the cooperative payoff distribution that any member j, (including player i himself) gets from coalition C is unique. Then, the Nash equilibrium of the coalition-choosing game exists, which in the virtual game of player i, the (mixed strategic) coalition equilibrium under the criterion of maximum expected cooperative payoff distribution exist. In the virtual non-cooperative coalition-choosing game of player i, since the saddle point may not exist, the equilibrium of the virtual coalition-choosing game of player i may be not unique. Hence, the coalition equilibrium in the virtual game of player i is not unique.

Theorem 5 In information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, assume that information is still asymmetric after the game is completed, and the assumption is common knowledge of all the players, the Nash equilibrium in the unallied bargaining game of each coalition is not exist. At the same time, there is no coalition equilibrium under the criterion of maximum expected cooperative payoff distribution that exist in information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party. Herein, that there is no coalition equilibrium under the criterion of maximum expected cooperative payoff distribution in information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party exist, or, there is no coalition equilibrium under the criterion of minimum expected escape payoff in the information asymmetric cooperative game with agreements implemented by a third party does not mean that there is no form of cooperative coalition in the game. Some players with a high degree of information symmetry (after the completion of the cooperative game) may still establish cooperative coalitions which aim at exploiting the synergies among them, and reach cooperative payoff distribution agreements with some kinds of compensation mechanisms. In addition, even if the degree of information asymmetry among the players is still high after the completion of the cooperative game, those who agree with each other on the synergy expectations and do not need distribution compensations (perhaps they can set up certain compensation mechanisms to benefit from the cooperation through the compensation mechanism they set up) may also reach some forms of distribution agreements and establish cooperative coalitions designed to take advantage of the synergy expectations among them. When the degree of information asymmetry among the players is still high after the completion of the cooperative game, in information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, there is at least a coalition situation shown as follows which is feasible.

Proof. Assume that information is still asymmetric after the game is completed, and that the assumption is common knowledge of all the players. In the virtual game of any player *i*, the unallied bargaining game of coalition *C*, which he belongs to, is an information asymmetric non-cooperative game. The coalition members' estimations of the threat-point vector of all the coalition members are different, indicating that, the coalition members cannot agree on the cooperative payoff distribution scheme. Therefore, there is no Nash equilibrium in the unallied bargaining game of the coalition.

According to the information set of player *i*, the equilibrium threat-point choices of the coalition members in the unallied bargaining game is the equilibrium distribution scheme of the cooperation payoff of coalition *C* (Chen, 2021) [2]. If there is no equilibrium in the above-mentioned bargaining game, the cooperative payoff distribution that some member gets is uncertain.

If according to the information set of player *i*, in some coalition situation *c*, the cooperative payoff distribution that any member (including player *i* himself) gets is uncertain, the Nash equilibrium of the coalition-choosing game does not exist. Which means, there is no coalition equilibrium exist under the criterion of maximum expected cooperative payoff distribution in information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party.

However, if information is still asymmetric after the game is completed, and there is no coalition equilibrium exist in information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$, it does not mean that there is no form of cooperative coalition in the game. If the sum of the threat-points of the coalition members is lower than the expected cooperative payoff, in the bargaining game, the cooperation agreement of the coalition is still possible. Even if the cooperation agreement among all the possible members of the coalition is not reached, some possible members may reach a cooperation agreement because of their lower threat-points. At this time, the behaviors of the players who withdraw from the coalition because of their high threat-points are not individually irrational. But, if some of the members always compromise in the bargaining game, their behavior does not conform to the criterion of individual rationality.

In an information asymmetric cooperative game with agreements implemented by a third party, the players search for possible partners according to their virtual games, and coalition members must agree on the cooperative payoff distribution scheme. If there is no agreement on the distribution of the cooperative payoff, even if the players agree with each other about the synergies among them, their cooperation agreement may not be reached. The players who withdraw from the coalition can continue to look for collaborators. However, even if his collaborators can be found, the synergy between him and the collaborators he later found is obviously not optimal in his view. In any case, the equilibrium of the information asymmetric cooperative game cannot be achieved, which means that the expected synergies in the society (the collection of all the players) are not fully utilized.

Theorem 6 In information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, assume that the information is still asymmetric after the game is completed, and the above assumption is common knowledge of all the players, assume that in the cooperative game there exists no compensation mechanism (or, the distribution of any member of each coalition is just the payoff that he gets in the game), the following coalition situation under the criterion of maximum expected payoff is feasible:

$$\forall i = 1, 2, \cdots, n, c_i^* = \begin{cases} i, & \text{if for any } c_i,\ u_i^{(i)}(i, c_{-i}^*) \geq u_i^{(i)}(c_i, c_{-i}^*), \\ \text{argmax } u_i^{(i)}(c_i, c_{-i}^*), & \text{if at least for a certain } c_i \neq i, u_i^{(i)}(c_i, c_{-i}^*) > u_i^{(i)}(i, c_{-i}^*). \end{cases}$$

The proof of Theorem 6 is omitted.

3.7 *Coalition equilibrium of the virtual coalition-choosing game under the criterion of minimum expected escape payoff*

From the above analysis, we can see that in an information asymmetric cooperative game with agreements implemented by a third party, if the information is symmetric after the game is completed, (and this is common knowledge of all the players), there exists the mixed strategic coalition equilibrium under the criterion of maximum expected cooperative payoff distribution. According to the above analysis, even if the coalition's strategic combination choice is not based on some member i's, judgment, he can still expect to obtain the cooperative payoff distribution that he "should" obtain under the strategic combination that he thinks is "correct". As after the game is completed, each member of the coalition must be responsible for their own misjudgement.

In an information asymmetric cooperative game with agreements implemented by a third party, the player's coalition-choosing strategy choice is aimed at his maximum expected cooperative payoff distribution. Thus, if the coalition's cooperative payoff distribution scheme meets the competitive distribution conditions, where the cooperative payoff distribution of any member is no less than his escape payoff, but no more than his contribution to the coalition, the player's goal is consistent with the goal of his minimum expected escape payoff. Therefore, if the distribution of the cooperative payoff of each coalition satisfies the competitive distribution condition, in the information asymmetric cooperative game with agreements implemented by a third party (when the coalition members are not allied in the bargaining game), the coalition equilibrium under the criterion of maximum expected cooperative payoff distribution is equivalent to the coalition equilibrium under the criterion of minimum expected escape payoff, if the coalition equilibrium does exist.

Defined the expected escape payoff of player i in coalition situation c, as the maximum value of the expected escape payoffs that player i can obtain when he withdraws from coalition C_j that he belongs to, and joins other coalitions. When he withdraws from coalition C_j, the expected escape payoff of player i is:

$$w_i^{(i_i)}(C_j) = \max_{\substack{C_h = C_1, \cdots, C_n \\ C_h \neq C_j}} Mv_i^{(i)}(C_j, C_h),$$

where $Mv_i^{(i)}(C_j, C_h)$ is the expected marginal contribution of player i to the target coalition C_h when player i withdraws from coalition C_j and joins another coalition C_h. This is the maximum expected cooperative payoff distribution that player i can obtain when he withdraws from coalition C_j and joins this coalition. In coalition situation c, assume that the coalition-choosing strategies of other players remain unchanged, player i will inevitably withdraw from coalition C_j and join coalition $C_{h^*} = \operatorname*{argmax}_{\substack{C_h = C_1, \cdots, C_n \\ C_h \neq C_j}} Mv_i^{(i)}(C_j, C_h)$ when he decides to implement escape.

Theorem 7 In an information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, assume that the information is symmetric after the game is completed, and that the above assumption is common knowledge of all the players; there exists the mixed strategic coalition equilibrium under the criterion of minimum expected escape payoff in the information asymmetric cooperative game with agreements implemented by a third party:

$$\forall i = 1, 2, \cdots, n, c_i^* = \begin{cases} i, & \text{if for any } c_i,\ w_i^{(i)}(i, c_{-i}^*) \le w_i^{(i)}(c_i, c_{-i}^*), \\ \operatorname{argmin} w_i^{(i)}(c_i, c_{-i}^*), & \text{if at least for a certain } c_i \ne i, w_i^{(i)}(c_i, c_{-i}^*) < w_i^{(i)}(i, c_{-i}^*). \end{cases}$$

If the distribution rule of each coalition meets the competitive distribution condition, the coalition equilibrium under the criterion of maximum expected cooperative payoff distribution is equivalent to the one under the criterion of minimum expected escape payoff. That is to say,

$$\forall i = 1, 2, \cdots, n, c_i^* = \begin{cases} i, & \text{if for any } c_i,\ x_i^{(i)}(i, c_{-i}^*) \ge x_i^{(i)}(c_i, c_{-i}^*), \\ \operatorname{argmax} x_i^{(i)}(c_i, c_{-i}^*), & \text{if at least for a certain } c_i \ne i,\ x_i^{(i)}(c_i, c_{-i}^*) > x_i^{(i)}(i, c_{-i}^*). \end{cases}$$

$$\Leftrightarrow \forall i = 1, 2, \cdots, n, c_i^* = \begin{cases} i, & \text{if for any } c_i,\ w_i^{(i)}(i, c_{-i}^*) \le w_i^{(i)}(c_i, c_{-i}^*), \\ \text{argmin}\, w_i^{(i)}(c_i, c_{-i}^*), & \text{if at least for a certain } c_i \ne i,\ w_i^{(i)}(c_i, c_{-i}^*) < w_i^{(i)}(i, c_{-i}^*). \end{cases}$$

Proof. Similar to the proof of Theorem 4, it is easy to prove that if information is symmetric after the game is completed, there exists a mixed strategic coalition equilibrium under the criterion of minimum expected escape payoff in the information asymmetric cooperative game with agreements implemented by a third party.

In the following, we will prove that the above-mentioned two coalition equilibrium are equivalent. If the distribution rules in all coalitions satisfy the competitive distribution condition,

$$c_i^* = i, \text{if for any } c_i, w_i^{(i)}(i, c_{-i}^*) \le w_i^{(i)}(c_i, c_{-i}^*) \Leftrightarrow c_i^* = i, \text{if for any } c_i, x_i^{(i)}(i, c_{-i}^*) \le x_i^{(i)}(c_i, c_{-i}^*).$$

So, what we needed to prove is that, if the distribution rules in all coalitions satisfy the competitive distribution condition,

$$c_i^* = \arg\min w_i^{(i)}(c_i, c_{-i}^*), \text{if for at least a certain } c_i \ne i, w_i^{(i)}(i, c_{-i}^*) > w_i^{(i)}(c_i, c_{-i}^*),$$

$$\Leftrightarrow c_i^* = \arg\max x_i^{(i)}(c_i, c_i), \text{if for at least a certain } c_i \ne i, x_i^{(i)}(i, c_{-i}^*) < x_i^{(i)}(c_i, c_{-i}^*).$$

Obviously,

$$w_i^{(i)}(i, c_{-i}^*) > w_i^{(i)}(c_i, c_{-i}^*) \Leftrightarrow x_i^{(i)}(i, c_{-i}^*) < x_i^{(i)}(c_i, c_{-i}^*).$$

What we need to do is to prove:

$$c_i^* = \arg\min w_i^{(i)}(c_i, c_{-i}^*) \Leftrightarrow c_i^* = \arg\max x_i^{(i)}(c_i, c_{-i}^*).$$

That is when any player i minimizes his escape payoff in the coalition equilibrium under the principle of minimum escape payoff, he could get maximum distribution from the coalition he chooses. At the same time, when player i maximizes his distribution from the coalition he chooses in the coalition equilibrium under the principle of maximum distribution, he could get his minimum escape payoff.

(1) If $c_i^* = \text{argmax}\, x_i^{(i)}$, (c_i^*, c_{-i}^*) is the coalition equilibrium under the principle of maximum distribution. In this coalition equilibrium, the distribution function $x_i^{(i)}(c_i, c_{-i}^*)$ of any player i satisfies the competitive distribution condition:

$$x_i^{(i)}(c_i, c_{-i}^*) \ge w_i^{(i)}(c_i, c_{-i}^*).$$

If the coalition-choosing strategy played by player i is $c_i, c_i \ne c_i^*$, according to the definition of the coalition equilibrium under the principle of maximum distribution, $x_i^{(i)}(c_i, c_{-i}^*) \le x_i^{(i)}(c_i, c_{-i}^*)$, player i has the motivation of choosing coalition-choosing strategy c_i^* in order to withdraw from C_{c_i} and join coalition $C_{c_i^*}$ in coalition situation (c_i, c_{-i}^*). When player i withdraws from coalition C_{c_i} and joins coalition $C_{c_i^*}$, his escape payoff

$$Mv_i^{(i)}(C_{c_i}) \le w_i^{(i)}(c_i, c_{-i}^*).$$

Coalition $C_{c_i^*}$ won't offer player i a distribution more than $Mv_i^{(i)}(C_{c_i})$, because $Mv_i^{(i)}(C_{c_i})$, is the contribution of player i to coalition $C_{c_i^*}$ when he withdraws from coalition $C_{c_i^*}$. According to the competitive distribution condition, we have:

$$x_i^{(i)}(c_i, c_{-i}^*) \le Mv_i^{(i)}(C_{c_i}).$$

So,

$$w_i^{(i)}(c_i, c_i) \le x_i^{(i)}(c_i, c_i) \le w_i^{(i)}(c_i, c_i).$$

Then,

$$w_i^{(i)}(c_i, c_{-i}^*) \le w_i^{(i)}(c_i, c_{-i}^*).$$

(2) If $c_i = \arg\min w_i^{(i)}(c_i, c_{-i}^*)$, (c_i^*, c_{-i}^*) is the coalition equilibrium under the principle of minimum escape payoff. In this coalition equilibrium, the escape payoff function $w_i^{(i)}(c_i, c_{-i}^*)$ of any player i satisfies the competitive distribution condition:

$$x_i^{(i)}(c_i, c_{-i}^*) \ge w_i^{(i)}(c_i, c_{-i}^*).$$

If the coalition-choosing strategy played by player i is $c_i, c_i \ne c_i^*$, according to the definition of the coalition equilibrium under the principle of minimum escape payoff,

$$w_i^{(i)}(c_i, c_{-i}^*) \ge w_i^{(i)}(c_i, c_{-i}^*),$$

in coalition situation (c_i, c_{-i}^*), the distribution that player i gets satisfies:

$$x_i^{(i)}(c_i, c_{-i}^*) \le w_i^{(i)}(c_i, c_{-i}^*).$$

Because $w_i^{(i)}(c_i, c_{-i}^*)$ is the marginal contribution that player i brings to coalition C_{c_i} when he withdraws from $C_{c_i^*}$, we have,

$$x_i^{(i)}(c_i, c_{-i}^*) \ge x_i^{(i)}(c_i, c_{-i}^*).$$

Therefore, we get:

$$x_i^{(i)}(c_i, c_{-i}^*) \ge x_i^{(i)}(c_i, c_{-i}^*) \Leftrightarrow w_i^{(i)}(c_i, c_{-i}^*) \le w_i^{(i)}(c_i, c_{-i}^*), \arg\min w_i^{(i)}(c_i, c_{-i}^*) \Leftrightarrow \arg\max x_i^{(i)}(c_i, c_{-i}^*) \ldots$$

4. Allied bargaining game and the information asymmetric cooperative games with agreements implemented by a third party

Next, we examine the information asymmetric cooperative games with agreements implemented by a third party, in which the members of each coalition are allied in the bargaining game on the distribution of its cooperative payoff.

If information is still asymmetric after the game is completed, there is no Nash equilibrium in the bargaining games of coalitions, therefore, there is no coalition equilibrium under the criterion of maximum expected cooperative payoff distribution in information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, or, there is no coalition equilibrium under the criterion of minimum expected escape payoff in the information asymmetric cooperative game with agreements implemented by a third party. So, in this section we only review the distribution of the cooperative payoff of a coalition in the case of information symmetry after the completion of the game.

4.1 *Coalition equilibrium*

An allied bargaining game means that the coalition members are allied in the bargaining game on the distribution of the cooperative payoff. If the coalition members are allied in the bargaining game, is there coalition equilibrium in information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party? In the allied bargaining game of some coalition, is there equilibrium of the cooperative payoff distribution?

Theorem 8 In information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, assume that information is symmetric after the game is completed, and the assumption is common knowledge of all the players. If members of each coalition are allied in the bargaining game on the distribution of its cooperative payoff, there exists the mixed strategic coalition equilibrium under the criterion of maximum expected cooperative payoff distribution in the information asymmetric cooperative game with agreements implemented by a third party:

$$\forall i = 1, 2, \cdots, n, c_i^* = \begin{cases} i, & \text{if for any } c_i,\ x_i^{(i)}(i, c_{-i}^*) \geq x_i^{(i)}(c_i, c_{-i}^*), \\ \operatorname{argmax} x_i^{(i)}(c_i, c_{-i}^*), & \text{if at least for a certain } c_i \neq i,\ x_i^{(i)}(c_i, c_{-i}^*) > x_i^{(i)}(i, c_{-i}^*). \end{cases}$$

Proof. Assume that information is symmetric after the game is completed, and assume that the above assumption is common knowledge of all the players. If the members of each coalition are allied in the bargaining game, in the virtual game of any player *i*, the bargaining game of the coalition he belongs to will be an information symmetric game. If the coalition members are allied in the bargaining game, this allied bargaining game is an information symmetric cooperative game. In this cooperative game, there must exist the coalition equilibrium under the criterion of maximum cooperative payoff distribution, and there must exist the Nash equilibrium in the information symmetric non-cooperative game among cooperative teams Chen [5].

According to the information set of player *i*, the equilibrium threat-point choices of the cooperative teams in the above-mentioned allied bargaining game is the distribution scheme of the cooperation payoff of the cooperative teams in the coalition. Due to the pure strategic equilibrium threat-point choices of teams in the bargaining game is one and only, the cooperative payoff distribution that any team gets is one and only. In the same way, in the distribution process of the cooperative payoff distribution obtained by any first-level cooperative team. The cooperative payoff distribution of any second-level team is unique and so on. Finally, the cooperative payoff distribution of any member is unique.

According to the final information set of player *i* before the formation of the coalition equilibrium, if any player *j*'s, (including player *i* himself) expected cooperative payoff distribution obtained from the coalition is unique in coalition situation *c*, then the Nash equilibrium of the coalition-choosing game exists, that is, in the virtual game of player *i* there exists the (mixed strategic) coalition equilibrium under the criterion of maximum expected cooperative payoff distribution.

Theorem 9 In information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, assume that information is symmetric after the game is completed, and that the above assumption is common knowledge of all the players; when members of each coalition are allied in the bargaining game on the distribution of its cooperative payoff, there exists the mixed strategic coalition equilibrium under the criterion of maximum expected cooperative payoff distribution in the information asymmetric cooperative game with agreements implemented by a third party; and the coalition equilibrium above is equivalent to the mixed strategic coalition equilibrium when members of each coalition are unallied in the bargaining game.

Proof. In an information symmetric bargaining game, whether the members of each coalition are allied or not, the cooperative payoff distribution that any member gets is no less than his escape payoff; therefore, with other players' coalition-choosing strategies unchanged, a player's optimal coalition-choosing strategy under the criterion of maximum cooperative payoff distribution and the one under the criterion of minimum escape payoff are the same. Therefore, the criterion of maximum cooperative payoff distribution is equivalent to the criterion of minimum escape payoff. When members of each coalition are allied in the bargaining game, the coalition equilibrium of the information symmetric bargaining game is just the one under the criterion of minimum escape payoff. When members of each coalition are unallied in the bargaining game, the coalition equilibrium of the information symmetric bargaining game is just the one under the criterion of minimum escape payoff too.

In an information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, assume that information is symmetric after the game is completed, and that the above assumption is common knowledge of all the players. Whether members of each coalition are allied in the bargaining game or not. In an information asymmetric cooperative game with agreements implemented by a third party, there exists a mixed strategic coalition equilibrium under the criterion of maximum expected cooperative payoff distribution:

$$\forall i = 1, 2, \cdots, n, c_i^* = \begin{cases} i, & \text{if for any } c_i,\ x_i^{(i)}(i, c_{-i}^*) \ge x_i^{(i)}(c_i, c_{-i}^*), \\ \operatorname{argmax} x_i^{(i)}(c_i, c_{-i}^*), & \text{if at least for a certain } c_i \ne i,\ x_i^{(i)}(c_i, c_{-i}^*) > x_i^{(i)}(i, c_{-i}^*). \end{cases}$$

At the same point, there exists a mixed strategic coalition equilibrium under the criterion of minimum expected escape payoff:

$$\forall i = 1, 2, \cdots, n, c_i^* = \begin{cases} i, & \text{if for any } c_i,\ w_i^{(i)}(i, c_{-i}^*) \le w_i^{(i)}(c_i, c_{-i}^*), \\ \operatorname{argmin} w_i^{(i)}(c_i, c_{-i}^*), & \text{if at least for a certain } c_i \ne i,\ w_i^{(i)}(c_i, c_{-i}^*) < w_i^{(i)}(i, c_{-i}^*). \end{cases}$$

If the distribution scheme of each coalition meets the competitive distribution condition, the coalition equilibrium under the criterion of maximum expected cooperative payoff distribution is equivalent to the coalition equilibrium under the criterion of minimum expected escape payoff. That is to say,

$$\forall i = 1, 2, \cdots, n, c_i^* = \begin{cases} i, & \text{if for any } c_i,\ x_i^{(i)}(i, c_{-i}^*) \ge x_i^{(i)}(c_i, c_{-i}^*), \\ \operatorname{argmax} x_i^{(i)}(c_i, c_{-i}^*), & \text{if at least for a certain } c_i \ne i,\ x_i^{(i)}(c_i, c_{-i}^*) > x_i^{(i)}(i, c_{-i}^*). \end{cases}$$

$$\Leftrightarrow \forall i = 1, 2, \cdots, n, c_i^* = \begin{cases} i, & \text{if for any } c_i,\ w_i^{(i)}(i, c_{-i}^*) \le w_i^{(i)}(c_i, c_{-i}^*), \\ \operatorname{argmin} w_i^{(i)}(c_i, c_{-i}^*), & \text{if at least for a certain } c_i \ne i,\ w_i^{(i)}(c_i, c_{-i}^*) < w_i^{(i)}(i, c_{-i}^*). \end{cases}$$

Theorem 10 In an information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, assume that information is still asymmetric after the game is completed, and that the above assumption is common knowledge of all the players. When members of each coalition are allied in the bargaining game on the distribution of the cooperative payoff, the Nash equilibrium of the allied bargaining game of the coalition does not exist.

Proof. When members of each coalition are allied in the bargaining game on the distribution of its cooperative payoff, the optimal coalition-choosing strategy of any player is just one when coalition members are unallied in the bargaining games, that is to say, when members of each coalition are allied in the bargaining game, the coalition equilibrium of the information asymmetric cooperative game with agreements implemented by a third party is just one when coalition members are unallied in the bargaining games if the coalition equilibrium does exist.

Assuming that information is still asymmetric after the game is completed and that the above assumption is common knowledge of all the players, the Nash equilibrium of the unallied bargaining game of any coalition *C* does not exist. In the allied bargaining game of coalition *C*, the competition among cooperative teams replaces the competition among members. Since information is still asymmetric among the cooperative teams after the game is completed, the Nash equilibrium of the allied bargaining game of coalition *C* does not exist either.

Theorem 11 In an information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, assume that information is still asymmetric after the game is completed and that the above assumption is common knowledge of all the players. When members of each coalition are allied in the bargaining game, there is no coalition equilibrium under the criterion of maximum expected cooperative payoff distribution or the coalition equilibrium under the criterion of minimum expected escape payoff. Herein, the fact that there is no coalition equilibrium under the criterion of maximum expected cooperative payoff distribution in the information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party or that there is no coalition equilibrium under the criterion of minimum expected escape payoff in the information asymmetric cooperative game with agreements implemented by a third party does not mean that there is no form of cooperative coalition in the game. Some players with a high degree of information symmetry (after the completion of the cooperative game) may still establish cooperative coalitions that aim at exploiting the synergies among them and reach cooperative payoff distribution agreements with some kinds of compensation mechanisms. In addition, even if the degree of information asymmetry among the players is still high after the completion of the cooperative game, those who agree with each other on the synergy expectations and do not need distribution compensations (perhaps they can set up certain compensation mechanisms to benefit from the cooperation through the compensation mechanism they set up) may also reach some forms of distribution agreements and establish cooperative coalitions designed to take advantage of the synergy expectations among them. When the degree of information asymmetry among the players is still high after the completion of the cooperative game, in an information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, there is at least a coalition situation shown as follows, which is feasible.

Proof. When members of each coalition are allied in the bargaining game based on the distribution of its cooperative payoff, the coalition equilibrium of an information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party is just the one when coalition members are unallied in the bargaining game, if the coalition equilibrium does exist.

Assume that information is still asymmetric after the game is completed. When members of each coalition are unallied in the bargaining game, there is no coalition equilibrium under the criterion of maximum expected cooperative payoff distribution or the coalition equilibrium under the criterion of minimum expected escape payoff in an information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party. Therefore, when members of each coalition are allied in its bargaining game, there is no coalition equilibrium under the criterion of maximum expected cooperative payoff distribution, nor is there a coalition equilibrium under the criterion of minimum expected escape payoff in an information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party.

Theorem 8 and Theorem 9 show that, assuming the information is symmetric after the game is completed and that the above assumption is common knowledge of all the players, whether members of each coalition are allied in the bargaining game or not, the coalition equilibrium in an information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party is the same. Now, our question is, after the formation of the coalition equilibrium of the information asymmetric cooperative game with agreements implemented by a third party, if the members of some coalition are allied in the bargaining game, how does the coalition equilibrium of its bargaining game be formed? How will the cooperative payoff of the coalition be distributed?

4.2 *Coalition equilibrium of the bargaining game of a coalition and the distribution of its cooperative payoff*

In an information asymmetric cooperative game with agreements implemented by a third party, assume that information is symmetric after the game is completed and that the above assumption is common knowledge of all the players. We will review the coalition equilibrium of the bargaining game of a coalition and the distribution of the cooperative payoff when members of the coalition are allied in the bargaining game on the distribution of its cooperative payoff.

Theorem 12 In an information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, assume that information is symmetric after the game is completed and that the above assumption is common knowledge of all the players. Whether members of each coalition are allied in the bargaining game or not, the equilibrium public choice of strategic combination of the coalition is the same.

Proof. Assume that information is symmetric after the game is completed and that the above assumption is common knowledge of all the players, whether members of each coalition are allied in the bargaining game or not. The coalition equilibrium of information in an asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party is the same.

Assume that the coalition equilibrium of the above cooperative game is c^*, and that the coalition members are unallied in the bargaining game of some coalition C; in the non-cooperative game among coalitions in coalition equilibrium c^*, the equilibrium public choice of strategic combination of coalition C is s_C^{*1},

$$s_C^{*1} = {}^{\text{argmax}}_{s_C} V_C^{(i)}\left(s_C, s_{-c}^{*(i)}\right).$$

Assume that in the bargaining game of coalition C, the members are allied, and in coalition equilibrium c^* of the information asymmetric cooperative game with agreements implemented by a third party, in the non-cooperative game among coalitions the equilibrium public choice of strategic combination of coalition C is s_C^{*2},

$$s_C^{*2} = \underset{s_C}{\operatorname{argmax}} \sum_{M \in C} V_C^{(M)}\left(s_C, s_{-c}^{*(M)}\right)$$

$$= \underset{s_C}{\operatorname{argmax}} \sum_{i \in C} V_C^{(i)}\left(s_C, s_{-c}^{*(i)}\right),$$

where M is any cooperative team of coalition C.

According to the definition of Nash equilibrium, we have:

$$s_C^{*1} = s_C^{*2}.$$

The above theorem shows that in an information asymmetric cooperative game with agreements implemented by a third party, whether members of each coalition are allied in its bargaining game or not, the equilibrium public choice of a strategic combination of each coalition is the same, and each coalition obtains the same cooperative payoff. However, in the bargaining game, when the coalition members are allied, the coalition's cooperative payoff distribution scheme is different from the one when coalition members are unallied.

If coalition members are allied in the bargaining game, in the virtual game of any player i, after the coalition equilibrium of the bargaining game is formed, the negotiation among the cooperative teams replaces the negotiation among the coalition members.

According to the above analysis of the virtual unallied bargaining game, on the basis of the information set of player k_1, we get the equilibrium coalition-choosing strategy of player k_1, (the coalition-choosing strategy he plays in the coalition equilibrium of the information asymmetric cooperative game with agreements implemented by a third party), strategic combination $s_{C_k}^{*(k_1)}$ that "should" be adopted by coalition C_k, and the equilibrium public choice of strategic combination $s_{C_k}^{*(k_1, C_k)}$ of coalition C_k. In the virtual game of player k_1, if all the members of coalition C_k must be responsible for their own misjudgements after the game is completed, for member k_1 who believes in his own information set, coalition C_k should be able to obtain cooperative payoff $V_{C_k}^{(k_1)}\left(s_{C_k}^{*(k_1)}, s_{-C_k}^{*(k_1)}\right)$ according to his "correct" judgment of the equilibrium situation $\left(s_{C_k}^{*(k_1)}, s_{-C_k}^{*(k_1)}\right)$, and member k_1 should get "correct" cooperative payoff distribution $V_{k_1}^{(k_1)}\left(s_{C_k}^{*(k_1)}, s_{-C_k}^{*(k_1)}\right)$.

In the virtual game of player k_1, in the allied bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ of coalition C_k (where M is the member set of coalition C_k, and m is the number of the members of the coalition), coalition situation $t = (t_1, t_2, ..., t_m)$ in allied bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ is the situation when any member i of coalition C_k chooses to join cooperative team No. $t_i (1 \le t_i \le m)$, that is, team T_{t_i}.

Apparently, according to the definition of coalition situation in allied bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ some coalition situations are duplicated because the teams in these coalition situations are just different in orders. So, we set special arrangement rules below on team-choosing which can guarantee that all the coalition situations in the coalition situation set are unique.

Rules on team-choosing. In allied bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ of coalition C_k, assume that all the members of the coalition follow the rule below when they choose their cooperative teams:

$$t_1 = 1;$$

$$t_2 = 1,\ 2;$$

$$t_3 = 1,\ 2,\ 3\ (\text{if } t_2 \ne 2,\ t_3 \ne 2);$$

$$\ldots\ldots$$

$$t_i = 1,\ 2,\ \cdots,\ i\ \ (\text{if } t_j \ne j,\ t_i \ne j, j < i);$$

$$\ldots\ldots$$

$$t_m = 1,\ 2,\ \cdots,\ m\ \ (\text{if } t_j \ne j,\ t_m \ne j, j < m);$$

Therefore, we get unrepeated coalition situations in game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ in the virtual game of player k_1. Of course, the rule above doesn't mean that the members of the coalition C_k are deprived of their choice of options. When the members of coalition C_k set up the cooperative teams in a coalition situation, the competition among the m members (m is the number of members of coalition C_k) of coalition C_k in the unallied bargaining game is replaced by the competition among the m cooperative teams in the allied bargaining game. In the following, we will define the common payoff of a k-team set $M_{m_1, m_2, \ldots, m_k}$, which is a subset of the team set M of coalition C_k in the allied bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$.

Let $M_{m_1, m_2, \ldots, m_k}$ denote a subset consisting of cooperative teams $m_1, m_2, \cdots, m_k$ of team set M of coalition C_k in coalition situation t in the virtual allied bargaining game of player k_1 $M_{m_1, m_2, \ldots, m_k} \subseteq M (k \le m)$, the common payoff $\delta^{(k_1)}(M_{m_1, m_2, \ldots, m_k})$ of team set $M_{m_1, m_2, \ldots, m_k}$ is defined as following:

$$\delta^{(k_1)}(M_{m_1, m_2, \cdots, m_k}) = V^{(k_1)}_{M_{m_1, m_2, \cdots, m_k}} - \sum_{i=1}^{k} w^{(k_1)}_{m_i} - \sum \delta^{(k_1)}_{(2)}(M_{m_1, m_2, \cdots, m_k}) - \cdots - \sum \delta^{(k_1)}_{(k-1)}(M_{m_1, m_2, \cdots, m_k}),$$

where $V^{(k_1)}_{M_{m_1, m_2, \cdots, m_k}}$ is the cooperative payoff of coalition C_k when all the members in other teams except those in set $M_{m_1, m_2, \ldots, m_k}$ have withdrawn from the coalition and join the same coalition as a whole to maximize their escape payoff, while members of other coalitions keep their coalition-choosing strategies unchanged in the virtual allied bargaining game of player k_1; $\delta^{(k_1)}_{(j)}(M_{m_1, m_2, \cdots, m_k})$ is the sum of common payoffs of all the j-team subsets of $M_{m_1, m_2, \ldots, m_k}$ in the virtual allied bargaining game of player k_1. $\sum_{i=1}^{k} w^{(k_1)}_{m_i}$ is the sum of escape payoffs of all the members of the k-teams in set $M_{m_1, m_2, \ldots, m_k}$ in the virtual allied bargaining game of player k_1.

According to Chen [5], in the coalition situation t of the bargaining game in the virtual game of player k_1, the Nash equilibrium of the bargaining game among teams $m_1, m_2, \cdots, m_k$ about the common payoff distribution $\delta^{(k_1)}(M_{m_1, m_2, \ldots, m_k})$ is:

$$y^{*(k_1)}_{m_1} = \frac{1}{k} \delta^{(k_1)}(M_{m_i, m_j, m_k}), i = 1, 2, \cdots, k.$$

That is to say, the teams which belong to set $M_{m_1, m_2, \ldots, m_k}$ will get the same common payoff distribution.

So, in the virtual game of player k_1 in any coalition situation t of the allied bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ of coalition C_k about the distribution of the cooperative payoff surplus, the distribution that some cooperative team m_i can get from coalition C_k is:

$$y^{(k_1)}_{m_i} = w^{(k_1)}_{m_i} + \frac{1}{2} \sum_{\substack{m_j = 1 \\ m_j \ne m_i}}^{m} \delta^{(k_1)}(M_{m_i, m_j}) + \frac{1}{3} \sum_{\substack{m_j = 1 \\ m_j \ne m_i}}^{m} \sum_{\substack{m_k = 1 \\ m_k \ne m_i}}^{m_j - 1} \delta^{(k_1)}(M_{m_i, m_j, m_k}) + \cdots + \frac{1}{m} \delta^{(k_1)}(M_{1, 2, \cdots, m}).$$

The total distribution that team m_i can get is:

$$\begin{aligned} x^{(k_1)}_{m_i} &= y^{(k_1)}_{m_i} + w^{(k_1)}_{m_i} \\ &= w^{(k_1)}_{m_i} + \frac{1}{2} \sum_{\substack{m_j = 1 \\ m_j \ne m_i}}^{m} \delta^{(k_1)}(M_{m_i, m_j}) + \frac{1}{3} \sum_{\substack{m_j = 1 \\ m_j \ne m_i}}^{m} \sum_{\substack{m_k = 1 \\ m_k \ne m_i}}^{m_j - 1} \delta^{(k_1)}(M_{m_i, m_j, m_k}) + \cdots + \frac{1}{m} \delta^{(k_1)}(M_{1, 2, \cdots, m}). \end{aligned}$$

Thus, we get the Nash equilibrium of the allied bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ about the distribution of the cooperative payoff surplus among the cooperative teams formed in any coalition situation t.

After obtaining the equilibrium of the allied bargaining game among cooperative teams of coalition C_k in coalition situation t of the bargaining game in the virtual game of player k_1, we will continue to analyze the coalition equilibrium of bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ of coalition C_k in the virtual game of player k_1.

In the coalition equilibrium c^* of information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, assume that information is symmetric after the game is completed and that the above assumption is common knowledge of all the players. In the virtual game of player k_1, if the coalition situation $t^{*(k_1)} = (t_1^{*(k_1)}, t_2^{*(k_1)}, \ldots, t_n^{*(k_1)})$ of virtual bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ of player k_1 is feasible and the team-choosing strategy of each coalition member is the best response to the collective actions of other coalition members, coalition situation $t^{*(k_1)} = (t_1^{*(k_1)}, t_2^{*(k_1)}, \cdots, t_n^{*(k_1)})$ is called the coalition equilibrium under the criterion of maximum expected cooperative payoff distribution, that is,

$$\forall i = 1, 2, \cdots, m,$$

$$t_i^{*(k_1)} = \begin{cases} i, & \text{if for any } t_i,\ x_i^{(k_1)}\left(i, t_{-i}^{*(k_1)}\right) \ge x_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right), \\ \operatorname{argmax} x_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right), & \text{if at least for a certain } t_i \ne i,\ x_i^{(k_1)}\left(i, t_{-i}^{*(k_1)}\right) < x_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right). \end{cases}$$

Theorem 13 In coalition equilibrium c^* of information asymmetric cooperative game $\Gamma(N, \{S_i\}, \{u_i\})$ with agreements implemented by a third party, assume that information is symmetric after the game is completed and that the above assumption is common knowledge of all the players. In the virtual game of player k_1, in allied bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ of coalition C_k there exists a (mixed strategic) coalition equilibrium under the criterion of maximum expected cooperative payoff distribution:

$$\forall i = 1, 2, \cdots, m,$$

$$t_i^{*(k_1)} = \begin{cases} i, & \text{if for any } t_i,\ x_i^{(k_1)}\left(i, t_{-i}^{*(k_1)}\right) \ge x_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right), \\ \operatorname{argmax} x_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right), & \text{if at least for a certain } t_i \ne i,\ x_i^{(k_1)}\left(i, t_{-i}^{*(k_1)}\right) < x_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right). \end{cases}$$

At the same time, in allied bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ of coalition C_k there exists the (mixed strategic) coalition equilibrium under the criterion of minimum expected escape payoff:

$$\forall i = 1, 2, \cdots, m,$$

$$t_i^{*(k_1)} = \begin{cases} i, & \text{if for any } t_i,\ w_i^{(k_1)}\left(i, t_{-i}^{*(k_1)}\right) \le w_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right), \\ \operatorname{argmin} w_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right), & \text{if at least for a certain } t_i \ne i, w_i^{(k_1)}\left(i, t_{-i}^{*(k_1)}\right) > w_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right). \end{cases}$$

If the distribution rules of all teams satisfy the competitive distribution condition, the coalition equilibrium under the criterion of maximum expected cooperative payoff distribution in allied bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ of coalition C_k is equivalent to the coalition equilibrium under the criterion of minimum expected escape payoff in game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$. That is,

$$\forall i = 1, 2, \cdots, m,$$

$$t_i^{*(k_1)} = \begin{cases} i, & \text{if for any } t_i,\ x_i^{(k_1)}\left(i, t_{-i}^{*(k_1)}\right) \ge x_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right), \\ \operatorname{argmax} x_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right), & \text{if at least for a certain } t_i \ne i,\ x_i^{(k_1)}\left(i, t_{-i}^{*(k_1)}\right) < x_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right). \end{cases}$$

$$\Leftrightarrow t_i^{*(k_1)} = \begin{cases} i, & \text{if for any } t_i,\ w_i^{(k_1)}\left(i, t_{-i}^{*(k_1)}\right) \le w_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right), \\ \operatorname{argmin} w_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right), & \text{if at least for a certain } t_i \ne i, w_i^{(k_1)}\left(i, t_{-i}^{*(k_1)}\right) > w_i^{(k_1)}\left(t_i, t_{-i}^{*(k_1)}\right). \end{cases}$$

The proof of Theorem 13 is omitted.

In the virtual game of player k_1, after the formation of coalition equilibrium $t^{*(k_1)}$ of allied bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ of coalition C_k, the competition among the m members (where m is the number of members of coalition C_k) of coalition C_k in the unallied bargaining game is replaced by the competition among the m cooperative teams in the allied bargaining game. In coalition equilibrium $t^{*(k_1)}$ of allied bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$

of coalition C_k, assuming that the team-choosing strategies of the m members of coalition C_k are respectively $t_1^{*(k_1)}$, $t_2^{*(k_1)}$, ..., $t_m^{*(k_1)}$ obviously, due to the different information sets, for a coalition member $-k_1$ $-k_1 \neq k_1$, usually,

$$\left(t_1^{*(k_1)}, t_2^{*(k_1)}, \cdots, t_m^{*(k_1)}\right) \neq \left(t_1^{*(-k_1)}, t_2^{*(-k_1)}, \cdots, t_m^{*(-k_1)}\right).$$

That is to say, for different members of coalition C_k, due to their different information sets, the coalition equilibrium of their virtual bargaining games are different. Obviously, if player k_1 finds that his estimation of the coalition equilibrium of the bargaining game is different from the actual possible coalition equilibrium, his information set is obviously not complete enough. Therefore, player k_1 is motivated to collect further information, which is helpful for increasing his expected cooperative payoff distribution. Thus, before the cooperative teams sign the cooperation agreements, all coalition members will collect further information, such that

$$\left(t_1'^{*(k_1)}, t_2'^{*(k_1)}, \cdots, t_m'^{*(k_1)}\right) = \left(t_1'^{*(-k_1)}, t_2'^{*(-k_1)}, \cdots, t_m'^{*(-k_1)}\right) = \left(t_1^*, t_2^*, \cdots, t_m^*\right).$$

If the information set of a coalition member that we refer to is his final information set before the cooperation agreements are signed by cooperative teams, then

$$\left(t_1^{*(k_1)}, t_2^{*(k_1)}, \cdots, t_m^{*(k_1)}\right) = \left(t_1^{*(-k_1)}, t_2^{*(-k_1)}, \cdots, t_m^{*(-k_1)}\right) = \left(t_1^*, t_2^*, \cdots, t_m^*\right).$$

That is,

$$t^{*(k_1)} = t^{*(-k_1)} = t^*.$$

In coalition equilibrium t^* of the bargaining game of coalition C_k, let team set $M_{m_1, m_2, ..., m_k}$ denote a subset of team set M^* of coalition C_k, which consists of cooperative teams $m_1^*, m_2^*, \cdots, m_k^*$, $M_{m_1^*, m_2^*, \cdots, m_k^*} \subseteq M^*(k \leq m)$, common payoff $\delta^{(k_1)}(M_{m_1^*, m_2^*, \cdots, m_k^*})$ of team set $M_{m_1^*, m_2^*, \cdots, m_k^*}$ is defined as following:

$$\delta^{(k_1)}(M_{m_1^*, m_2^*, m_k^*}) = V^{(k_1)}_{M_{m_1^*, m_2^*, \cdots, m_k^*}} - \sum\nolimits_{i=1}^{k} w^{(k_1)}_{m_i^*} - \sum \delta^{(k_1)}_{(2)}(M_{m_1^*, m_2^*, m_k^*}) - \cdots - \sum \delta^{(k_1)}_{(k-1)}(M_{m_1^*, m_2^*, m_k^*}),$$

where $V^{(k_1)}_{M_{m_1^*, m_2^*, \cdots, m_k^*}}$ is the cooperative payoff of coalition C_k when all the members in other teams except those in the teams of set $M_{m_1^*, m_2^*, \cdots, m_k^*}$ have withdrawn from the coalition and join the same coalition as a whole to maximize their escape payoff, while members of other coalitions keep their coalition-choosing strategies unchanged; $\sum \delta^{(k_1)}_{(j)}(M_{m_1^*, m_2^*, \cdots, m_k^*})$ is the sum of common payoffs of all the j-team subsets of set $M_{m_1^*, m_2^*, \cdots, m_k^*}$; $\sum\nolimits_{i=1}^{k} w^{(k_1)}_{m_i^*}$ is the sum of the escape payoffs of all the members of the k-teams in set $M_{m_1^*, m_2^*, \cdots, m_k^*}$.

According to Chen [5], we can get the common payoff distribution rule of team set $M_{m_1^*, m_2^*, \cdots, m_k^*}$ in player k_1's virtual bargaining game of coalition C_k, which is shown below.

$$y^{*(k_1)}_{m_i^*} = \frac{1}{k} \delta^{(k_1)}(M_{m_1^*, m_2^*, m_k^*}), i = 1, 2, \cdots, k.$$

Therefore, in player k_1's virtual bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ of coalition C_k, the cooperative payoff distribution surplus that some cooperative team m_i^* of coalition C_k can get is:

$$y^{(k_1)}_{m_i^*} = \frac{1}{2} \sum\nolimits_{\substack{m_j^*=1 \\ m_j^* \neq m_i^*}}^{m} \delta^{(k_1)}(M_{m_i^*, m_j^*}) + \frac{1}{3} \sum\nolimits_{\substack{m_j^*=1 \\ m_j^* \neq m_i^*}}^{m} \sum\nolimits_{\substack{m_k^*=1 \\ m_k^* \neq m_i^*}}^{m_j^*-1} \delta^{(k_1)}(M_{m_i^*, m_j^*, m_k^*}) + \cdots + \frac{1}{m} \delta^{(k_1)}(M_{1, 2, \cdots, m}).$$

The total cooperative payoff distribution that some cooperative team m_i^* of coalition C_k can get is:

$$x_{m_i^*}^{(k_1)} = y_{m_i^*}^{(k_1)} + w_{m_i^*}^{(k_1)}$$

$$= w_{m_i^*}^{(k_1)} + \frac{1}{2}\sum_{\substack{m_j^*=1 \\ m_j^* \neq m_i^*}}^{m} \delta^{(k_1)}(M_{m_i^*, m_j^*}) + \frac{1}{3}\sum_{\substack{m_j^*=1 \\ m_j^* \neq m_i^*}}^{m} \sum_{\substack{m_k^*=1 \\ m_k^* \neq m_i^*}}^{m_j^*-1} \delta^{(k_1)}(M_{m_i^*, m_j^*, m_k^*}) + \cdots + \frac{1}{m}\delta^{(k_1)}(M_{1, 2, \cdots, m}).$$

Thus, we get the Nash equilibrium of coalition C_k's allied bargaining game $\Gamma^{(k_1)}(M, \{T_i\}, \{x_i^{(k_1)}\})$ about the distribution of its cooperative payoff, in the coalition equilibrium t^* of the allied bargaining game of coalition C_k in player k_1's virtual game.

If the members of a cooperative team are no less than 3, after the discussion of the cooperative payoff distribution process at the coalition level (that is, at the first level) bargaining game, we need to extend the above-mentioned distribution process and analyze the second level, third level, fourth level of bargaining games, step by step, before finally we'll get the distribution vector of a coalition with limited members.

We have player k_1's estimation of the coalition equilibrium of the allied bargaining game of a coalition, and his estimation of the cooperative payoff distributions obtained by cooperative teams in the coalition equilibrium of the bargaining game of the coalition in player k_1's virtual game. However, player k_1's estimation of the cooperative payoff distribution obtained by the cooperative team he belongs to is not the final actual distribution. After the game is completed, each cooperative team in the coalition C_k should be responsible for their misjudgement.

Assume that after the game is completed, the information among all players, including all the members of coalition C_k, is symmetric, and that all the members or teams of each coalition must be responsible for their own misjudgements. In coalition equilibrium c^* of the information asymmetric cooperative game with agreements implemented by a third party, assume that the equilibrium public choice of strategic combination of coalition C_k is $s_{C_k}^*$, and the actual public choices of strategic combination of other coalitions are denoted as $s_{-C_k}^{*T}$. The cooperative payoff actually obtained by coalition C_k after the game is completed is $V_{C_k}\left(s_{C_k}^*, s_{-C_k}^{*T}\right) = \sum_{i=1}^{m} u_i\left(s_{C_k}^*, s_{-C_k}^{*T}\right)$. This cooperative payoff is what coalition C_k actually can distribute.

Obviously, if there is no information asymmetry among the players after the game is completed, when all the members or teams of coalition C_k have no misjudgement of the equilibrium public choices of strategic combinations of other coalitions, namely:

$$s_{C_k}^* = s_{C_k}^{**} = {}^{\text{argmax}}_{s_{C_k}} V_{C_k}\left(s_{C_k}, s_{-C_k}^{*T}\right),$$

the distribution of the cooperative payoff of coalition C_k will be carried out according to the distribution rule in the information symmetric allied bargaining game of coalition C_k.

$$x_{m_i^*}^{**} = w_{m_i^*}^{**} + \frac{1}{2}\sum_{\substack{m_j^*=1 \\ m_j^* \neq m_i^*}}^{m} \delta^{**}(M_{m_i^*, m_j^*}) + \frac{1}{3}\sum_{\substack{m_j^*=1 \\ m_j^* \neq m_i^*}}^{m} \sum_{\substack{m_k^*=1 \\ m_k^* \neq m_i^*}}^{m_j^*-1} \delta^{**}(M_{m_i^*, m_j^*, m_k^*}) + \cdots + \frac{1}{m}\delta^{**}(M_{1, 2, \cdots, m}),$$

where $w_{m_i^*}^{**}$ is the escape payoff of cooperative team m_i^*, $\delta^{**}(M_{m_i^*, m_i^*})$, $\delta^{**}(M_{m_i^*, m_i^*, m_k^*})$, $\cdots$, $\delta^{**}(M_{1, 2, \cdots, m})$ are respectively the common payoffs of cooperative teams $M_{m_i^*, m_j^*}, M_{m_i^*, m_j^*, m_k^*}, \cdots, M_{1, 2, \cdots, m}$.

If not all the estimations of coalition C_k's members of the equilibrium public choices of a strategic combination of other coalitions are correct, then $s_{C_k}^* \neq s_{C_k}^{**}$ (correspondingly, $V_{C_k}\left(s_{C_k}^*, s_{-C_k}^{*T}\right) \neq V_{C_k}\left(s_{C_k}^{**}, s_{-C_k}^{*T}\right)$), coalition C_k's cooperative payoff $V_{C_k}\left(s_{C_k}^*, s_{-C_k}^{*T}\right)$ can be regarded as the "cooperation" result of coalition C_k starting from strategic combination $s_{C_k}^{**}$ and through the misjudgements of coalition C_k's equilibrium public choice of strategic combination and the equilibrium public choices of strategic combination of other coalitions. The distribution of the cooperative payoff $V_{C_k}\left(s_{C_k}^*, s_{-C_k}^{*T}\right)$ should be based on the distribution of the cooperative payoff $V_{C_k}\left(s_{C_k}^{**}, s_{-C_k}^{*T}\right)$ with an additional distribution of the "cooperative" payoff $V_{C_k}\left(s_{C_k}^*, s_{-C_k}^{*T}\right) - V_{C_k}\left(s_{C_k}^{**}, s_{-C_k}^{*T}\right)$ which is caused by the misjudgement "cooperation" of the members of coalition C_k.

Similar to the distribution of the "cooperative" payoff brought by misjudgement cooperation in the unallied bargaining game discussed in the previous section, in the allied bargaining game of coalition C_k, the cooperative payoff distribution of any member k_1 of coalition C_k in the misjudgement "cooperation" is:

$$x_{k_1}^* = w_{k_1}^* + \frac{1}{2}\sum_{\substack{q_j=1 \\ q_j \neq k_1}}^{m} \delta^*(M_{k_1, q_j}) + \frac{1}{3}\sum_{\substack{q_j=1 \\ q_j \neq k_1}}^{m} \sum_{\substack{q_k=1 \\ q_k \neq k_1}}^{q_j-1} \delta^*(M_{k_1, q_j, q_k}) + \cdots + \frac{1}{m}\delta^*(M_{1, 2, \cdots, m}).$$

Therefore, the total cooperative payoff distribution that cooperative team m_i^* obtains is:

$$x_{m_i^*} = x_{m_i^*}^{**} + \sum_{i \in m_i^*} x_i^*.$$

In the above, we have examined the coalition equilibrium of an information asymmetric cooperative game with agreements implemented by a third party and the cooperative payoff distribution scheme of a coalition, including the one formed in the unallied bargaining game and the one formed in the allied bargaining game, if information is symmetric among the players after the game is completed.

If information is still asymmetric among the players after the game is completed, there is no coalition equilibrium in an information asymmetric cooperative game with agreements implemented by a third party, and there is no Nash equilibrium in the information asymmetric bargaining game of any coalition, whether the coalition members are allied in the bargaining game or not.

5. Discussions

To explain cooperative behaviors when agreements are implemented by a third party, game theory should model the formation of cooperative coalitions, the determination of the strategic combinations of the coalitions, and the distribution of the cooperative payoffs of the coalitions under the following different circumstances:

(1) In a one-time cooperative game with agreements implemented by a third party, all the players are rational (that is, they always pursue the maximization of their expected utilities) and perfectly intelligent, and the information in the game is symmetric.

(2) In a one-time cooperative game with agreements implemented by a third party, all players are rational and perfectly intelligent, but the information in the game is asymmetric.

(3) In an infinitely-repeated cooperative game with agreements implemented by a third party, all the players are rational and perfectly intelligent, the players are asymptotically information-symmetric in the repeating process of the stage games, although the information is initially asymmetric.

(4) In an evolutionary cooperative game with agreements implemented by a third party, all the players are rational but imperfectly intelligent, and information is asymmetric in the game.

At present, the literature mainly focuses on information symmetric cooperative games with agreements implemented by a third party, especially the cooperative payoff distribution scheme of a coalition. Chen [1] provided a new way of analyzing an information symmetric cooperative game with agreements implemented by a third party. He examined the coalition formation in the game, defined and provided the existence proof of the coalition equilibrium, which is formed when the coalition-choosing strategy of each player who aims to maximize his cooperative payoff distribution from the coalition he belongs to is the best response to the collective actions of other players; Chen inherited Nash's idea that the distribution scheme of the cooperative payoff in a coalition is the equilibrium of the bargaining game among the coalition members, and defined the common payoff of a member set of the coalition, which is a part of the marginal contribution of this member set to the coalition and will disappear completely when any member or team of the member set withdraws from the coalition, and proved that the equilibrium of the bargaining game on the common payoff distribution is that each member obtains the same distribution share. Chen shows that in the bargaining game on the cooperative payoff distribution of a coalition, the cooperative payoff distribution obtained by each coalition member is the sum of the common payoff distributions he obtains from the different member sets he belongs to.

An information asymmetric cooperative game is quite different from an information symmetric one. Since the information set of a player is incomplete, he must estimate the strategy sets and payoff functions of other players on the basis of his own information set, so as to form his own virtual game. On this basis, he can estimate the strategic combination of each coalition and the cooperative payoff distribution he gets from the coalition he belongs to, and thus determine his coalition-choosing strategy. However, the basic methodology proposed by Chen [1] can still be well applied to the analysis of information asymmetric cooperative games with agreements implemented by a third party: The formation of the coalition equilibrium is the result of the choices of the players who pursue the maximization of their expected utilities, and the (expected) cooperative payoff of a coalition can always be decomposed into the common payoffs of different member sets. The equilibrium of the bargaining game of a coalition on the expected cooperative payoff distribution can easily be obtained by applying the distribution rule of common payoffs.

This paper applies the methodology for studying coalition formation and cooperative payoff distribution proposed by Chen [1] to analyze information asymmetric cooperative games with agreements implemented by a third party. Based on the analysis of the virtual games of the players, the main achievements of this paper include:

(1) Due to the different information sets of the players, each member of a coalition has different estimations of the optimal strategic combination of the coalition. This paper defines a coalition's public choice game on the strategic combination choice, proposes and provides the existence proof of the equilibrium of this public choice game.

(2) This paper defines the virtual game of an arbitrary player, and defines the equilibrium of the virtual game.

(3) This paper investigates the condition for the existence of the equilibrium of the bargaining game of a coalition on the distribution of its cooperative payoff and that of the coalition equilibrium in the information asymmetric cooperative game with agreements implemented by a third party, and proposes the existence proof of the coalition equilibrium when it does exist.

(4) This paper examines the distribution of the cooperative payoff of a coalition when information in the game is symmetric after the game is completed. Assuming that members are unallied in the bargaining game of a coalition on the distribution of its cooperative payoff, this paper presents the equilibrium of the unallied bargaining game and provides the proof of its existence. Assume that members are allied in the bargaining game, this paper presents the coalition equilibrium of the allied bargaining game, provides the proof of its existence, and at the same time shows the equilibrium scheme of the cooperative payoff of the coalition.

Although in this paper we have established theoretical models of a one-time information asymmetric cooperative game with agreements implemented by a third party, the discussion on information asymmetric cooperative games with agreements implemented by a third party is far from over. In an infinitely-repeated information asymmetric cooperative game, or, in an evolutionary cooperative game, can the coalition equilibrium, the equilibrium strategic combinations of the coalitions, and the equilibrium of the bargaining games of the coalitions on the distribution of their cooperative payoffs be achieved? Obviously, seeking cooperation equilibrium in a repeated game or an evolutionary game is precisely the reason for the long-term survival of many cooperative organizations.

6. Conclusions and future extensions

In this paper, we have examined a one-time information asymmetric cooperative game with agreements implemented by a third party, investigated the public choice game of a coalition on the strategic combination choice, the virtual game of an arbitrary player, the coalition formation, and the bargaining game on the distribution of the cooperative payoff of a coalition.

In the virtual game of a player, the optimal strategic combination of the coalition to which he belongs depends on his estimation of the strategic combination choices of other coalitions, and the strategic combination choice of another coalition, which is determined by the coalition members, depends on the members' estimations of the optimal strategic combinations of other coalitions. By backward induction, from the final level virtual game to the first level virtual game, we can finally get this player's estimation of the public choice of strategic combination of the coalition he belongs to in his virtual game, as well as his estimation of the public choice of strategic combination of other coalitions. That's to say, we get the equilibrium of his virtual game.

The members' estimations of the optimal strategic combination of the coalition they belong to are different because of their different information sets. In an information asymmetric cooperative game with agreements implemented by a third party, the strategic combination of a coalition is determined by its members through a public choice game under the unanimity rule on the strategic combination choice of the coalition. The Nash equilibrium of this public choice game is the strategic combination that maximizes the sum of the estimated cooperative payoffs of their coalition in the virtual games of all members.

In an information asymmetric cooperative game with agreements implemented by a third party, assume that information is symmetric after the game is completed. Whether the coalition members are allied in the bargaining games or not, there exists a mixed strategic Nash equilibrium in the bargaining game of each coalition, a mixed strategic coalition equilibrium under the criterion of maximum expected cooperative payoff distribution, and a mixed strategic coalition equilibrium under the criterion of minimum expected escape payoff in the information asymmetric cooperative game. If the distribution rule of each coalition meets the competitive distribution condition, the coalition equilibrium under the criterion of maximum expected cooperative payoff distribution is equivalent to the one under the criterion of minimum expected escape payoff.

Assume that information is still asymmetric after the game is completed; whether the coalition members are allied in the bargaining games or not, there exist no Nash equilibrium in the bargaining game of any coalition. At the same time, there is no coalition equilibrium in the information asymmetric cooperative game. Of course, this does not mean that there is no form of cooperative coalition in the information asymmetric cooperative game.

When members are unallied in the bargaining game, at the coalition equilibrium of the information asymmetric cooperative game if it does exist (that is to say, the information is symmetric after the game is completed), the actual cooperative payoff of a coalition can be regarded as the “cooperation” result of the members of the coalition starting from the coalition’s actual optimal strategic combination and through the misjudgements of the equilibrium public choice of strategic combination of the coalition and those of other coalitions. The distribution of the coalition’s actual cooperative payoff should be based on the distribution of the maximum cooperative payoff at the optimal strategic combination that the coalition “should” adopt, with an additional distribution of the “cooperative” payoff caused by the misjudgement of “cooperation” between the members. If information is still asymmetric after the game is completed, the equilibrium of the unallied bargaining game of the coalition does not exist.

In an information asymmetric cooperative game with agreements implemented by a third party, assume that information is symmetric after the game is completed. When coalition members are allied in the bargaining games, there exists the coalition equilibrium under the criterion of minimum expected escape payoff, or the coalition equilibrium under the criterion of minimum expected escape payoff which is equivalent to the former. In the bargaining game among the allied teams of a coalition, the distribution of the coalition’s actual cooperative payoff should similarly be based on the distribution of the maximum cooperative payoff at the optimal strategic combination that the coalition “should” adopt, with an additional distribution of the “cooperative” payoff caused by the misjudgement of “cooperation” between all the teams. If information is still asymmetric after the game is completed, the coalition equilibrium and the distribution equilibrium of the allied bargaining game of a coalition do not exist.

An information asymmetric cooperative game with agreements implemented by a third party may be one-time or infinitely repeated. An important future extension of this paper should be the infinitely-repeated information symmetric cooperative game models. At the same time, the players in the game are not always intelligent. In an evolutionary cooperative game, how are the coalitions formed? What strategic combination would a coalition choose? And, how is the cooperative payoff of a coalition distributed? Another future extension of this paper should be the evolutionary cooperative game models.

In a cooperative game, agreements aren’t always implemented by a third party, and may be implemented by the coalition members themselves. Therefore, an important direction of future research is towards the study of cooperative games with agreements self-implemented, when the information in the game is symmetric or asymmetric, or, when the game is one-time or infinitely repeated.

Acknowledgments

The author is very grateful to the reviewers for their detailed and constructive comments which have helped improve the presentation of the paper.

Conflict of interest

There is no conflict of interest in this study.

References

[1] Harsanyi JC. Games with incomplete information played by "Bayesian" players, I-III. Part I. The basic model. *Management Science*. 1967; 14(3): 159-182. Available from: http://www.jstor.org/stable/2628393.

[2] Harsanyi JC. Games with incomplete information played by "Bayesian" players, I-III. Part II. Bayesian Equilibrium Points. *Management Science*. 1968; 14(5): 320-334. Available from: http://www.jstor.org/stable/2628673.

[3] Harsanyi JC. Games with incomplete information played by "Bayesian" players, I-III. Part III. The basic probability distribution of the game. *Management Science*. 1968; 14(7): 486-502. Available from: https://www.jstor.org/stable/2628894.

[4] Kadane J, Larkey P. Subjective probability and the theory of games. *Management Science*. 1982; 28(2): 113-219. Available from: https://doi.org/10.1287/mnsc.28.2.113.

[5] Chen J. Cooperative game with agreements implemented by a third party. *International Game Theory Review*. 2021; 23(3): 1-50. Available from: https://doi.org/10.1142/S0219198920500218.

[6] Aumann RJ, Maschler M, Stearns RE. *Repeated Games of Incomplete Information*. 1st ed. Massachusetts: MIT Press; 1995.

[7] Mertens JF, Sorin S, Zamir S. *Repeated games*. Cambridge: Cambridge University Press; 2015. Available from: https://doi.org/10.1017/CBO9781139343275.

[8] Novikov D, Chkhartishvili A. Models of reflexive games in control problems of ecological-economic system. In: Mazalov V, Novikov D, Ougolnitsky G, Petrosyan L. (eds.) *Game-theoretic Models in Mathematical Ecology*. New York: Nova Science Publishers; 2015. p.167-174. Available from: https://www.researchgate.net/publication/285601155.

[9] Konishi H, Ray D. Coalition formation as a dynamic process. *Journal of Economic Theory*. 2003; 110(1): 1-41. Available from: https://doi.org/10.1016/S0022-0531(03)00004-8.

[10] Hyndman K, Ray D. Coalition formation with binding agreements. *The Review of Economic Studies*. 2007; 74(4): 1125-1147. Available from: https://doi.org/10.1111/j.1467-937X.2007.00450.x.

[11] Gomes AR, Jehiel P. Dynamic processes of social and economic interactions: on the persistence of inefficiencies. *Journal of Political Economy*. 2005; 113(3): 626-667. Available from: https://doi.org/10.2139/ssrn.279303.

[12] Gomes A. Multilateral contracting with externalities. *Econometrica*. 2005; 73(4): 1329-1350. Available from: https://doi.org/10.1111/j.1468-0262.2005.00617.x.

[13] Aumann RJ. Repeated games. In: Feiwel GR. (ed.) *Issues in Contemporary Microeconomics and Welfare*. London: Palgrave Macmillan; 1985. p.209-242. Available from: https://doi.org/10.1007/978-1-349-06876-0_5.

[14] Noguchi M. Alpha cores of games with nonatomic asymmetric information. *Journal of Mathematical Economics*. 2018; 75: 1-12. Available from: https://doi.org/10.1016/j.jmateco.2017.12.005.

[15] Kamishiro Y. *New Approaches to Cooperative Game Theory: Core and Value.* PhD thesis. Brown University; 2010.

[16] Myerson RB. Cooperative games with incomplete information. In: Glycopantis D, Yannelis NC. (eds.) *Differential Information Economies*. Studies in Economic Theory. Vol. 19. Heidelberg: Springer, Berlin, Heidelberg; 2005. p.481-505. Available from: https://doi.org/10.1007/3-540-26979-7_25.

[17] Masuya S. The Shapley value on a class of cooperative games under incomplete information. In: Czarnowski I, Caballero A, Howlett R, Jain L. (eds.) *Intelligent Decision Technologies*. Smart innovation, system, and technologies (SIST). Vol. 56. Switzerland: Springer, Cham; 2016. p.129-139. Available from: https://doi.org/10.1007/978-3-319-39630-9_11.

[18] Masuya S, Inuiguchi M. Toward the theory of cooperative games under incomplete information. In: Torra V, Narukawa Y, Inuiguchi M. (eds.) *Modeling Decisions for Artificial Intelligence*. Lecture Notes in Computer Science. Vol. 5861. Heidelberg: Springer, Berlin; 2009. p.102-113. Available from: https://doi.org/10.1007/978-3-642-04820-3_10.

[19] Masuya S. The Shapley value and consistency axioms of cooperative games under incomplete information.

In: Czarnowski I, Howlett R, Jain L. (eds.) *Intelligent Decision Technologies*. Smart innovation, systems and technologies (SIST). Vol 72. Switzerland: Springer, Cham; 2018. p.3-12. Available from: https://doi.org/10.1007/978-3-319-59421-7_1.

[20] Lin J, Zhang Q. The L-nucleolus and I-Shapley value for cooperative games under incomplete information. *Journal of Beijing University of Technology*. 2015; 35(7): 767-770. Available from: http://doi.org/10.15918/j.tbit1001-0645.2015.07.021.

[21] Petrosjan LA. Cooperative dynamic games with incomplete information. In: Haurie A, Zaccour G. (eds.) *Dynamic games: Theory and Applications*. Massachusetts: Springer, Boston; 2005. p.111-134. Available from: https://doi.org/10.1007/0-387-24602-9_6.

[22] Aumann RJ, Maschler M. The bargaining set for cooperative games. *Advances in Game Theory*. Vol. 52. Princeton: Princeton University Press; 1964. p.443-476. Available from: https://doi.org/10.1515/9781400882014-022.

[23] Gillies DB. S*ome Theorems on N-Person Games*. PhD thesis. Princeton University; 1953.

[24] Shapley LS. A value for N-person games. In: Kuhn H, Tucker AW. (eds.) *Contributions to the Theory of Games*. Vol. 2. Princeton: Princeton University Press; 1953. p.307-317. Available from: https://doi.org/10.1515/9781400881970-018.

[25] Von Neumann J, Morgenstern O. *Theory of Games and Economic Behavior*. Princeton: Princeton University Press; 1944. Available from: https://psycnet.apa.org/record/1945-00500-000.

[26] Harsanyi JC. An equilibrium-point interpretation of stable sets and a proposed alternative definition. *Management Science*. 1974; 20(11): 1413-1496. Available from: https://doi.org/10.1287/mnsc.20.11.1472.

[27] Aumann R, Myerson R. Endogenous formation of links between players and of coalitions: An application of the Shapley value. In: Roth A. (ed.) *The Shapley Value: Essays in Honor of Lloyd S. Shapley*. Cambridge: Cambridge University Press; 1988. p.175-192. Available from: https://doi.org/10.1017/CBO9780511528446.013.

[28] Chwe M. Farsighted coalitional stability. *Journal of Economic Theory*. 1994; 63(2): 299-325. Available from: https://doi.org/10.1006/jeth.1994.1044.

[29] Ray D, Vohra R. Equilibrium binding agreements. *Journal of Economic Theory*. 1997; 73(1): 30-78. Available from: https://doi.org/10.1006/jeth.1996.2236.

[30] Diamantoudi E, Xue L. Farsighted stability in hedonic games. *Social Choice and Welfare*. 2003; 21: 39-61. Available from: https://doi.org/10.1007/s00355-003-0200-7.

[31] Shapley LS. *Notes on the N-person Game, III: Some Variants of the Von Neumann-Morgenstern Definition of Solution*. The RAND Corporation; 1952. Available from: https://doi.org/10.7249/RM817.

[32] Schmeidler D. The nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics*. 1969; 17(6): 1163-1170. Available from: https://doi.org/10.1137/0117107.

[33] Nash JF. Two-person cooperative games. *Econometrica*. 1953; 21(1): 128-140. Available from: https://doi.org/10.2307/1906951.

[34] Shang Y. A system model of three-body interactions in complex networks: Consensus and conservation. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*. 2022; 478(2258): 20210564. Available from: https://doi.org/10.1098/rspa.2021.0564.

[35] Shang Y. Resilient consensus for robust multiplex networks with asymmetric confidence intervals. *IEEE Transactions on Network Science and Engineering*. 2001; 8(1): 65-74. Available from: https://doi.org/10.1109/TNSE.2020.3025621.

[36] Qi Q, Qiu Z, Wang X, Ji Z. Asymmetric information control for stochastic systems with different intermittent observations. *Applied Mathematics and Computation*. 2022; 412(1): 126544. Available from: https://doi.org/10.1016/j.amc.2021.126544.