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Research Article

On Impulsive Nonlocal Nonlinear Fuzzy Integro-Differential Equations in Banach Space

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Abstract: The aim of this article is to investigate the existence, uniqueness and other qualitative properties of the solution of first-order nonlocal impulsive nonlinear fuzzy integro-differential equations in Banach space by using the concept of fuzzy numbers whose values are normal, upper semicontinuous, compact, and convex. The result is attained by utilizing a modified version of the Banach contraction principle. We offer an example as an application of the results.

***Keywords*:** nonlocal condition, fixed point, fuzzy nonlinear integro-differential equations, mild solution

MSC: 34A07, 47H10

1. Introduction

The theory and implementation of fuzzy systems have advanced significantly in many areas since Zadeh [1] first introduced the idea of fuzzy sets in 1965, particularly in the theory of fuzzy control systems. Fuzzy differential equations have been explored by many authors [2–5]. Fuzzy differential equations were studied for the first time by Kaleva [6]. In addition to presenting the existence and uniqueness theorem for a fuzzy differential equation solution, he also addressed the characteristics of differentiable fuzzy set value mappings. Zadeh's extension of a function with regard to a parameter and the independent variable is the fuzzy optimization problem, which is an objective function in [7].

Fuzzy integro-differential equations have earned notable in the theory of fuzzy analysis, which has made them occupy a valuable place in theory, application, measurement theory and control theory. Impulsive functional differential equations represent a significant area of study because these equations provide a suitable foundation for the mathematical modeling of many phenomena and real processes explored in electronics, optimal control, economics and other fields [8–11]. However, a nonlocal condition is better at describing natural events compared to a classical initial condition. In recent years, the Cauchy problem with the nonlocal condition has also attracted a lot of interest [12–15].

In Ramesh et al. [16] studied the existence and uniqueness of a solution of the fuzzy impulsive differential equation

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$$\rho'(\kappa) = \mathscr{P}(\kappa,\, \rho_\kappa)$$

$$\rho(\kappa_0) = \rho_0 \in \mathbb{X}^n,$$

$$\Delta\rho(\kappa_n) = \mathscr{I}_n\rho(\kappa_n),\, \kappa \neq t_n, \quad n = 1,\, 2,\, 3,\, \ldots,\, k,$$

by using the method of successive approximation. Then Benchohra et al. [17] studied existence of impulsive fuzzy differential equations by using a fixed point theorem for absolute retract.

In Vengataasalam et al. [18] studied the existence and uniqueness of the nonlocal impulsive fuzzy differential equation

$$\rho'(\kappa) = \mathscr{A}\rho(\kappa) + \mathscr{P}(\kappa,\, \rho_\kappa), \quad \kappa \in [0,\, a]$$

$$\Delta\rho(\kappa_n) = \mathscr{I}_n\rho(\kappa_n), \quad \kappa \neq t_n, \quad n = 1,\, 2,\, 3,\, \ldots,\, k,$$

$$\rho(0) = \mathfrak{h}(\kappa_1,\, \kappa_2,\, \ldots,\, \kappa_q,\, \rho(.)) + \rho_0,$$

by using the Banach fixed point theorem.

Motivated by the above work, in this article, we study the fuzzy nonlocal impulsive integro-differential equations as form:

$$\rho'(\kappa = \mathscr{A}\rho(\kappa) + \mathscr{P}(\kappa,\, \rho_\kappa,\, \int_0^\kappa \mathscr{H}(\kappa,\, \mu,\, \rho_\mu)d\mu), \quad \kappa \in (0,\, \mathscr{K}]$$

$$\Delta\rho(\kappa_n) = \mathscr{I}_n\rho(\kappa_n), \quad \kappa \neq t_n, \quad n = 1,\, 2,\, 3,\, \ldots,\, k,$$

$$\rho(\kappa) + \mathfrak{h}(\rho_{\sigma_1},\, \rho_{\sigma_2},\, \ldots,\, \rho_{\sigma_q})(\kappa) = \psi(\kappa), \quad \kappa \in [-\mathfrak{r},\, 0], \tag{1}$$

where $\mathscr{A}$: $[0,\, \mathscr{K}] \to \mathbb{X}^n$ is the fuzzy coefficient, $\mathbb{X}^n$ is the set of all normal, convex, and upper semicontinuous fuzzy numbers with bounded α-levels, $\mathscr{P}$: $[0,\, \mathscr{K}] \times \mathscr{C}([-\mathfrak{r},\, 0],\, \mathbb{X}^n) \times \mathbb{X}^n \to \mathbb{X}^n$, $\mathscr{H}$: $[0,\, \mathscr{K}] \times [0,\, \mathscr{K}] \times \mathbb{X}^n \to \mathbb{X}^n$ and $\mathfrak{h}$: $(\mathscr{C}([-\mathfrak{r},\, 0],\, \mathbb{X}^n)^q \to \mathbb{X}^n$ are regular fuzzy nonlinear functions, $\mathscr{I}_n \in \mathscr{C}(\mathbb{X}^n,\, \mathbb{X}^n)$, and ψ: $[-\mathfrak{r},\, 0] \to \mathbb{X}^n$ are bounded functions. $\Delta\rho(\kappa_n) = \rho(\kappa_n^+) - \rho(\kappa_n^-)$, $\rho(\kappa_n^+) = \lim_{h\to 0^+} \rho(t_n + h)$, $\rho(\kappa_n^-) = \lim_{h\to 0^+} \rho(t_n - h)$ represents the left and right limits of $\rho(\kappa)$ at $\kappa = t_n$, respectively, $n = 1,\, 2,\, \ldots\, k$. For any function ρ defined on $[-\mathfrak{r},\, \mathscr{K}]$ and any $\kappa \in [0,\, \mathscr{K}]$, we denote ρ_κ the element of $\mathscr{C}([-\mathfrak{r},\, 0],\, \mathbb{X}^n)$ defined by $\rho_\kappa(w) = \rho(\kappa + w)$; $w \in [-\mathfrak{r},\, 0]$. Here, $\rho_\kappa(.)$ represents the history of the state from time $\kappa - \mathfrak{r},$ up to the present time κ.

The objective of this article is to obtain the existence and uniqueness of a mild solution to equation (1). Note that here we are generalizing and improving the results mentioned in [16–18]. Also we are achieving better results by using a modified version of the Banach contraction theorem and impulsive inequality. Like in paper [17], hypothesis $(\mathscr{A}_1)$ is not required if we use our method.

The remainder of the article is structured as follows: In Section 2, we give the preliminaries and hypotheses. In Sections 3 and 4, we prove the existence, uniqueness, nearness, and convergence of the solution of first-order nonlocal impulsive nonlinear fuzzy integro-differential equations. In Section 5, we give an illustrative application of our results, and we conclude the results in Section 6.

2. Preliminaries and hypotheses

Let $P_r(\mathfrak{R}^n)$ be the family consisting of all nonempty, convex, and compact subsets of $\mathfrak{R}^n$. Denote by $\mathbb{X}^n = \{\vartheta\colon \mathfrak{R}^n \to [0, 1]$ such that ϑ satisfy (1)–(4) as bellow.

1) ϑ is normal, that is, there exists $\rho_0 \in \mathfrak{R}^n$ such that $\vartheta(\rho_0) = 1$.

2) ϑ is fuzzy convex, that is, for $\rho,\, \nu \in \mathfrak{R}^n$ and $0 < \lambda \leq 1$, $\vartheta(\lambda\rho + (1-\lambda)\nu) \geq \min\{\vartheta(\rho),\, \vartheta(\nu)\}$.

3) ϑ is upper semicontinuous.

4) $[\vartheta]^0 = \overline{\{\rho \in \mathfrak{R}^n\colon \vartheta(\rho) > 0\}}$ is compact.

For $0 < \alpha \leq 1$, $[\vartheta]^\alpha = \{\rho \in \mathfrak{R}^n\colon \vartheta(\rho) \geq \alpha\}$. Then from (1)–(4), it follows that the α- level sets $[\vartheta]^\alpha \in P_r(\mathfrak{R}^n)$.

If $\mathfrak{h}\colon \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ is a function, then by using Zadeh's extension principle, we can extend $\mathfrak{h}$ to $\mathbb{X}^n \times \mathbb{X}^n \to \mathbb{X}^n$ by the equation $[\mathfrak{h}(\vartheta,\, \sigma)(w)] = \sup_{w=\mathfrak{h}(\rho,\, \nu)} \min\{\vartheta(\rho),\, \sigma(\nu)\}$.

It is well knowledge that $[\mathfrak{h}(\vartheta,\, \sigma)]^\alpha = \mathfrak{h}([\vartheta]^\alpha,\, [\sigma]^\alpha)$, $\forall \vartheta,\, \sigma \in \mathbb{X}^n$, $0 \leq \alpha \leq 1$ and the function $\mathfrak{h}$ is a continuous. In addition, we have

$$[\vartheta + \sigma]^\alpha \quad = \quad [\vartheta]^\alpha + [\sigma]^\alpha, \quad [\mathfrak{a}\vartheta]^\alpha = \mathfrak{a}[\vartheta]^\alpha,$$

where

$$\vartheta,\, \sigma \in \mathbb{X}^n, \;\; 0 \leq \alpha \leq 1, \;\; \mathfrak{a} \in \mathfrak{R}.$$

Let $\Xi_1,\, \Xi_2 \neq \phi$ be bounded subsets of $\mathfrak{R}^n$. The Hausdorff metric is defined as follows

$$\mathscr{H}_d^*(\Xi_1,\, \Xi_2) \quad = \quad \max\left\{ \sup_{\xi_1 \in \Xi_1} \inf_{\xi_2 \in \Xi_2} \|\xi_1 - \xi_2\|,\, \sup_{\xi_2 \in \Xi_2} \inf_{\xi_1 \in \Xi_1} \|\xi_1 - \xi_2\| \right\}$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathfrak{R}^n$. Then $(P_r(\mathfrak{R}^n),\, \mathscr{H}_d^*)$ is a separable and complete metric space [19].

We define the complete metric d_∞ on $\mathbb{X}^n$ by

$$d_\infty^*(\vartheta,\, \sigma) \quad = \quad \sup_{0<\alpha\leq 1} \mathscr{H}_d^*([\vartheta]^\alpha,\, [\sigma]^\alpha) = \sup_{0<\alpha\leq 1} [\vartheta_l^\alpha - \sigma_l^\alpha,\, \vartheta_r^\alpha - \sigma_r^\alpha]$$

for all $\vartheta,\, \sigma \in \mathbb{X}^n$. $(\mathbb{X}^n,\, d_\infty^*)$ is a complete metric space. Also $\forall \vartheta,\, \sigma,\, \mu \in \mathbb{X}^n$ and $\lambda \in \mathfrak{R}$, we have $d_\infty^*(\vartheta + \mu,\, \sigma + \mu) = d_\infty^*(\vartheta,\, \sigma)$ and $d_\infty^*(\lambda\vartheta,\, \lambda\sigma) = |\lambda| d_\infty^*(\vartheta,\, \sigma)$.

We define $\hat{0} \in \mathbb{X}^n$ as $\hat{0}(\rho) = 1$ if $\rho = 0$ and $\hat{0}(\rho) = 0$ if $\rho \neq 0$. The supremum metric $\mathscr{H}_1$ on $C([0, 1],\, \mathbb{X}^n)$ is defined by

$$\mathscr{H}_1(\vartheta, \sigma) = \sup_{0\leq\kappa\leq\mathscr{K}} d_\infty^*(\vartheta(\kappa), \sigma(\kappa))$$

Hence $(\mathscr{C}([0, 1], \mathbb{X}^n), \mathscr{H}_1)$ is a complete metric space.

Definition 1 A family of functions $(\mathscr{D}(\kappa))_{\kappa\geq 0}$ of continuous linear operators on $\mathbb{X}^n$ is called fuzzy $\mathscr{C}_0$-semigroup if

1. For all $\rho \in \mathbb{X}^n$ the mapping $\mathscr{D}(\kappa)(\rho)$: $\mathfrak{R}^+ \to \mathbb{X}^n$ is continuous with respect to $\kappa \geq 0$,
2. $\mathscr{D}(\kappa+\mu) = \mathscr{D}(\kappa)\mathscr{D}(\mu)\ \ \forall\kappa, \mu \in \mathfrak{R}^+$,
3. $\mathscr{D}(0) = I$ where I is the identity operator on $\mathbb{X}^n$.

Definition 2 A continuous function $\rho(\kappa)$: $[0, \mathscr{K}] \to \mathbb{X}^n$ is said to be a mild solution of equation (1) if

$$\rho(\kappa) = \mathscr{D}(\kappa)[\psi(0) - \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(0)] + \int_0^\kappa \mathscr{D}(\kappa-\mu)\mathscr{P}(\mu, \rho_\mu, \int_0^\mu \mathscr{H}(\mu, \sigma, \rho_\sigma)d\sigma)d\mu$$

$$+ \sum_{0<\sigma_n<\kappa} \mathscr{D}(\kappa-\sigma_n)\mathscr{I}_n\rho(\sigma_n), \quad \kappa \in (0, \mathscr{K}]$$

$$\rho(\kappa) + \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(\kappa) = \psi(\kappa), \quad \kappa \in [-\mathfrak{r}, 0].$$

Lemma 1 ([20], p.12) Let a nonnegative piece-wise continuous function $\rho(\kappa)$ satisfies $\kappa \geq \kappa_0$ the inequality

$$\nu(\kappa) \leq \mathscr{B} + \int_{\kappa_0}^\kappa \rho(\mu)\nu(\mu)d\mu + \sum_{0<\sigma_n<\kappa} \beta_n\rho(\sigma_n)$$

where $\mathscr{B} \geq 0$, $\beta_n \geq 0$, $\rho(\kappa) > 0$, σ_n are the first kind discontinuity points of the function $\rho(\kappa)$. Then the following estimate holds for the function $\rho(\kappa)$,

$$\rho(\kappa) \leq \mathscr{B} \prod_{\kappa_0<\sigma_n<\kappa} (1+\beta_n)\exp(\int_{\kappa_0}^\kappa \rho(\mu)d\mu).$$

Lemma 2 ([21], p.196) Let $\mathscr{E}$ be a Banach space. Let $\mathscr{D}$: $\mathscr{E} \to \mathscr{E}$ be an operator which maps the elements of $\mathscr{E}$ into itself for which $\mathscr{D}^r$ is a contraction, where r is a positive integer. Then $\mathscr{D}$ has a unique fixed point.

We introduce the following hypotheses:

($\mathscr{A}_1$) The linear and continuous operator $\mathscr{A}$ generates a $\mathscr{C}_0$ semigroup $(\mathscr{D}(\kappa))_{\kappa\geq 0}$ on $\mathbb{X}^n$ such that $\|\mathscr{D}(\kappa)\|_{\mathbb{X}^n} \leq \mathscr{M}, \forall\kappa \geq 0$ with $\mathscr{M} > 0$.

($\mathscr{A}_2$) Let $\mathscr{P}$: $[0, \mathscr{K}] \times \mathscr{C}([-\mathfrak{r}, 0], \mathbb{X}^n) \times \mathbb{X}^n \to \mathbb{X}^n$ such that for every $\eta, \zeta \in \mathscr{C}$, $\kappa \in [0, \mathscr{K}]$, $\rho, \nu \in \mathbb{X}^n$ and there exists $L_\mathscr{P} > 0$ such that

$$d_\infty(\mathscr{P}(\kappa, \eta, \rho), \mathscr{P}(\kappa, \zeta, \nu)) \leq L_\mathscr{P}[d_\infty(\eta, \zeta) + d_\infty(\rho, \nu)]$$

($\mathscr{A}_3$) Let $\mathscr{H}$: $[0, \mathscr{K}] \times [0, \mathscr{K}] \times \mathbb{X}^n \to \mathbb{X}^n$ such that for every $\rho, \nu \in \mathbb{X}^n$, $\kappa, \mu \in [0, \mathscr{K}]$ and there exists $L_\mathscr{H} > 0$ such that

$$d_\infty(\mathscr{H}(\kappa,\mu,\rho),\mathscr{H}(\kappa,\mu,\nu))\le L_{\mathscr{H}} d_\infty(\rho,\nu)$$

$(\mathscr{A}_4)$ Let $\mathfrak{h}$: $(\mathscr{C}[-\mathfrak{r},0],\mathbb{X}^n)^q\to\mathbb{X}^n$ and there exists $\mathscr{Q}$ such that

$$d_\infty(\mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\ldots,\rho_{\sigma_q})(\kappa),\mathfrak{h}(\nu_{\sigma_1},\nu_{\sigma_2},\ldots,\nu_{\sigma_q})(\kappa))\le\mathscr{Q}d_\infty(\rho,\nu),\quad\forall\kappa\in[-\mathfrak{r},0]$$

$(\mathscr{A}_5)$ Let I_n: $\mathbb{X}^n\to\mathbb{X}^n$ such that for every $\rho,\nu\in\mathbb{X}^n$, $\kappa\in[0,\mathscr{K}]$, $n=1,2,3,\ldots,k$ and there exists L_n such that

$$d_\infty(\mathscr{I}_n\rho(\kappa_n),\mathscr{I}_n\nu(\kappa_n))\le L_n d_\infty(\rho,\nu).$$

3. Main result

Theorem 1 Suppose that the hypotheses $(\mathscr{A}_1)-(\mathscr{A}_5)$ are satisfied. Then the equation (1) has a unique mild solution ρ on $[-\mathfrak{r},\mathscr{K}]$.

Proof. Consider $\rho(\kappa)$ be a mild solution of the equation (1) then it satisfies the equivalent integral equation

$$\rho(\kappa)=\mathscr{D}(\kappa)[\psi(0)-\mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\ldots,\rho_{\sigma_q})(0)]+\int_0^\kappa\mathscr{D}(\kappa-\mu)\mathscr{P}(\mu,\rho_\mu,\int_0^\mu\mathscr{H}(\mu,\kappa,\rho_\sigma)d\sigma)d\mu$$

$$+\sum_{0<\sigma_n<\kappa}\mathscr{D}(\kappa-\sigma_n)\mathscr{I}_n\rho(\sigma_n),\qquad\kappa\in(0,\mathscr{K}]$$

$$\rho(\kappa)+\mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\ldots,\rho_{\sigma_q})(\kappa)]=\psi(\kappa),\quad\kappa\in[-\mathfrak{r},0].\tag{2}$$

Now we rewrite equation (1) as follows:
For $\psi\in\mathscr{C}([-\mathfrak{r},0],\mathbb{X}^n)$, define $\hat{\psi}\in\mathbb{X}^n$ by

$$\hat{\psi}(\kappa)=\begin{cases}\psi(\kappa)-\mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},...,\rho_{\sigma_q})(\kappa) & \text{if}\quad\kappa\in[-\mathfrak{r},0]\\ \mathscr{D}(\kappa)[\psi(0)-\mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\ldots,\rho_{\sigma_q})(0)] & \text{if}\quad\kappa\in[0,\mathscr{K}]\end{cases}$$

If $w\in\mathbb{X}^n$ and $\rho(\kappa)=w(\kappa)+\hat{\psi}(\kappa)$, $\kappa\in[-\mathfrak{r},\mathscr{K}]$, so that it is clear that w satisfies

$$w(\kappa)=\begin{cases}0 & \text{if}\quad\kappa\in[-\mathfrak{r},0]\\ \int_0^\kappa\mathscr{D}(\kappa-\mu)\mathscr{P}(\mu,w_\mu+\hat{\psi}_\mu,\int_0^\mu\mathscr{H}(\mu,\sigma,w_\sigma+\hat{\psi}_\sigma)d\sigma)d\mu & \\ +\sum_{0<\sigma_n<\kappa}\mathscr{D}(\kappa-\sigma_n)\mathscr{I}_n(w_\sigma+\hat{\psi}(\sigma_n)) & \text{if}\quad\kappa\in[0,\mathscr{K}]\end{cases}\tag{3}$$

if and only if $\rho(\kappa)$ satisfies the equation (1) We define the operator Λ: $\mathbb{X}^n \to \mathbb{X}^n$ by

$$(\Lambda w)(\kappa) = \begin{cases} 0 & \text{if} \quad \kappa \in [-\mathfrak{r}, 0] \\ \int_0^\kappa \mathscr{D}(\kappa-\mu)\mathscr{P}(\mu, w_\mu + \hat{\psi}_\mu, \int_0^\mu \mathscr{H}(\mu, \sigma, w_\sigma + \hat{\psi}_\sigma)d\sigma)d\mu & \\ + \sum_{0<\sigma_n<\kappa} \mathscr{D}(\kappa-\sigma_n)\mathscr{I}_n(w_\sigma + \hat{\psi}(\sigma_n)) & \text{if} \quad \kappa \in [0, \mathscr{K}] \end{cases} \tag{4}$$

From the definition of an operator Λ defined by the equation (4), It should be mentioned that the equation (3) can be expressed as $w = \Lambda w$.

We now demonstrate that Λ^n is a contraction on $\mathbb{X}^n$ for some positive integer n. Let $w, \nu \in \mathbb{X}^n$ and using hypotheses $(\mathscr{A}_1) - (\mathscr{A}_5)$ we get

$$\begin{aligned} d_\infty^*((\Lambda w)(\kappa), (\Lambda \nu(\kappa))) \leq\ & d_\infty^*\Bigg(\int_0^\kappa \mathscr{D}(\kappa-\mu)\mathscr{P}(\mu, w_\mu + \hat{\psi}_\mu, \int_0^\mu \mathscr{H}(\mu, \sigma, w_\sigma + \hat{\psi}_\sigma)d\sigma)d\mu \\ & + \sum_{0<\sigma_n<\kappa} \mathscr{D}(\kappa-\sigma_n)\mathscr{I}_n(w_\sigma + \hat{\psi}(\sigma_n)), \int_0^\kappa \mathscr{D}(\kappa-\mu)\mathscr{P}(\mu, \nu_\mu + \hat{\psi}_\mu, \\ & \int_0^\mu \mathscr{H}(\mu, \sigma, \nu_\sigma + \hat{\psi}_\sigma)d\sigma)d\mu + \sum_{0<\sigma_n<\kappa} \mathscr{D}(\kappa-\sigma_n)\mathscr{I}_n(\nu_\sigma + \hat{\psi}(\sigma_n))\Bigg) \\ \leq\ & d_\infty^*\Bigg(\int_0^\kappa \mathscr{D}(\kappa-\mu)[\mathscr{P}(\mu, w_\mu + \hat{\psi}_\mu, \int_0^\mu \mathscr{H}(\mu, \sigma, w_\sigma + \hat{\psi}_\sigma)d\sigma), \\ & \mathscr{P}(\mu, \nu_\mu + \hat{\psi}_\mu, \int_0^\mu \mathscr{H}(\mu, \sigma, \nu_\sigma + \hat{\psi}_\sigma)d\sigma]d\mu\Bigg) \\ & + d_\infty^*\Bigg(\sum_{0<\sigma_n<\kappa} \mathscr{D}(\kappa-\sigma_n)[\mathscr{I}_n(w_\sigma + \hat{\psi}(\sigma_n)), \mathscr{I}_n(\nu_\sigma + \hat{\psi}(\sigma_n))]\Bigg) \\ \leq\ & \int_0^\kappa \|\mathscr{D}(\kappa-\mu)\|_{\mathbb{X}^n}[d_\infty^*\Bigg(\mathscr{P}(\mu, w_\mu + \hat{\psi}_\mu, \int_0^\mu \mathscr{H}(\mu, \sigma, w_\sigma + \hat{\psi}_\sigma)d\sigma), \\ & \mathscr{P}(\mu, \nu_\mu + \hat{\psi}_\mu, \int_0^\mu \mathscr{H}(\mu, \sigma, \nu_\sigma + \hat{\psi}_\sigma)d\sigma]d\mu\Bigg) \\ & + \sum_{0<\sigma_n<\kappa} \|\mathscr{D}(\kappa-\sigma_n)\| d_\infty^*\Bigg(\mathscr{I}_n(w_\sigma + \hat{\psi}(\sigma_n)), \mathscr{I}_n(\nu_\sigma + \hat{\psi}(\sigma_n))\Bigg) \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^{\kappa} \mathscr{M} L_{\mathscr{P}} [d_\infty^*(w_\mu, \nu_\mu) + \int_0^{\mu} L_{\mathscr{H}} d_\infty^*(w_\sigma, \nu_\sigma) d\sigma] d\mu \\
&\quad + \sum_{0<\sigma_n<\kappa} \mathscr{M} L_n d_\infty^*(w, \nu) \\
&\leq \int_0^{\kappa} \mathscr{M} L_{\mathscr{P}} d_\infty^*(w_\mu, \nu_\mu) d\mu + \int_0^{\kappa} \mathscr{M} L_{\mathscr{P}} \int_0^{\mu} L_{\mathscr{H}} d_\infty^*(w_\sigma, \nu_\sigma) d\sigma d\mu \\
&\quad + \sum_{0<\sigma_n<\kappa} \mathscr{M} l_n d_\infty^*(w, \nu) \\
&\leq \mathscr{M} L_{\mathscr{P}} d_\infty^*(w, \nu) \kappa + \mathscr{M} L_{\mathscr{P}} L_{\mathscr{H}} d_\infty^*(w, \nu) \frac{\kappa^2}{2} + \sum_{0<\sigma_n<\kappa} \mathscr{M} L_n d_\infty^*(w, \nu) \\
&\leq \mathscr{M} L_{\mathscr{P}} d_\infty^*(w, \nu) \kappa + \mathscr{M} L_{\mathscr{P}} L_{\mathscr{H}} d_\infty^*(w, \nu) \kappa \mathscr{K} + \sum_{0<\sigma_n<\kappa} \mathscr{M} L_n d_\infty^*(w, \nu) \\
&\leq \left[\mathscr{M} L_{\mathscr{P}} (1 + L_{\mathscr{H}} \mathscr{K}) \kappa + \sum_{0<\sigma_n<\kappa} \mathscr{M} L_n \right] d_\infty^*(w, \nu) \\
&\leq \left[\mathscr{M} L_{\mathscr{P}} (1 + L_{\mathscr{H}} \mathscr{K}) \kappa + \sum_{0<\sigma_n<\kappa} \mathscr{M} L_n \right] \mathscr{H}_1(w, \nu)
\end{aligned}
$$

$$
\begin{aligned}
d_\infty^*((\Lambda^2 w)(\kappa), (\Lambda^2 \nu(\kappa))) &= d_\infty^*(\Lambda(\Lambda w)(\kappa), \Lambda(\Lambda \nu(\kappa))) \\
&\leq \int_0^{\kappa} \|\mathscr{D}(\kappa - \mu)\|_{\mathbb{X}^n} d_\infty^* \bigg(\mathscr{P}(\mu, \Lambda w_\mu + \hat{\psi}_\mu, \int_0^{\mu} \mathscr{H}(\mu, \sigma, \Lambda w_\sigma + \hat{\psi}_\sigma) d\sigma), \\
&\quad \mathscr{P}(\mu, \Lambda \nu_\mu + \hat{\psi}_\mu, \int_0^{\mu} \mathscr{H}(\mu, \sigma, \Lambda \nu_\sigma + \hat{\psi}_\sigma) d\sigma) \bigg) d\mu \\
&\quad + \sum_{0<\sigma_n<\kappa} \|\mathscr{D}(\kappa - \sigma_n)\| d_\infty^* \bigg(\mathscr{I}_n(\Lambda w_\sigma + \hat{\psi}(\sigma_n)), \mathscr{I}_n(\Lambda \nu_\sigma + \hat{\psi}(\sigma_n)) \bigg) \\
&\leq \int_0^{\kappa} \mathscr{M} L_{\mathscr{P}} [d_\infty^*(\Lambda w_\mu, \Lambda \nu_\mu) + \int_0^{\mu} L_{\mathscr{H}} d_\infty^*(\Lambda w_\sigma, \Lambda \nu_\sigma) d\sigma] d\mu \\
&\quad + \sum_{0<\sigma_n<\kappa} \mathscr{M} d_\infty^*(\Lambda w, \Lambda \nu)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^{\kappa} \mathscr{M} L_{\mathscr{P}} d_{\infty}^*(\Lambda w, \Lambda \nu) d\mu + \int_0^{\kappa} \mathscr{M} L_{\mathscr{P}} \int_0^{\mu} L_{\mathscr{H}} d_{\infty}^*(\Lambda w, \Lambda \nu) d\sigma d\mu \\
&\quad + \sum_{0<\sigma_n<\kappa} \mathscr{M} L_n d_{\infty}^*(\Lambda w, \Lambda \nu) \\
&\leq \mathscr{M} L_{\mathscr{P}} [\mathscr{M} L_{\mathscr{P}} [1 + L_{\mathscr{H}} \mathscr{K}]] \mathscr{H}_1(w, \nu) [\int_0^{\kappa} \mu d\mu + \int_0^{\kappa} L_{\mathscr{H}} \int_0^{\mu} \sigma d\sigma d\mu] \\
&\quad + [\mathscr{M} \sum_{0<\sigma_n<\kappa} L_n]^2 \mathscr{H}_1(w, \nu) \\
&\leq \mathscr{M}^2 L_{\mathscr{P}}^2 [1 + L_{\mathscr{H}} \mathscr{K}] \mathscr{H}_1(w, \nu) [\frac{\kappa^2}{2} + L_{\mathscr{H}} \frac{\kappa^3}{3}] + [\mathscr{M} \sum_{0<\sigma_n<\kappa} L_n]^2 \mathscr{H}_1(w, \nu) \\
&\leq \mathscr{M}^2 L_{\mathscr{P}}^2 [1 + L_{\mathscr{H}} \mathscr{K}] \mathscr{H}_1(w, \nu) [\frac{\kappa^2}{2!} + L_{\mathscr{H}} \mathscr{K} \frac{\kappa^2}{2!}] + [\mathscr{M} \sum_{0<\sigma_n<\kappa} L_n]^2 \mathscr{H}_1(w, \nu) \\
&\leq \mathscr{M}^2 L_{\mathscr{P}}^2 [1 + L_{\mathscr{H}} \mathscr{K}] \mathscr{H}_1(w, \nu) [1 + L_{\mathscr{H}} \mathscr{K}] \frac{\kappa^2}{2!} + [\mathscr{M} \sum_{0<\sigma_n<\kappa} L_n]^2 \mathscr{H}_1(w, \nu) \\
&\leq \mathscr{M}^2 L_{\mathscr{P}}^2 [1 + L_{\mathscr{H}} \mathscr{K}]^2 \mathscr{H}_1(w, \nu) \frac{\kappa^2}{2!} + [\mathscr{M} \sum_{0<\sigma_n<\kappa} L_n]^2 \mathscr{H}_1(w, \nu) \\
&\leq \{\frac{\mathscr{M} L_{\mathscr{P}} [1 + L_{\mathscr{H}} \mathscr{K}] \kappa]^2}{2!} + [\mathscr{M} \sum_{0<\sigma_n<\kappa} L_n]^2\} \mathscr{H}_1(w, \nu).
\end{aligned}
$$

Continuing in this way, we get

$$
\begin{aligned}
d_{\infty}^*((\Lambda^n w)(\kappa), (\Lambda^n \nu(\kappa))) &\leq \left\{ \frac{[\mathscr{M} L_{\mathscr{P}} [1 + L_{\mathscr{H}} \mathscr{K}] \kappa]^n}{n!} + [M \sum_{0<\sigma_n<\kappa} L_n]^n \right\} \mathscr{H}_1(w, \nu) \\
&\leq \left\{ \frac{[\mathscr{M} L_{\mathscr{P}} [1 + L_{\mathscr{H}} \mathscr{K}] \mathscr{K}]^n}{n!} + [\mathscr{M} \sum_{0<\sigma_n<\kappa} L_n]^n \right\} \mathscr{H}_1(w, \nu).
\end{aligned}
$$

For n large enough, $\frac{[\mathscr{M} L_{\mathscr{P}} [1 + L_{\mathscr{H}} \mathscr{K}] \mathscr{K}]^n}{n!} + [\mathscr{M} \sum_{0<\sigma_n<\kappa} L_n]^n < 1$. Thus there exists a positive integer n such that Λ^n is a contraction in $\mathbb{X}^n$. By virtue of lemma (2) the operator Λ has a unique fixed point $\tilde{w}$ in $\mathbb{X}^n$. Then $\tilde{\rho} = \tilde{w} + \hat{\psi}$ is a solution of the equation (1). □

4. Nearness and convergence of solutions

Consider the fuzzy impulsive nonlocal equation (1), along with the fuzzy impulsive nonlocal equation

$$\rho'(\kappa) = \mathscr{A}\rho(\kappa) + \tilde{\mathscr{P}}(\kappa, \rho_\kappa, \int_0^\kappa \mathscr{H}(\kappa, \mu, \rho_\mu)d\mu), \quad \kappa \in (0, \mathscr{K}]$$

$$\Delta\rho(\kappa_n) = \tilde{\mathscr{I}}_n\rho(\kappa_n), \kappa \neq t_n, \quad n = 1, 2, 3, \ldots, k,$$

$$\rho(\kappa) + \tilde{\mathfrak{h}}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(\kappa) = \tilde{\psi}(\kappa), \quad \kappa \in [-\mathfrak{r}, 0], \tag{5}$$

where $\mathscr{H}$ is as given in (1), $\tilde{\mathscr{P}}$: $[0, \mathscr{K}] \times \mathscr{C}([-\mathfrak{r}, 0], \mathbb{X}^n) \times \mathbb{X}^n \to \mathbb{X}^n$, $\tilde{\mathfrak{h}}$: $(\mathscr{C}([-\mathfrak{r}, 0], \mathbb{X}^n))^q \to \mathbb{X}^n$, $\tilde{\mathscr{I}}_n \in \mathscr{C}(\mathbb{X}^n, \mathbb{X}^n)$, and $\tilde{\psi} \in \mathscr{C}([-\mathfrak{r}, 0], \mathbb{X}^n)$.

Theorem 2 Assume that the function $\mathscr{P}$, $\mathscr{H}$, $\mathfrak{h}$, $\mathscr{I}_n$ in equation (1) satisfy hypotheses $(\mathscr{A}_0) - (\mathscr{A}_3)$ and there exists nonnegative constants ε_1, ε_2, ε_3, ε_4 such that

$$d_\infty^*\left(\mathscr{P}(\kappa, \eta, \rho), \tilde{\mathscr{P}}(\kappa, \eta, \rho)\right) \leq \varepsilon_1$$

$$d_\infty^*\left(\mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(\kappa), \tilde{\mathfrak{h}}((\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(\kappa)\right) \leq \varepsilon_2$$

$$d_\infty^*\left(\psi(\kappa), \tilde{\psi}(\kappa)\right) \leq \varepsilon_3$$

$$d_\infty^*\left(\mathscr{I}_n\rho(\kappa_n), \tilde{\mathscr{I}}_n\rho(\kappa_n)\right) \leq \varepsilon_4 \tag{6}$$

Let $\rho(\kappa)$ and $\nu(\kappa)$ be respectively solutions of (1) and (5) on $[-\mathfrak{r}, \mathscr{K}]$. Then the following inequality holds:

$$\mathscr{H}_1(\rho, \nu) \leq \frac{\mathscr{M}[\prod_{0<\sigma<\kappa}(1+\mathscr{M}L_n)\exp(\mathscr{M}L_{\mathscr{P}}\mathscr{K})]}{[1-\Gamma\prod_{0<\sigma<\kappa}(1+\mathscr{M}L_n)\exp(\mathscr{M}L_{\mathscr{P}}\mathscr{K})]}[\varepsilon_1\mathscr{K} + \varepsilon_2 + \varepsilon_3 + \varepsilon_4]$$

Proof. Using the facts that $\rho(\kappa)$ and $\nu(\kappa)$ be respectively solutions of (1) and (5) and hypotheses $(\mathscr{A}_0) - (\mathscr{A}_3)$ we obtain, for $\kappa \in [\mathfrak{r}, 0]$

$$
\begin{aligned}
d_\infty^*\Big(\rho(\kappa),\, \nu(\kappa)\Big) &= d_\infty^*\Big(\psi(\kappa) - \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(\kappa),\, \tilde{\psi}(\kappa) - \tilde{\mathfrak{h}}(\nu_{\sigma_1}, \nu_{\sigma_2}, \ldots, \nu_{\sigma_q})(\kappa)\Big) \\
&\leq d_\infty^*\Big(\psi(\kappa),\, \tilde{\psi}(\kappa)\Big) + d_\infty^*\Big(\mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(\kappa),\, \tilde{\mathfrak{h}}(\nu_{\sigma_1}, \nu_{\sigma_2}, \ldots, \nu_{\sigma_q})(\kappa)\Big) \\
&\leq d_\infty^*\Big(\psi(\kappa),\, \tilde{\psi}(\kappa)\Big) + d_\infty^*\Big(\mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(\kappa),\, \mathfrak{h}(\nu_{\sigma_1}, \nu_{\sigma_2}, \ldots, \nu_{\sigma_q})(\kappa)\Big) \\
&\quad + d_\infty^*\Big(\mathfrak{h}(\nu_{\sigma_1}, \nu_{\sigma_2}, \ldots, \nu_{\sigma_q})(\kappa),\, \tilde{\mathfrak{h}}(\nu_{\sigma_1}, \nu_{\sigma_2}, \ldots, \nu_{\sigma_q})(\kappa)\Big) \\
&\leq \varepsilon_3 + \mathscr{Q}\mathscr{H}_1(\rho,\, \nu) + \varepsilon_2
\end{aligned} \tag{7}
$$

For $\kappa \in [0,\, \mathscr{K}]$

$$
\begin{aligned}
d_\infty^*\Big(\rho(\kappa),\, \nu(\kappa)\Big) &\leq \|\mathscr{D}(\kappa)\|_{\mathbb{X}^n}\Big[d_\infty^*\Big(\psi(\kappa),\, \tilde{\psi}(\kappa)\Big) + d_\infty^*\Big(\mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(\kappa), \\
&\quad \tilde{\mathfrak{h}}(\nu_{\sigma_1}, \nu_{\sigma_2}, \ldots, \nu_{\sigma_q})(\kappa)\Big)\Big] + \int_0^\kappa \|\mathscr{D}(\kappa - \mu)\|_{\mathbb{X}^n} d_\infty^*\Big(\mathscr{P}(\mu,\, \rho_\mu, \\
&\quad \int_0^\mu \mathscr{H}(\mu,\, \sigma,\, \rho_\sigma)d\sigma),\, \tilde{\mathscr{P}}(\mu,\, \nu_\mu,\, \int_0^\mu \mathscr{H}(\mu,\, \sigma,\, \nu_\sigma)d\sigma)\Big)d\mu \\
&\quad + \sum_{0<\sigma_n<\kappa} \|\mathscr{D}(\kappa - \sigma_n)\|_{\mathbb{X}^n} d_\infty^*\Big(\mathscr{I}_n\rho(\kappa_n),\, \tilde{\mathscr{I}}_n\nu(\sigma_n)\Big) \\
&\leq \mathscr{M}[\varepsilon_3 + \mathscr{Q}d_\infty^*(\rho,\, \nu) + \varepsilon_2] + \mathscr{M}L_{\mathscr{P}} \int_0^\kappa [d_\infty^*(\rho_\mu,\, \nu_\mu) \\
&\quad + L_{\mathscr{H}} \int_0^\mu d_\infty^*(\rho_\sigma,\, \nu_\sigma)d\sigma]d\mu + \mathscr{M}\varepsilon_1\mathscr{K} \\
&\quad + \sum_{0<\sigma_n<\kappa} \mathscr{M}L_n d_\infty^*(\rho(\sigma_n),\, \nu(\sigma_n)) + \mathscr{M}\varepsilon_4
\end{aligned}
$$

$$\sup_{0\le\kappa\le\mathscr{K}}(d_\infty^*(\rho(\kappa),\,\nu(\kappa)) \le \sup_{0\le\kappa\le\mathscr{K}}\Big\{\mathscr{M}[\varepsilon_1+\mathscr{Q}d_\infty^*(\rho,\,\nu)+\varepsilon_2]+\mathscr{M}L_{\mathscr{P}}\int_0^\kappa[d_\infty^*(\rho_\mu,\,\nu_\mu)$$

$$+L_{\mathscr{H}}\int_0^\mu d_\infty^*(\rho_\sigma,\,\nu_\sigma)d\sigma]d\mu+\mathscr{M}\varepsilon_3\mathscr{K}$$

$$+\sum_{0<\sigma_n<\kappa}\mathscr{M}L_n d_\infty^*(\rho(\sigma_n),\,\nu(\sigma_n))+\mathscr{M}\varepsilon_4\Big\}$$

$$\mathscr{H}_1(\rho,\,\nu) \le \mathscr{M}[\varepsilon_1\kappa+\varepsilon_2+\varepsilon_3+\varepsilon_4+[\mathscr{Q}+L_{\mathscr{P}}l_{\mathscr{H}}\frac{\kappa^2}{2}]]\mathscr{H}_1(\rho,\,\nu)+\int_0^\kappa\mathscr{M}L_{\mathscr{P}}\mathscr{H}_1(\rho,\,\nu)d\mu$$

$$+\sum_{0<\sigma_n<\kappa}\mathscr{M}L_n\mathscr{H}_1(\rho,\,\nu). \tag{8}$$

Let $\Gamma=\mathscr{Q}+L_{\mathscr{P}}L_{\mathscr{H}}\frac{\mathscr{K}^2}{2}$, define the function w: $[-\mathfrak{r},\,\mathscr{K}]\to\mathbb{X}^n$ by $w(\kappa)=\sup\{d_\infty(\rho(\mu),\,\nu(\mu))\colon -\mathfrak{r}\le\mu\le\kappa\}$, $\kappa\in[0,\,\mathscr{T}]$. Let $\kappa^*\in[-\mathfrak{r},\,\kappa]$ be such that $w(\kappa)=d_\infty(\rho(\kappa^*),\,\nu(\kappa^*))$. If $\kappa^*\in[0,\,\kappa]$ then from inequality (8) we have

$$w(\kappa)=\mathscr{H}_1(\rho(\kappa^*),\,\nu(\kappa^*)) \le \mathscr{M}[\varepsilon_1\mathscr{K}+\varepsilon_2+\varepsilon_3+\varepsilon_4+\Gamma\mathscr{H}_1(\rho,\,\nu)]+\int_0^{\kappa^*}\mathscr{M}L_{\mathscr{P}}\mathscr{H}_1(\rho,\,\nu)d\mu$$

$$+\sum_{0<\sigma_n<\kappa}\mathscr{M}L_n\mathscr{H}_1(\rho,\,\nu). \tag{9}$$

Now applying lemma (1) to the inequality (9) we get

$$\mathscr{H}_1(\rho,\,\nu) \le \Big(\mathscr{M}[\varepsilon_1\mathscr{K}+\varepsilon_2+\varepsilon_3+\varepsilon_4+\Gamma\mathscr{H}_1(\rho,\,\nu)]\Big)\prod_{0<\sigma<\kappa}(1+\mathscr{M}L_n)\exp(\mathscr{M}L_{\mathscr{P}}\mathscr{K}).$$

Hence we get

$$\mathscr{H}_1(\rho,\,\nu) \le \frac{\mathscr{M}[\prod\limits_{0<\sigma<\kappa}(1+\mathscr{M}L_n)\exp(\mathscr{M}L_{\mathscr{P}}\mathscr{K})]}{[1-\Gamma\prod\limits_{0<\sigma<\kappa}(1+\mathscr{M}L_n)\exp(\mathscr{M}L_{\mathscr{P}}\mathscr{K})]}[\varepsilon_1\mathscr{K}+\varepsilon_2+\varepsilon_3+\varepsilon_4].$$

Remark 1 The result given in the above theorem, relates the solutions of equations (1) and (5) in the sense that if $\mathscr{P}$ and $\tilde{\mathscr{P}}$, $\psi(\kappa)$ is close to $\tilde{\psi}(\kappa)$ and $\mathfrak{h}$ is close to $\tilde{\mathfrak{h}}$. Then not only the solutions of equations (1) and (5) are close to each other, but also depend continuously on the functions involved therein.

Consider

$$v'(\kappa) = \mathscr{A}v(\kappa) + \mathscr{P}_m(\kappa, v_\kappa, \int_0^\kappa \mathscr{H}(\kappa, \mu, v_\kappa)d\mu), \quad \kappa \in (0, \mathscr{K}]$$

$$\Delta v(\kappa_m) = \mathscr{I}_m v(\kappa_m), \ \kappa \neq t_m, \quad m = 1, 2, 3, \ldots, k,$$

$$v(\kappa) + \mathfrak{h}_m(v_{\sigma_1}, v_{\sigma_2}, \ldots, v_{\sigma_q})(\kappa) = \psi_m(\kappa), \quad \kappa \in [-\mathfrak{r}, 0], \tag{10}$$

where $\mathscr{H}$ is given in equation (1), $\mathscr{P}_m$: $[0, \mathscr{K}] \times \mathscr{C}([-\mathfrak{r}, 0], \mathbb{X}^n) \times \mathbb{X}^n \to \mathbb{X}^n$, $\mathfrak{h}_m$: $\mathscr{C}([-\mathfrak{r}, 0], \mathbb{X}^n)^q \to \mathbb{X}^n$ and $\psi_m(\kappa)$ is a sequence in $\mathbb{X}^n$.

We have the following corollary as an immediate consequence of the aforementioned theorem:

Corollary 1 Suppose that the following $\mathscr{P}$, $\mathscr{H}$, $\mathfrak{h}$, $\mathscr{I}_n$ in 1 satisfy the hypotheses $(\mathscr{A}_0) - (\mathscr{A}_4)$ and there exists nonnegative constants ε_m, ε'_m, δ_m, δ'_m such that

$$d^*_\infty\Big(\mathscr{P}(\kappa, \eta, \rho), \mathscr{P}_m(\kappa, \eta, \rho)\Big) \leq \varepsilon_m$$

$$d^*_\infty\Big(\mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, ..., \rho_{\sigma_q})(\kappa), \mathfrak{h}_m((\rho_{\sigma_1}, \rho_{\sigma_2}, ..., \rho_{\sigma_q})(\kappa)\Big) \leq \varepsilon'_m$$

$$d^*_\infty\Big(\psi(\kappa), \psi_m(\kappa)\Big) \leq \delta_m$$

$$d^*_\infty\Big(\mathscr{I}_n\rho(\kappa_n), \mathscr{I}_m\rho(\kappa_n)\Big) \leq \delta'_m \tag{11}$$

where $\varepsilon_m \to 0$, $\varepsilon'_m \to 0$, $\delta_m \to 0$, $\delta'_m \to 0$ as $m \to \infty$. If $\rho(\kappa)$ and $v_m(\kappa)$, $m = 1, 2, ...$ be respectively solutions of equations (1) and (10) on $[-\mathfrak{r}, \mathscr{K}]$. Then as $m \to \infty$, $v_m(\kappa) \to \rho(\kappa)$ on $[-\mathfrak{r}, \mathscr{K}]$.

Remark 2 The result obtained in this corollary provides sufficient conditions that ensure solutions of equations (10) will converge to solutions of equation (1).

5. Application

Consider the following nonlinear fuzzy partial functional differential equation, to clarify the result mentioned in Section 3 of the type

$$\frac{\partial}{\partial \kappa} v(v, \kappa) = \frac{\partial^2}{\partial v^2} v(v, \kappa) + \mathscr{Q}\Big(\kappa, v(v, \kappa - r), \int_0^\kappa \mathscr{W}(\kappa, \mu, v(\mu - r))d\mu\Big),$$

$$v \in [0, \pi], \ \kappa \in [0, \mathscr{K}] \tag{12}$$

$$\nu(0,\,\kappa) = \nu(\pi,\,\kappa) = 0, \quad 0 \le \kappa \le \mathscr{K} \tag{13}$$

$$\nu(v,\,\kappa) + \sum_{n=1}^{q} \nu(v,\,\kappa_n + \kappa) = \psi(v,\,\kappa), \quad 0 \le v \le \pi, \quad -\mathfrak{r} \le \kappa \le 0 \tag{14}$$

$$\Delta\nu(v,\,t_n) = \mathscr{I}_n(\nu(v,\,t_n)), \quad n = 1,\,2,\,3,\,\ldots,\,k, \tag{15}$$

where $\mathscr{Q}$: $[0,\,\mathscr{K}] \times \mathbb{X}^n \times \mathbb{X}^n \to \mathbb{X}^n$, $\mathscr{W}$: $[0,\,\mathscr{K}] \times \mathbb{X}^n \to \mathbb{X}^n$, $\mathscr{I}_n$: $\mathbb{X}^n \to \mathbb{X}^n$ are continuous. We assume that the functions $\mathscr{Q}$, $\mathscr{W}$ and $\mathscr{I}_n$ satisfy the following conditions:

i. $\forall\ \kappa \in [0,\,\mathscr{K}]$ and $\rho,\,\nu \in \mathbb{X}^n$, $\exists\ L_{\mathscr{Q}} > 0$ such that:

$$d_\infty^*\Big(\mathscr{Q}(\kappa,\,v,\,\rho),\,\mathscr{Q}(\kappa,\,w,\,\nu)\Big) \quad \le \quad L_{\mathscr{Q}}\Big(d_\infty^*(w,\,v) + d_\infty^*(\rho,\,\nu)\Big)$$

ii. $\forall\ \kappa \in [0,\,\mathscr{K}]$ and $\rho,\,\nu \in \mathbb{X}^n$, there exists $L_{\mathscr{W}} > 0$ such that:

$$d_\infty^*\Big(\mathscr{W}(\kappa,\,\mu,\,\rho),\,\mathscr{W}(\kappa,\,\mu,\,\nu)\Big) \quad \le \quad L_{\mathscr{W}}\,d_\infty^*(\rho,\,\nu)$$

iii. $\exists\ c_n,\,d > 0$ such that:

$$d_\infty^*(\mathscr{I}_n(\rho),\,\mathscr{I}_n(\nu)) \le c_n d_\infty^*(\rho,\,\nu), \quad n = 1,\,2,\,3,\,\ldots,\,k,$$

$$\sum_{n=1}^{q} d_\infty^*(\nu(v,\,\kappa_n + \kappa),\,\nu(w,\,\kappa_n + \kappa)) \le d$$

We define the operator $\mathscr{A}: \mathbb{X}^n \to \mathbb{X}^n$ by $\mathscr{A}w = w''$ with domain $D(\mathscr{A}) = \{w \in \mathbb{X}^n$: w and w' are absolutely continuous, $w'' \in \mathbb{X}^n$ and $w(0) = w(\pi) = 0\}$. Then the operator $\mathscr{A}$ can be written as

$$\mathscr{A}w \quad = \quad \sum_{m=1}^{\infty} -m^2(w,\,w_m)w_m, \quad w \in D(\mathscr{A}),$$

where $w_m(v) = (\sqrt{\frac{2}{\pi}})\sin(mv)$, $m = 1,\,2,\,3,\,\ldots$ is the orthogonal set of eigenvectors of $\mathscr{A}$ and $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup $\mathscr{D}(\kappa)$, $\kappa \ge 0$ and is given by

$$\mathscr{D}(\kappa)w \quad = \quad \sum_{m=1}^{\infty} \exp(-m^2\kappa)(w,\,w_m)w_m, \;\; w \in \mathbb{X}^n.$$

Now, the analytic semigroup $\mathscr{K}(\kappa)$ being compact, there exists $\mathscr{M}$ such that $|\mathscr{K}(\kappa)| \le \mathscr{M}\ \forall\ \kappa \in [0,\ \mathscr{K}]$. Define the functions $\mathscr{P}$: $[0,\ \mathscr{K}] \times \mathscr{C}([-\mathfrak{r},\ 0],\ \mathbb{X}^n) \times \mathbb{X}^n \to \mathbb{X}^n$, $\mathscr{H}$: $[0,\ \mathscr{K}] \times [0,\ \mathscr{K}] \times \mathscr{C}([-\mathfrak{r},\ 0],\ \mathbb{X}^n) \to \mathbb{X}^n$, $\mathscr{I}_n$: $\mathbb{X}^n \to \mathbb{X}^n$ as follows

$$\begin{aligned} \mathscr{P}(\kappa,\ \eta,\ \rho)(v) &= \mathscr{Q}(\kappa,\ \eta(-\mathfrak{r})v,\ \rho(v)), \\ \mathscr{H}\kappa,\ \mu,\ \zeta) &= \mathscr{W}(\kappa,\ \mu,\ \zeta(-\mathfrak{r})v), \end{aligned}$$

where $\kappa \in [0,\ \mathscr{K}]$, $\eta,\ \zeta \in \mathscr{C}([-\mathfrak{r},\ 0],\ \mathbb{X}^n)$, $\rho \in \mathbb{X}^n$ and $0 \le v \le \pi$. With these choices of the functions the equations $(12)-(15)$ can be formulated as an fuzzy integro-differential equations in $\mathbb{X}^n$

$$\rho'(\kappa) = \mathscr{A}\rho(\kappa) + \mathscr{P}(\tau,\ \rho_\kappa,\ \int_0^\kappa \mathscr{H}(\kappa,\ \mu,\ \rho_\mu)d\mu), \quad \kappa \in (0,\ \mathscr{K}]$$

$$\Delta\rho(\kappa_n) = \mathscr{I}_n\rho(\kappa_n),\ \kappa \ne t_n, \quad n = 1,\ 2,\ 3,\ \ldots,\ k,$$

$$\rho(\kappa) + \mathfrak{h}(\rho_{\sigma_1},\ \rho_{\sigma_2},\ \ldots,\ \rho_{\sigma_q})(\kappa) = \psi(\kappa), \quad \kappa \in [-\mathfrak{r},\ 0].$$

Since all the hypotheses of the theorem (1) are satisfied, the theorem (1), can be used to warranty that a mild solution exists $\nu(v,\ \kappa) = \rho(\kappa)v$, $\kappa \in [0,\ \mathscr{K}]$, $v \in [0,\ \pi]$, of the nonlinear fuzzy partial integro-differential equations $(12)-(15)$.

6. Conclusions

In this article, the modified version of the Banach contraction principle was employed to get the existence and other qualitative properties of nonlocal impulsive fuzzy solutions for nonlinear integro-differential equations. An application example was given to prove the validity of our result.

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Conflict of interest

The authors declare no competing financial interest.

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