Contemporary Mathematics
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UNIVERSAL WISER PUBLISHER

Research Article

Application of Sumudu Transform and New Iterative Method to Solve Certain Nonlinear Ordinary Caputo Fractional Differential Equations

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Received: 25 February 2023; **Revised:** 25 April 2023; **Accepted:** 17 May 2023

Abstract: By combining the new iteration method (NIM) and Sumudu transform (ST) methods, together known as the new iteration ST method (NISTM), a semi-analytical or series solution for various fractional differential equations, in which new iteration method (NIM) is used to decompose the nonlinear operator prior to performing the ST, is produced in this script. Using the Caputo sense to account for the fractional derivative, this method computes series solutions for a variety of nonlinear fractional differential equation instances. While in many convergence issues a solution is achieved for just a small number of series terms, our series solution merges with the exact solution differential equation of fractional order in many example problems.

Keywords: sumudu transformation, new iteration method, fractional ordinary differential equations

MSC: 34A99

1. Introduction

Fractional calculus is an extension of differentiation as well as integration to pseudo-integer orders [1]. Laplace defined a fractional derivative of an arbitrary order; later on, many mathematicians provided the definition of a noninteger order of derivative or integral, the most popular of which was the Riemann-Liouville and Caputo definition. The majority of fractional differential equations, however, are extremely challenging for us to solve precisely, necessitating the use of numerical and approximation approaches. Integral transformation, including Laplace, Fourier, Mellin, and Hankel, is one of the approaches that have so far been employed to solve linear and nonlinear fractional differential equations. The Sumudu transform (ST), a novel integral transform introduced by Watugala [2, 3], was later used to solve issues with ordinary differential equations in control engineering. For more information and characteristics regarding STs, see [4–11] and a number of other sources. Numerous writers have employed the Sumudu transformation to solve differential equations of rational order, including both ordinary [12, 13] and partial differential equations [14–17].

The ST and new iteration method (NIM) have been incorporated by Kumar and Daftardar-Gejji [18] to precisely solve fractional partial differential equations. By employing the ST iterative approach, Wang and Liu [19] were able to resolve the time-fractional Cauchy reaction-diffusion problem.

Motivated and inspired by the current studies in this field, certain of Caputo's fractional ordinary differential equations will be solved in this script using a novel approximate technique called the new iteration ST method (NISTM).

DOI: https://doi.org/10.37256/cm.5420242561

The advantage of this new method, which we proposed, is that it makes the calculation simple and highly accurate to approximate the exact solution. The research is divided into five parts: (1) introduction; (2) preliminary material on fractional calculus; (3) description of the technique; (4) application to numerical problems; and (5) conclusion.

2. Preliminary material on fractional calculus

We provide some definitions as well as properties of fractional calculus and ST at a glance.

2.1 *Definition 1*

The fractional derivative of order β of $f(t) \in C^n[a,\ b]$ defined in Caputo's sense as

$$D^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-p)^{n-\beta-1} f^n(p) dp,\ n-1 < \beta \leq n,\ n \in N, \tag{1}$$

and in Riemann-Liouville's sense as

$$D^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \left(\int_0^t (t-p)^{n-\beta-1} f(p) dp \right),\ n-1 < \beta \leq n,\ n \in N. \tag{2}$$

Where N and N_0 denote set of positive integers and nonnegative integers, respectively, and $C^n[a,\ b],\ (-\infty < a < b < +\infty)$ be the class of all n times (where $n \in N_0$) continuously differentiable functions.

Reimann-Liouville's and Caputo derivatives are related in the following way:

$$D^{\beta} f(t) = D_c^{\beta} f(t) + \sum_{k=0}^{n-1} \frac{t^{k-\beta}}{\Gamma(k-\beta+1)} f^k\left(0^+\right). \tag{3}$$

2.2 *Definition 2*

Let the set

$$X := \left\{ f(t) :\ \exists M,\ \tau_1,\ \tau_2 > 0,\ |f(t)| < Me^{\frac{|t|}{\tau_i}},\ \text{ if },\ t \in (-1)^i * [0,\ \infty) \right\}. \tag{4}$$

Now, define ST over the set of functions X as

$$\mathscr{S}[f(t)] = \bar{f}(u) = \int_0^{\infty} f(ut) e^{-t} dt,\ u \in (-\tau_1,\ \tau_2). \tag{5}$$

ST of n th order derivative where $n \in N$ is

$$\mathscr{S}\left[D^n f(t)\right] = \frac{1}{u^n} \left(\bar{f}(u) - \sum_{k=0}^{n-1} u^k D^n f(t) \Bigg|_{t=0} \right). \tag{6}$$

2.3 *Definition 2*

The ST of Caputo derivative of order β where β lies between $n-1<\beta\leq n$ is provided below:

$$\mathscr{S}\left[D^{\beta}f(t)\right]=\left(\frac{\bar{f}(u)}{u^{\beta}}-\sum_{k=0}^{n-1}\frac{f^{k}(0^{+})}{u^{\beta-k}}\right),\ -1<n-1<\beta\leq n. \tag{7}$$

The ST of Riemann-Liouville derivative of order β where β lies between $n-1<\beta\leq n$ is provided below:

$$\mathscr{S}\left[D^{\beta}f(t)\right]=\frac{1}{u^{\beta}}\left(\bar{f}(u)-\sum_{k=1}^{n}u^{\beta-k}\left[D^{\beta-k}f(t)\right]_{t=0}\right). \tag{8}$$

2.4 *Definition 4: Mittag-leffler function*

$$\varepsilon_{\alpha,\ \beta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)},\ z,\ \beta\in C,\ \text{Real}(\alpha)>0. \tag{9}$$

The n th order derivative of Mittag-Leffler functions is given below:

$$\frac{d^{n}}{dz}\varepsilon_{\alpha,\ \beta}(z)=n!\ \varepsilon_{\alpha,\ \beta+2n}^{n+1}(z)\cdot n\in Z. \tag{10}$$

3. Proposed method's algorithm

Let us consider the fractional differential equation in the following form:

$$D^{\beta}f(t)+\mathscr{R}f(t)+\mathscr{N}f(t)=h(t),\ \beta>0 \tag{11}$$

$$f^{k}(t)\Big|_{t=0}=a_{k},\ k=0,\ 1,\ 2\ldots\ n-1,\ n-1<\beta\leq n. \tag{12}$$

Here, $\mathscr{N}f(t)$ is considered the nonlinear part of the given fractional differential equation, while $\mathscr{R}f(t)$ is the remaining part containing linear order operator terms. $h(t)$ is another function over the set X. In order to find the solution to the above equation, take ST, and we obtain

$$\mathscr{S}\left[D^{\beta}f(t)+\mathscr{R}f(t)+\mathscr{N}f(t)\right]=\mathscr{S}[h(t)],$$

$$\mathscr{S}\left[D^{\beta}f(t)\right]+\mathscr{S}[Rf(t)]+\mathscr{S}[\mathscr{N}f(t)]=\mathscr{S}[h(t)],$$

$$\frac{1}{u^{\beta}}(f(u)-a)+\mathscr{S}[\mathscr{R}f(t)]+\mathscr{S}[\mathscr{N}f(t)]=\mathscr{S}[h(t)],$$

where, $a = \sum_{k=0}^{n-1} \frac{f^k(0^+)}{u^{-k}}$, by using (7)

$$\Rightarrow f(u) = a + u^\beta (\mathscr{S}[h(t)] - \mathscr{S}[\mathscr{R} f(t)] - \mathscr{S}[\mathscr{N} f(t)]),$$

$$f(u) = a + u^\beta \mathscr{S}[h(t)] - u^\beta \mathscr{S}[\mathscr{R} f(t)] - u^\beta \mathscr{S}[\mathscr{N} f(t)].$$

Applying inverse ST to the above, we have

$$f(t) = \mathscr{S}^{-1}(a) + \mathscr{S}^{-1}\left(u^\beta \mathscr{S}[h(t)]\right) - \mathscr{S}^{-1}\left(u^\beta \mathscr{S}[\mathscr{R} f(t)]\right) - \mathscr{S}^{-1}\left(u^\beta \mathscr{S}[\mathscr{N} f(t)]\right). \tag{13}$$

Analyzing the answer by breaking it down into series components, i.e.,

$$f(t) = \sum_{i=0}^{\infty} f_i(t).$$

We decompose the term $\mathscr{N} f(t)$ by new iterative method as $G_0 = \mathscr{N}(f_0)$ and

$$\mathscr{N} f(t) = \sum_{n=0}^{\infty} G_n = \sum_{n=0}^{\infty} \left(\mathscr{N}\left(\sum_{i=0}^{n} f_i\right) - \mathscr{N}\left(\sum_{i=0}^{n-1} f_i\right)\right),\ n = 1,\ 2,\ \ldots.$$

Then, equation (13) becomes

$$\sum_{i=0}^{\infty} f_i(t) = \mathscr{S}^{-1}(a) + \mathscr{S}^{-1}\left(u^\beta \mathscr{S}[h(t)]\right) - \mathscr{S}^{-1}\left(u^\beta \mathscr{S}\left[\mathscr{R}\sum_{i=0}^{\infty} f_i(t)\right]\right) - \mathscr{S}^{-1}\left(u^\beta \mathscr{S}\left[\sum_{n=0}^{\infty} G_n\right]\right). \tag{14}$$

Recursively, the following relationship was found:

$$\begin{aligned} &f_0(t) = K(t), \\ &f_n(t) = -\mathscr{S}^{-1}\left(u^\beta \mathscr{S}\left[\mathscr{R} f_{n-1}(t)\right]\right) - \mathscr{S}^{-1}\left(u^\beta \mathscr{S}\left[G_{n-1}\right]\right),\ n = 1,\ 2,\ \ldots. \end{aligned} \tag{15}$$

In order to make it more convenient, we are interested in further decomposing the initial approximation $f_0(t)$ into $K_1(t)$ and $K_2(t)$, i.e., $f_0(t) = K(t) = K_1(t) + K_2(t)$. Consequently, a revised iteration plan is shown below:

$$\begin{aligned} &f_0(t) = K_1(t) \\ &f_n(t) = K_2(t) - \mathscr{S}^{-1}\left(u^\beta \mathscr{S}\left[\mathscr{R} f_{n-1}(t)\right]\right) - \mathscr{S}^{-1}\left(u^\beta \mathscr{S}\left[G_{n-1}\right]\right),\ n = 1,\ 2,\ \ldots \end{aligned} \tag{16}$$

4. Application to numerical problems

4.1 *Example 1*

Consider the following fractional order initial value problem, whose exact solution is $\tanh(t)$ for $\alpha = 1$.

$$\frac{d^{\alpha}y}{dt^{\alpha}} = 1 - y^2,\ y(0) = 0,\ t \geq 0,\ 0 < \alpha \leq 1. \tag{17}$$

Apply ST on both sides of equation (17)

$$\mathscr{S}\left[D^{\alpha}y(t)\right] = \mathscr{S}[1] - \mathscr{S}\left[y^2\right],$$

$$\frac{y(u)}{u^{\alpha}} - \frac{y(0)}{u^{\alpha}} = 1 - \mathscr{S}[\mathscr{N}(y)], \tag{18}$$

$$y(u) = u^{\alpha} - u^{\alpha}\mathscr{S}[\mathscr{N}(y)],$$

where $\mathscr{N}(y)$ represents nonlinear term of above equation. Now, taking inverse ST

$$y(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}[\mathscr{N}(y)]\right]. \tag{19}$$

Decomposing of nonlinear terms $\mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}[\mathscr{N}(y)]\right]$ is done by NIM, as follows. Let this equation has series solution as:

$$y(t) = y_n = \sum_{i=0}^{\infty} y_i(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[\sum_{n=0}^{\infty} G_n\right]\right]. \tag{20}$$

Let $y_0 = \dfrac{t^{\alpha}}{\Gamma(\alpha+1)}$

$$y_{n+1}(t) = -\mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}[G_n]\right],\ n = 0,\ 1,\ 2\ldots \tag{21}$$

$$y_1 = -\mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}[G_0]\right] = -\mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}[\mathscr{N}y_0]\right] = -\mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[\frac{t^{2\alpha}}{(\Gamma(\alpha+1))^2}\right]\right] \tag{22}$$

$$= -\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$y_2 = \mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[G_1\right]\right] = \mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[N\left(y_0+y_1\right)-N\left(y_0\right)\right]\right]$$

$$= \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha} - \frac{(\Gamma(2\alpha+1))^2\Gamma(6\alpha+1)}{(\Gamma(\alpha+1))^4(\Gamma(3\alpha+1))^2\Gamma(7\alpha+1)}t^{7\alpha}.$$

Hence, the series solution can be written as:

$$\begin{aligned} y(t) = y_0+y_1+y_2+\ldots = & \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ & + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha} \\ & - \frac{(\Gamma(2\alpha+1))^2\Gamma(6\alpha+1)}{(\Gamma(\alpha+1))^4(\Gamma(3\alpha+1))^2\Gamma(7\alpha+1)}t^{7\alpha} + \ldots. \end{aligned} \tag{23}$$

Put $\alpha = 1$ in the above equation,

$$y(t) = t - \frac{t^3}{3} + \frac{2t^5}{15} - \ldots = \tanh(t), \text{ exact solution.} \tag{24}$$

4.2 *Example 1*

$$D^3y(t) + D^{\frac{5}{2}}y(t) + y^2(t) = t^4,\ y(0) = y'(0) = 0,\ y''(0) = 2,\ D = d/(dt). \tag{25}$$

Apply ST on the above equation,

$$\mathscr{S}\left[D^3y(t)\right] + \mathscr{S}\left[D^{\frac{5}{2}}y(t)\right] + \mathscr{S}\left[y^2(t)\right] = \mathscr{S}\left[t^4\right]$$

$$\left(\frac{y(u)}{u^{\frac{5}{2}}} - \frac{y(0)}{u^{\frac{5}{2}}} - \frac{y'(0)}{u^{\frac{3}{2}}} - \frac{y''(0)}{u^{\frac{1}{2}}}\right) + \mathscr{S}\left[D^3y(t)\right] + \mathscr{S}\left[y^2(t)\right] = 5!u^4$$

$$y(u) = 2u^2 + 5!u^{\frac{13}{2}} - u^{\frac{5}{2}}\mathscr{S}\left[D^3y(t)\right] - u^{\frac{5}{2}}\mathscr{S}[N(y)].$$

Taking the inverse transform, we get

$$y(t) = t^2 + \frac{5!}{\Gamma\frac{13}{2}}t^{\frac{13}{2}} - \mathscr{S}^{-1}\left[u^{\frac{5}{2}}\mathscr{S}\left[D^3y(t)\right]\right] - \mathscr{S}^{-1}\left[u^{\frac{5}{2}}\mathscr{S}[\mathscr{N}(y)]\right]. \tag{26}$$

Using decomposition in series with NIM:

$$\sum_{i=0}^{\infty} y_i(t) = t^2 + \frac{5!}{\Gamma\frac{13}{2}} t^{\frac{13}{2}} - \mathscr{S}^{-1}\left[u^{\frac{5}{2}} \mathscr{S}\left[\sum_{i=0}^{\infty} D^3 y_i(t)\right] + \mathscr{S}^{-1}\left[u^{\frac{5}{2}} \mathscr{S}\left[\sum_{n=0}^{\infty} G_n\right]\right]\right. \tag{27}$$

Let $y_0 = t^2$. The recurrence formula for other terms:

$$y_{n+1}(t) = \frac{5!}{\Gamma\frac{13}{2}} t^{\frac{13}{2}} - \mathscr{S}^{-1}\left[u^{\frac{5}{2}} \mathscr{S}\left[D^3 y_n(t)\right] + \mathscr{S}^{-1}\left[u^{\frac{5}{2}} \mathscr{S}\left[G_n\right]\right], \; n = 0,\; 1,\; 2\ldots\right. \tag{28}$$

$$y_1 = \frac{5!}{\Gamma\frac{13}{2}} t^{\frac{13}{2}} - \mathscr{S}^{-1}\left[u^{\frac{5}{2}} \mathscr{S}\left[D^3 y_0(t)\right] + \mathscr{S}^{-1}\left[u^{\frac{5}{2}} \mathscr{S}\left[G_0\right]\right]\right.$$

$$= \frac{5!}{\Gamma\frac{13}{2}} t^{\frac{13}{2}} - \mathscr{S}^{-1}\left[u^{\frac{5}{2}} \mathscr{S}\left[D^3\left(t^2\right)\right] - \mathscr{S}^{-1}\left[u^{\frac{5}{2}} \mathscr{S}\left[t^4\right]\right]\right.$$

$$= \frac{5!}{\Gamma\frac{13}{2}} t^{\frac{13}{2}} - \frac{5!}{\Gamma\frac{13}{2}} t^{\frac{13}{2}} = 0$$

$$y_2 = 0 = y_3 = y_4 = \ldots \tag{29}$$

$$y(t) = \sum_{0}^{\infty} y_i = t^2.$$

4.3 *Example 3*

Consider relaxation oscillation equation [19], whose exact solution is $\varepsilon_a(-\omega t)^a$.

$$D^\alpha y(t) + \omega^\alpha y(t) = 0,\; y(0) = 1,\; y'(0) = 0,\; t > 0,\; \omega > 0,\; 0 < \alpha < 2. \tag{30}$$

Applying ST on above equation:

$$\mathscr{S}\left[D^\alpha y(t)\right] + \mathscr{S}\left[\omega^\alpha y(t)\right] = 0,$$

$$\left(\frac{y(u)}{u^\alpha} - \frac{y(0)}{u^\alpha} - \frac{y'(0)}{u^{\alpha-1}}\right) + \mathscr{S}\left[\omega^\alpha y(t)\right] = 0,$$

$$y(u) + u^\alpha \mathscr{S}\left(\omega^\alpha y(t)\right) = 1.$$

Taking inverse ST and decomposing solution into series, we have

$$y=\sum_{i=0}^{\infty}y_i(t)=1-\mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[\omega^{\alpha}\sum_{i=0}^{\infty}y_i(t)\right]\right]. \tag{31}$$

The recurrence formula is given below:

$$y_0=1,$$

$$y_1=-\frac{\omega^{\alpha}t^{\alpha}}{\Gamma(\alpha+1)}=-\mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[\omega^{\alpha}y_0(t)\right.\right.$$

$$y_2=-\mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[\omega^{\alpha}y_1(t)\right]\right]=\frac{\omega^{2\alpha}t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$y(t)=y_0+y_1(t)+y_2(t)+\ldots=1-\frac{\omega^{\alpha}t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\omega^{2\alpha}t^{2\alpha}}{\Gamma(2\alpha+1)}-\ldots.$$

Therefore, as $n\to\infty,\ y_n(t)$ tends to the exact solution $t^4\mathscr{E}_{\alpha,\,5}(-t)^{\alpha}$, where ε is Mittag-Leffler function.

4.4 *Example 4*

$$D^{\alpha}y(t)=\frac{t^{4-\alpha}}{\Gamma(5-\alpha)}-y(t),\ 0<\alpha<2,\ t>0,\ y(0)=0. \tag{32}$$

Taking ST of the above equation:

$$\begin{aligned}&\mathscr{S}\left[D^{\alpha}y(t)\right]=\mathscr{S}\left[\frac{t^{4-\alpha}}{\Gamma(5-\alpha)}\right]-\mathscr{S}[y(t)]\\&\left(\frac{y(u)}{u^{\alpha}}-\frac{y(0)}{u^{\alpha}}\right)=u^{4-\alpha}-\mathscr{S}[y(t)]\\&y(u)=u^{\alpha}u^{4-\alpha}-u^{\alpha}\mathscr{S}[y(t)].\end{aligned} \tag{33}$$

Taking inverse ST and decomposing solution into series, we have:

$$y=\sum_{i=0}^{\infty}y_i(t)=\frac{t^4}{\Gamma(5)}-\mathscr{S}\left[\sum_{i=0}^{\infty}y_i(t)\right]. \tag{34}$$

Set the recurrence formula as:

$$y_0 = \frac{t^4}{\Gamma(5)}$$

$$y_{n+1}(t) = -\mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[y_n(t)\right]\right]$$

$$y_1 = -\mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[y_0(t)\right]\right] = -\frac{t^{4+\alpha}}{\Gamma(5+\alpha)} \tag{35}$$

$$y_2 = -\mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[y_1(t)\right]\right] = \frac{t^{4+2\alpha}}{\Gamma(5+2\alpha)}$$

$$y(t) = y_0 + y_1(t) + y_2(t) + \ldots = \frac{t^4}{\Gamma(5)} - \frac{t^{4+\alpha}}{\Gamma(5+\alpha)} + \frac{t^{4+2\alpha}}{\Gamma(5+2\alpha)} - \ldots.$$

Therefore, as $n \to \infty$, $y_n(t)$ tends to the exact solution $t^4\varepsilon_{\alpha,\,5}(-t)^{\alpha}$, where ε is Mittag-Leffler function.

4.5 *Example 5*

$$D^{\alpha}y(t) = 1 - y^2 + 2t,\ y(0) = 0,\ t \geq 0,\ 0 < \alpha \leq 1. \tag{36}$$

Taking ST on both sides, we get:

$$\mathscr{S}\left[D^{\alpha}y(t)\right] = \mathscr{S}[1] - \mathscr{S}\left[y^2\right] + \mathscr{S}[2t]$$

$$\frac{y(u)}{u^{\alpha}} - \frac{y(0)}{u^{\alpha}} = 1 - \mathscr{S}[\mathscr{N}(y)] + 2u$$

$$y(u) = u^{\alpha} + 2u^{\alpha+1} - u^{\alpha}\mathscr{S}[\mathscr{N}(y)].$$

Taking inverse ST

$$y(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}[\mathscr{N}(y)]\right]. \tag{37}$$

Decomposing by NIM, we get

$$\sum_{i=0}^{\infty} y_i(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[\sum_{n=0}^{\infty} G_n\right]\right].$$

Let $y_0 = \dfrac{t^{\alpha}}{\Gamma(\alpha+1)}$

$$y_{n+1}(t) = 2\frac{t^{\alpha+1}}{\Gamma(\alpha+1)} - \mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[G_n\right]\right],\ n = 0,\ 1,\ 2\ldots$$

$$y_1 = 2\frac{t^{\alpha+1}}{\Gamma(\alpha+1)} - \mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[G_0\right]\right]$$

$$y_1 = 2\frac{t^{\alpha+1}}{\Gamma(\alpha+1)} - \mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[\mathscr{N}y_0\right]\right]$$

$$y_1 = 2\frac{t^{\alpha+1}}{\Gamma(\alpha+1)} - \mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[\frac{t^{(2\alpha)}}{(\Gamma(\alpha+1))^2}\right]\right].$$

Solving above, we get

$$\begin{aligned} y_1 &= -\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ y_2 &= -\mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[G_1\right]\right] \\ y_2 &= -\mathscr{S}^{-1}\left[u^{\alpha}\mathscr{S}\left[N\left(y_0+y_1\right)-N\left(y_0\right)\right]\right]. \end{aligned} \tag{38}$$

Solving above, we get

$$\begin{aligned} y_2 = &-\frac{4}{\Gamma(\alpha+2)^2}\frac{\Gamma(2\alpha+3)}{\Gamma(2\alpha+4)}t^{3\alpha+2} - \frac{(\Gamma(2\alpha+1))^2\Gamma(6\alpha+1)}{(\Gamma(\alpha+1))^4(\Gamma(3\alpha+1))^2\Gamma(7\alpha+1)}t^{7\alpha} \\ &-\frac{4}{\Gamma(\alpha+1)\Gamma(\alpha+2)}\frac{\Gamma(2\alpha+2)}{\Gamma(3\alpha+2)}t^{3\alpha+1} \\ &+\frac{4\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(\alpha+2)\Gamma(5\alpha+1)}t^{5\alpha+1} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha}, \end{aligned}$$

and so on.

$$
\begin{aligned}
y(t) =& y_0 + y_1(t) + y_2(t) + \dots \\
=& \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{4}{\Gamma(\alpha+2)^2}\frac{\Gamma(2\alpha+3)}{\Gamma(2\alpha+4)}t^{3\alpha+2} \\
&- \frac{(\Gamma(2\alpha+1))^2\Gamma(6\alpha+1)}{(\Gamma(\alpha+1))^4(\Gamma(3\alpha+1))^2\Gamma(7\alpha+1)}t^{7\alpha} \\
&- \frac{4}{\Gamma(\alpha+1)\Gamma(\alpha+2)}\frac{\Gamma(2\alpha+2)}{\Gamma(3\alpha+2)}t^{3\alpha+1} + \frac{4\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(\alpha+2)\Gamma(5\alpha+1)}t^{5\alpha+1} \\
&+ \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha} + \dots.
\end{aligned}
$$

In this instance, a graphic representation for $\alpha = 1$ has been supplied by the closeness of the solution series' components to the precise solution (see Figure 1).

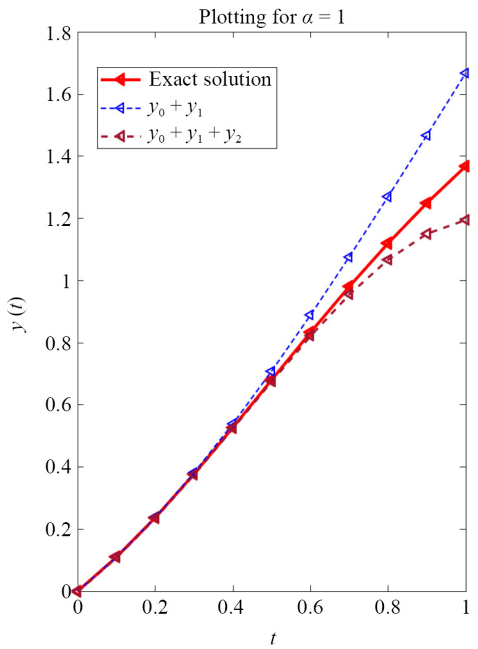

Figure 1. The proximity of series solution components towards the exact solution for $\alpha = 1$ (Example 5)

5. Conclusion and future work

ST with NIM is found to be a very effective semi-analytical method, in which series solutions are found, for solving nonlinear fractional differential equations. When $\alpha = 1$, we discovered that as the number of series components rises, the approximate solution's resemblance to the precise solution increases in Example 5 (see Figure 1). Therefore, it has been presented that utilizing NIM to decompose the nonlinear portion of the provided differential equation has increased the proximity of the approximate solution to the precise solution as well as rendering it easier to manage the nonlinearity. For further work, we will expand this method to higher-order nonlinear fractional differential equations, which commonly arise in the fields of engineering, physics, etc.

Conflict of interest

The authors declare no competing financial interest.

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