Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Caristi-Type Fixed-Point Theorems in the Framework *w*-Distances

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Received: 28 February 2023; **Revised:** 29 March 2023; **Accepted:** 13 April 2023

Abstract: The aim of this paper is to generalize Caristi's theorem, Bollenbacher and Hicks' theorem, and also Hicks and Rhoades' theorem by substituting the continuity assumption with relatively weaker conditions of *k*-continuity or $(C; k)$ condition in a complete metric space with a *w*-distance. Another proof for Caristi's theorem using Zorn's lemma will be given. Some examples will confirm the novelty and usefulness of our results.

***Keywords*:** Caristi's fixed-point theorem, *w*-distance, weakly orbitally continuous mapping, *k*-continuous mapping

MSC: 47H10, 47H09

1. Introduction

Caristi [1] proved the following important theorem:

Theorem 1.1. [1] Suppose $(\mathcal{M}, d)$ is a complete metric space, $\mathbb{S}: \mathcal{M} \to \mathcal{M}$ is a self-map, and $\varphi: \mathcal{M} \to [0, \infty)$ is a lower semi-continuous such that for all $\mathfrak{x} \in \mathcal{M}$,

$$d(\mathfrak{x}, \mathbb{S}\mathfrak{x}) \leq \varphi(\mathfrak{x}) - \varphi(\mathbb{S}\mathfrak{x}).$$

Then, $\mathbb{S}$ has a unique fixed point.

Caristi-type fixed-point theorems are of interest in the field of fixed-point theory and show other faces for the Banach contraction principle. This theorem has been investigated in several general structures of metric spaces. For example, Abodayeh et al. [2] generalized Caristi-Kirki's theorem from partial metric spaces to *M*-metric spaces. Interested readers can see the generalizations of Caristi-Kirk's theorem on partial metric spaces by Karapinar [3], and for other generalizations, see [4]. Some generalizations of Caristi's fixed-point theorem with applications to the fixed-point theory of weakly contractive set-valued maps and the minimization problem were obtained by Amini-Harandi [5]. Pant et al. [6] gave a new improvement to this theorem when the mapping φ is not lower semi-continuous in general. Zakeri et al. [7] proved some new fixed point theorems in this field for convex contraction mappings by replacing weaker conditions of *k*-continuity or $\mathbb{S}$-orbitally lower semi-continuity on a complete metric space with a *w*-distance.

With this motivation, we will prove some generalizations to Caristi's fixed-point theorem under the existence of the *w*-distance structure on a complete metric space. We will use the *k*-continuity or $(C; k)$ condition in place of the continuity to obtain our generalizations.

DOI: https://doi.org/10.37256/cm.5120242584

Kada et al. [8] introduced the concept of a w-distance on a metric space for generalizing the fixed point theorem by Caristi, Ekeland's ϵ-variational principle, and the non-convex minimization theorem of Takahashi in a complete metric space with a w-distance.

Definition 1. [8] Let $\mathcal{M}$ be a metric space endowed with a metric d. A function $\varrho : \mathcal{M} \times \mathcal{M} \to [0,\infty)$ is called a w-distance on $\mathcal{M}$ if it satisfies the following properties:

(1) $\varrho(\mathfrak{x},\mathfrak{z}) \le \varrho(\mathfrak{x},\mathfrak{y}) + \varrho(\mathfrak{y},\mathfrak{z})$ for any $\mathfrak{x},\mathfrak{y},\mathfrak{z} \in \mathcal{M}$;

(2) ϱ is lower semi-continuous in its second variable; i.e., if $\mathfrak{x} \in \mathcal{M}$ and $\mathfrak{y}_n \to \mathfrak{y}$ in $\mathcal{M}$, then $\varrho(\mathfrak{x},\mathfrak{y}) \le \liminf_n \varrho(\mathfrak{x},\mathfrak{y}_n)$

(3) For each $\epsilon > 0$, there exists a $\delta > 0$ such that $\varrho(\mathfrak{z},\mathfrak{x}) \le \delta$ and $\varrho(\mathfrak{z},\mathfrak{y}) \le \delta$ imply $d(\mathfrak{x},\mathfrak{y}) \le \epsilon$.

Example 1. [8]

(i) Let $(\mathcal{M},\|\cdot\|)$ be a normed space. Then, the functions $\varrho : \mathcal{M}\times\mathcal{M} \to [0,\infty)$ defined by $\varrho(\mathfrak{x},\mathfrak{y}) = \|\mathfrak{x}\| + \|\mathfrak{y}\|$ and $\varrho(\mathfrak{x},\mathfrak{y}) = \|\mathfrak{y}\|$ for each $\mathfrak{x},\mathfrak{y} \in \mathcal{M}$ are two w-distances on $\mathcal{M}$.

(ii) Let $(\mathcal{M}, d)$ be a metric space and let $\mathbb{S} : \mathcal{M} \to \mathcal{M}$ be a continuous map. Then, the function $\varrho : \mathcal{M}\times\mathcal{M} \to [0,\infty)$ defined by

$$\varrho(\mathfrak{x},\mathfrak{y}) = \max\{d(\mathbb{S}\mathfrak{x},\mathfrak{y}), d(\mathbb{S}\mathfrak{x},\mathbb{S}\mathfrak{y})\},$$

for every $\mathfrak{x},\mathfrak{y} \in \mathcal{M}$, is a w-distance on $\mathcal{M}$.

The following lemma has a crucial role in proving our theorems in this paper.

Lemma 1. [8] Let $(\mathcal{M}, d)$ be a metric space and ϱ be a w-distance on $\mathcal{M}$.

(i) Suppose that $\{\mathfrak{x}_n\}$ is a sequence in $\mathcal{M}$ such that $\lim_n \varrho(\mathfrak{x}_n,\mathfrak{x}) = \lim_n \varrho(\mathfrak{x}_n,\mathfrak{y}) = 0$, then $\mathfrak{x} = \mathfrak{y}$. Particularly, from $\varrho(\mathfrak{z},\mathfrak{x}) = \varrho(\mathfrak{z},\mathfrak{y}) = 0$, we have $\mathfrak{x} = \mathfrak{y}$.

(ii) If $\varrho(\mathfrak{x}_n,\mathfrak{y}_n) \le \alpha_n \varrho(\mathfrak{x}_n,\mathfrak{y}) \le \beta_n$, for any $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,\infty)$ converging to 0, then $\{\mathfrak{y}_n\}$ converges to $\mathfrak{y}$.

(iii) Suppose that ϱ is a w-distance on a metric space $(\mathcal{M},d)$, and $\{\mathfrak{x}_n\}$ is a sequence in $\mathcal{M}$ such that for all $\varepsilon > 0$, there exists an $N_\varepsilon \in N$ such that $m > n > N_\varepsilon$ implies $\varrho(\mathfrak{x}_n,\mathfrak{x}_m) < \varepsilon$ (or $\lim_{m,n} \varrho(\mathfrak{x}_n,\mathfrak{x}_m) = 0$). Then, the sequence $\{\mathfrak{x}_n\}$ is Cauchy.

Remark 1.1. Let $\varrho(\mathfrak{a},\mathfrak{b}) = \varrho(\mathfrak{b},\mathfrak{a}) = 0$. Then, by triangular inequality, we have $\varrho(\mathfrak{a},\mathfrak{a}) \le \varrho(\mathfrak{a},\mathfrak{b}) + \varrho(\mathfrak{b},\mathfrak{a}) = 0$ and so $\varrho(\mathfrak{a},\mathfrak{a}) = 0$ and from Lemma 1, we get $\mathfrak{a} = \mathfrak{b}$.

Let us now describe the plan of this paper. In Section 2, we will give some preliminaries that are necessary for other sections. In Section 3, we will give a generalization of Caristi's [1] theorem, Bollenbacher and Hicks' [9] theorem, and also Hicks and Rhoades' [10] theorem using the substitution of the continuity assumption with relatively weaker conditions of k-continuity or $(C; k)$ condition in a complete metric space with a w-distance. (See Theorems 3.1-3.3 below.) In Section 4, using the concept of w-distance, we will give some new versions of Caristi's theorem. (See Theorem 4.1 below.) In the final section, we will obtain another proof for Caristi's theorem using Zorn's lemma, which is one of the important results in this paper. (See Theorem 5.1 below.)

2. Preliminaries

We start this section with some useful definitions.

The concept of k-continuity on a metric space $(\mathcal{M},d)$ defined by Pant & Pant [11] in as follows:

Definition 2. [11] A self-mapping $\mathbb{S}$ on a metric space $(\mathcal{M},d)$ is said to be k-continuous, $(k = 1, 2, 3, \ldots)$. If for every sequence $\{\mathfrak{x}_n\}$ in $\mathcal{M}$ from $\mathbb{S}^{k-1}\mathfrak{x}_n \to \mathfrak{z}$, one can conclude that $\mathbb{S}^k\mathfrak{x}_n \to \mathbb{S}\mathfrak{z}$.

Example 2. On the metric space $(\mathcal{M},d) = ([1,5],\|\cdot\|)$, define $\mathbb{S} : \mathcal{M} \to \mathcal{M}$ by

$$\mathbb{S}(\mathfrak{x}) = \begin{cases} \frac{\mathfrak{x}}{3} + 1 & 1 \le \mathfrak{x} \le 3; \\ 3 & 3 < \mathfrak{x} \le 5. \end{cases}$$

Then, $\mathbb{S}$ is not k-continuous for each $k \geq 0$. Indeed,

$$\mathbb{S}^k(\mathfrak{x}) = \begin{cases} \dfrac{\mathfrak{x} + \Sigma_{i=1}^{k} 3^i}{3^k} + 1 & 1 \leq \mathfrak{x} \leq 3; \\ \dfrac{2 + \Sigma_{i=1}^{k-2} 3^i}{3^{k-2}} & 3 < \mathfrak{x} \leq 5. \end{cases}$$

Note that $\lim_{n\to\infty} \mathbb{S}^k(\mathfrak{x}) \neq \mathbb{S}^k 3$ for every $k \geq 0$.

Example 3. On the metric space $(\mathcal{M}, d) = ([2,4], \|\cdot\|)$, define $\mathbb{S} : \mathcal{M} \to \mathcal{M}$ by

$$\mathbb{S}\mathfrak{x} = 2 \text{ if } 2 \leq \mathfrak{x} \leq 3,$$

$$\mathbb{S}\mathfrak{x} = 0 \text{ if } 3 < \mathfrak{x} \leq 4.$$

So, $\lim_{n\to\infty} \mathbb{S}\mathfrak{x}_n = z$ implies $\lim_{n\to\infty} \mathbb{S}^2\mathfrak{x}_n = z$. On the other hand, $\lim_{n\to\infty} \mathbb{S}\mathfrak{x}_n = \mathfrak{z}$ can be verified as $\mathfrak{z} = 0$ or $\mathfrak{z} = 2$. Also, for every n, we have $\mathbb{S}^2\mathfrak{x}_n = 2$. Therefore, $\lim_{n\to\infty} \mathbb{S}^2\mathfrak{x}_n = 2 = \mathbb{S}\mathfrak{z}$. Thus, $\mathbb{S}$ is 2-continuous, but $\mathbb{S}$ is not continuous at $\mathfrak{x} = 3$.

Definition 3. For the metric space $(\mathcal{M}, d)$, $\mathfrak{x} \in \mathcal{M}$, and the self-map $\mathbb{S} : \mathcal{M} \to \mathcal{M}$, we define the $\mathbb{S}$-orbit of $\mathfrak{x}$ as follows:

$$O(\mathfrak{x}; \infty) = \{\mathfrak{x}, \mathbb{S}\mathfrak{x}, \mathbb{S}^2\mathfrak{x}, \ldots\}.$$

The mapping $\mathbb{S}$ is said to be orbitally continuous at $\mathfrak{x}$ if for each sequence $\{\mathfrak{x}_n\}$ in $O(\mathfrak{x}; \infty)$, which is convergent to $\mathfrak{u}$, then $\mathbb{S}\mathfrak{x}_n$ is convergent to $\mathbb{S}\mathfrak{u}$, as $n \to \infty$ [10].

$\mathbb{S}$ is called weakly orbitally continuous if whenever the set $\{\mathfrak{x} \in \mathcal{M} : \lim_i \mathbb{S}^{m_i}\mathfrak{x} = \mathfrak{u}\}$ is non-empty implies the set $\{\mathfrak{y} \in \mathcal{M} : \lim_i \mathbb{S}^{m_i}\mathfrak{y} = \mathfrak{u} \Rightarrow \lim_i \mathbb{S}\mathbb{S}^{m_i}\mathfrak{y} = \mathbb{S}\mathfrak{u}\}$ is non-empty [6].

Suppose that $\mathcal{T} : \mathcal{M} \to [0, +\infty)$ is a function. Then, $\mathcal{T}$ is called the $\mathbb{S}$-orbitally is called the lower semi-continuous at $\mathfrak{x}$ if for every sequence $\{\mathfrak{x}_n\}$ in $O(\mathfrak{x}; \infty)$ that $\mathfrak{x}_n \to \mathfrak{u}$, one can conclude that $\mathcal{T}(\mathfrak{u}) \leq \liminf_{n\to\infty} \mathcal{T}(\mathfrak{x}_n)$ [12].

This is clear to see that every k-continuous mapping $\mathbb{S}$ is weakly orbitally continuous, but the converse is not true in general.

Example 4. Let $\mathcal{M} = [0,3]$ equipped with the Euclidean metric. Define $\mathbb{S} : M \to M$ by

$$\mathbb{S}\mathfrak{x} = \frac{\mathfrak{x}+1}{2} \text{ if } 1 \leq \mathfrak{x} < 2, \qquad \mathbb{S}\mathfrak{x} = 0 \text{ if } 2 \leq \mathfrak{x} < 3, \qquad \mathbb{S}3 = 3.$$

If $2 \leq \mathfrak{x} < 3$, then $\lim_{n\to\infty} \mathbb{S}^n\mathfrak{x} = \dfrac{1}{2}$; while $\lim_{n\to\infty} \mathbb{S}^{n+1}\mathfrak{x} = \dfrac{1}{2} \neq \dfrac{3}{4} = \mathbb{S}\dfrac{1}{2}$. Thus, $\mathbb{S}$ is not orbitally continuous.

On the other hand, if we get $\mathfrak{x} = 3$, it implies $\lim_{n\to\infty} \mathbb{S}^n 3 = 3$ and $\lim_{n\to\infty} \mathbb{S}(\mathbb{S}^n 3) = 3 = \mathbb{S}3$. So, $\mathbb{S}$ is a weakly orbitally continuous mapping.

We suppose the sequence $\{\mathbb{S}^n 0\}$ such that for every $k \in \mathbb{N}$, this implies $\lim_{n\to\infty} \mathbb{S}^{k-1}(\mathbb{S}^n 0) = 1$ and $\lim_{n\to\infty} \mathbb{S}^k(\mathbb{S}^n 0) = 1 \neq \mathbb{S}1$. Hence, $\mathbb{S}$ is not k-continuous.

Definition 4. [13] For the metric space $(\mathcal{M}, d)$, the mapping $\mathbb{S} : \mathcal{M} \to \mathcal{M}$ satisfies the condition $(C; k)$ if there exists a non-negative constant k such that for each sequence $\{\mathfrak{x}_n\}$ in $\mathcal{M}$,

$$\mathfrak{x}_n \to \mathfrak{u} \in \mathcal{M} \text{ implies } \mathbb{G}(\mathfrak{u}) \leq k \cdot \limsup \mathbb{G}(\mathfrak{x}_n),$$

where $\mathbb{G}(\mathfrak{x}) = d(\mathfrak{x}, \mathbb{S}\mathfrak{x})$, $\mathfrak{x} \in \mathcal{M}$. The condition of $(C; 1)$ was introduced by Ćirić [14].

3. Main results

In this section, we will give one of the main results of this paper. The following theorem is a generalization of the Caristi [1] fixed-point theorem and ([6], Theorem 2.10) in the setting of w-distance; also, this theorem improves Theorem 2 in [8] for a great class of the mapping $\varphi : \mathcal{M} \to [0, \infty)$, which is not necessarily proper lower semi-continuous and bounded from below.

Theorem 3.1. Suppose that $(\mathcal{M}, d)$ is a complete metric space endowed with a w-distance ϱ on it. Let $\mathbb{S} : \mathcal{M} \to \mathcal{M}$ be a self-map and let $\varphi : \mathcal{M} \to [0, \infty)$, be a function such that for each $\mathfrak{m} \in \mathcal{M}$, we have

$$\varrho(\mathfrak{m}, \mathbb{S}\mathfrak{m}) \leq \varphi(\mathfrak{m}) - \varphi(\mathbb{S}\mathfrak{m}). \tag{1}$$

Suppose that either of the following holds:

(a) the mapping $\mathbb{S}$ is k-continuous or $\mathbb{S}^k$ is continuous;

(b) for every $\mathfrak{w} \in \mathcal{M}$ with $\mathfrak{w} \neq \mathbb{S}\mathfrak{w}$, we have $\inf\{\varrho(\mathfrak{x}, \mathfrak{w}) + \varrho(\mathfrak{x}, \mathbb{S}\mathfrak{x}) : \mathfrak{x} \in \mathcal{M}\} > 0$;

(c) the map $\mathbb{G}(\mathfrak{x}) = d(\mathfrak{x}, \mathbb{S}\mathfrak{x})$ is $\mathbb{S}$-orbitally lower semi-continuous;

(d) the mapping $\mathbb{S}$ satisfies the condition $(C; k)$

Then, the mapping $\mathbb{S}$ has a fixed-point $\mathfrak{u}$. Moreover, if $\mathfrak{u}_\circ = \mathbb{S}\mathfrak{u}_\circ$, we have $\varrho(\mathfrak{u}_\circ, \mathfrak{u}_\circ) = 0$.

Proof. **Step 1.** $\lim_{n\to\infty} \varrho(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}+1}) = 0$.

Suppose $\mathfrak{x}_0 \in \mathcal{M}$. Define a sequence $\{\mathfrak{x}_\mathfrak{n}\}$ by $\mathfrak{x}_1 = \mathbb{S}\mathfrak{x}_0, \mathfrak{x}_2 = \mathbb{S}\mathfrak{x}_1, \ldots, \mathfrak{x}_\mathfrak{n} = \mathbb{S}\mathfrak{x}_{\mathfrak{n}-1}, \ldots$, that is, $\mathfrak{x}_\mathfrak{n} = \mathbb{S}^n \mathfrak{x}_0$ for every $\mathfrak{n} \geq 1$. Then, from (7), we get $\varrho(\mathfrak{x}_0, \mathfrak{x}_1) = \varrho(\mathfrak{x}_0, \mathbb{S}\mathfrak{x}_0) \leq \varphi(\mathfrak{x}_0) - \varphi(\mathbb{S}\mathfrak{x}_0) = \varphi(\mathfrak{x}_0) - \varphi(\mathfrak{x}_1)$. Again using (7), and applying the inductive way, we have

$$\varrho(\mathfrak{x}_1, \mathfrak{x}_2) \leq \varphi(\mathfrak{x}_1) - \varphi(\mathfrak{x}_2)$$

$$\varrho(\mathfrak{x}_2, \mathfrak{x}_3) \leq \varphi(\mathfrak{x}_2) - \varphi(\mathfrak{x}_3)$$

$$\varrho(\mathfrak{x}_3, \mathfrak{x}_4) \leq \varphi(\mathfrak{x}_3) - \varphi(\mathfrak{x}_4)$$

$$\varrho(\mathfrak{x}_4, \mathfrak{x}_5) \leq \varphi(\mathfrak{x}_4) - \varphi(\mathfrak{x}_5)$$

$$\ldots$$

$$\varrho(\mathfrak{x}_{n-1}, \mathfrak{x}_n) \leq \varphi(\mathfrak{x}_{n-1}) - \varphi(\mathfrak{x}_n)$$

$$\varrho(\mathfrak{x}_n, \mathfrak{x}_{n+1}) \leq \varphi(\mathfrak{x}_n) - \varphi(\mathfrak{x}_{n+1})$$

So, by adding inequalities implies that

$$\varrho(\mathfrak{x}_0, \mathfrak{x}_1) + \varrho(\mathfrak{x}_1, \mathfrak{x}_2) + \varrho(\mathfrak{x}_2, \mathfrak{x}_3) + \ldots + \varrho(\mathfrak{x}_{\mathfrak{n}-1}, \mathfrak{x}_\mathfrak{n}) + \varrho(\mathfrak{x}_n, \mathfrak{x}_{n+1}) \leq \varphi(\mathfrak{x}_0) - \varphi(\mathfrak{x}_{\mathfrak{n}+1}) \leq \varphi(\mathfrak{x}_0).$$

Taking the limit as $n \to \infty$, the above inequality yields

$$\sum_{\mathfrak{n}=0}^{\infty} \varrho(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}+1}) \leq \varphi(\mathfrak{x}_0). \tag{2}$$

Thus, the series in (2) is convergent. Therefore, $\lim_{n\to\infty} \varrho(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}+1}) = 0$.

Step 2. We claim that

$$\lim_{\mathfrak{m},\mathfrak{n}} \varrho(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_\mathfrak{m}) = 0. \tag{3}$$

From (2), we get $\sum_{\mathfrak{n}=0}^{\infty} \varrho(\mathfrak{x}_\mathfrak{n}, \mathfrak{x}_{\mathfrak{n}+1}) \leq \infty$. Let $\varepsilon > 0$ and $\mathfrak{m} > \mathfrak{n}$. So, there exists an $\mathfrak{n}_0$ such that for each $\mathfrak{m}, \mathfrak{n} \geq \mathfrak{n}_0$, we have

$$\varrho(\mathfrak{x}_n,\mathfrak{x}_m) \le \sum_{i=n}^{m-1}\varrho(\mathfrak{x}_i,\mathfrak{x}_{i+1}) < \varepsilon.$$

So, (3) holds. By Lemma 1, $\{\mathfrak{x}_n\}$ is a Cauchy sequence. Since $\mathcal{M}$ is complete, there exists a $\mathfrak{u} \in \mathcal{M}$ such that $\mathfrak{x}_n \to u$ in $\mathcal{M}$. Also, for any positive integer k, we get $\mathbb{S}^{\mathfrak{k}}_{\mathfrak{x}_n} \to \mathfrak{u}$.

Step 3. We will prove that $\mathbb{S}\mathfrak{u} = \mathfrak{u}$ under either of the conditions (a)-(d).

Case (a). Suppose that $\mathbb{S}$ is a weakly orbitally continuous mapping. We have $\{\mathbb{S}^n\mathfrak{x}_0\}$ converges for every $\mathfrak{x}_0$ in $\mathcal{M}$. So, weak orbital continuity yields that there exists $\mathfrak{y}_0 \in \mathcal{M}$ such that $\lim_{n\to\infty}\mathbb{S}^n\mathfrak{y}_0 = u$ and $\lim_{n\to\infty}\mathbb{S}^{n+1}\mathfrak{y}_0 = \mathbb{S}\mathfrak{u}$ for some $\mathfrak{u}$ in $\mathcal{M}$. Hence, $\mathfrak{u} = \mathbb{S}\mathfrak{u}$, and therefore, $\mathfrak{u}$ is a fixed point of $\mathbb{S}$. If $\mathbb{S}$ is a k-continuous mapping or $\mathbb{S}^k$ is continuous mapping for some positive integer k, then $\mathbb{S}$ is weakly orbitally continuous and the proof follows.

Case (b). Assume that for $\mathfrak{w} \in \mathcal{M}$ with $\mathfrak{w} \neq \mathbb{S}\mathfrak{w}$ we have $\inf\{\varrho(\mathfrak{x},\mathfrak{w}) + \varrho(\mathfrak{x},\mathbb{S}\mathfrak{x}) : x \in \mathcal{M}\} > 0$, and $\mathfrak{n} \in \mathbb{N}$ be fixed. We know that $\{\mathfrak{x}_n\}$ converges to $\mathfrak{u}$ and $\varrho(\mathfrak{x}_n,.)$ is lower semi-continuous. So,

$$\varrho(\mathfrak{x}_n,\mathfrak{u}) \le \liminf_{m\to\infty}\varrho(\mathfrak{x}_n,\mathfrak{x}_m) \le \varphi(\mathfrak{x}_0).$$

Let $\mathfrak{u} \neq \mathbb{S}\mathfrak{u}$. Thus, by hypothesis, we get

$$0 < \inf\left\{\varrho(\mathfrak{x},\mathfrak{u}) + \varrho(\mathfrak{x},\mathbb{S}\mathfrak{x}) : \mathfrak{x} \in \mathcal{M}\right\} \le \inf\left\{\varrho(\mathfrak{x}_n,\mathfrak{u}) + \varrho(\mathfrak{x}_n,\mathfrak{x}_{n+1}) : \mathfrak{n} \in \mathbb{N}\right\} = 0.$$

This is a contradiction. So, we have $\mathfrak{u} = \mathbb{S}\mathfrak{u}$, and therefore, $\mathbb{S}$ has a fixed point.

Case (c). Define $\mathbb{G} : \mathcal{M} \to \mathbb{R}$ by $\mathbb{G}(\mathfrak{x}) = \varrho(\mathfrak{x},\mathbb{S}\mathfrak{x})$ for each $\mathfrak{x} \in \mathcal{M}$ is $\mathbb{S}$-orbitally lower semi-continuous. Suppose that $\{\mathfrak{x}_n\}$ is a sequence in $O(\mathfrak{x}_0;\infty)$ defined by $\mathfrak{x}_0 = \mathfrak{x}$ and $\mathfrak{x}_{n+1} = \mathbb{G}\mathfrak{x}_n = \mathbb{G}^{n+1}\mathfrak{x}_0$ for each $\mathfrak{n} \in \mathbb{N} \cup \{0\}$. Suppose that $\mathfrak{x}_n = \mathbb{S}^n\mathfrak{x} \to \mathfrak{u}$. Therefore,

$$0 \le \varrho(\mathfrak{u},\mathbb{S}\mathfrak{u}) = \mathbb{G}(\mathfrak{u}) \le \liminf_{n\to\infty}\mathbb{G}(\mathfrak{x}_n) = \liminf_{n\to\infty}\varrho(\mathbb{G}^n x,\mathbb{G}^{n+1}x) = \liminf_{n\to\infty}\varrho(\mathfrak{x}_{n+1},\mathfrak{x}_{n+2}) = 0.$$

Thus, we get $\mathfrak{u} = \mathbb{S}\mathfrak{u}$.

Case (d). Suppose that mapping $\mathbb{S} : \mathcal{M} \to \mathcal{M}$ satisfies the condition $(C;k)$. Then, there exists a non-negative constant k such that for each sequence $\{\mathfrak{x}_n\}$ in $\mathcal{M}$ from $\mathfrak{x}_n \to \mathfrak{x}_0 \in \mathcal{M}$, one can obtain that $\mathbb{G}(\mathfrak{x}_0) \le k \cdot \limsup\mathbb{G}(\mathfrak{x}_n)$, where $\mathbb{G}(\mathfrak{x}) = \varrho(\mathfrak{x},\mathbb{S}\mathfrak{x}), \mathfrak{x} \in \mathcal{M}$.

So, $\mathfrak{u} = \mathbb{S}\mathfrak{u}$.

Therefore, we showed that under either of the conditions (a)-(d), the self-map $\mathbb{S}$ has a fixed point.

Finally, we prove $\varrho(\mathfrak{u},\mathfrak{u}) = 0$. Assuming the theorem, we have

$$\varrho(\mathfrak{u},\mathfrak{u}) = \varrho(\mathfrak{u},\mathbb{S}\mathfrak{u}) \le \varphi(\mathfrak{u}) - \varphi(\mathbb{S}\mathfrak{u}) = \varphi(\mathfrak{u}) - \varphi(\mathfrak{u}) = 0.$$

Thus, $\varrho(\mathfrak{u},\mathfrak{u}) = 0$. Therefore, we get the desired result.

In the following example, we will show that our generalization of the Caristi fixed point theorem [1] and Theorem 2.10 in [6] are real; i.e., it shows that a Caristi contraction with respect to metric d may not be a Caristi contraction with respect to w-distance ϱ. Also, Theorem 1.1 and Theorem 2.10 in [6] do not work, while Theorem 3.1 gives us a fixed point.

Example 5. Assume $\mathcal{M} = [0,+\infty)$ equipped with usual metric d. Let a function $\varrho : \mathcal{M}\times\mathcal{M} \to [0,+\infty)$ defined by $\varrho(\mathfrak{m},\mathfrak{n}) = \mathfrak{n}$ be a w-distance on a complete metric space $(\mathcal{M},d)$. Also, we consider $\mathbb{S} : \mathcal{M} \to \mathcal{M}$ by

$$\mathbb{S}\mathfrak{m} \overset{\text{def}}{=} \begin{cases} 0 & \text{if } 0 \le \mathfrak{m} \le 1, \\ 1 & \text{if } \mathfrak{m} > 1. \end{cases} \tag{4}$$

Then, $\mathbb{S}$ is not continuous, but it is a 2-continuous mapping.

Now, we define $\varphi : \mathcal{M} \to [0,+\infty)$ as

$$\varphi(\mathfrak{m}) \overset{\text{def}}{=} \begin{cases} 0 & \text{if } \mathfrak{m} = 0, \\ 1-\mathfrak{m} & \text{if } 0 < \mathfrak{m} \le 1, \\ \mathfrak{m} & \text{if } \mathfrak{m} > 1. \end{cases} \tag{5}$$

Therefore, for $0 \ge \mathfrak{m} \le 1$, we have

$$\varrho(\mathfrak{m}, \mathbb{S}\mathfrak{m}) = \mathbb{S}\mathfrak{m} = 0 \le 1-\mathfrak{m} = 1-\mathfrak{m}-0 = \varphi(\mathfrak{m}) - \varphi(0) = \varphi(\mathfrak{m}) - \varphi(\mathbb{S}\mathfrak{m}).$$

So, it implies $\varrho(\mathfrak{m}, \mathbb{S}\mathfrak{m}) \le \varphi(\mathfrak{m}) - \varphi(\mathbb{S}\mathfrak{m})$. In addition, for $m > 1$, we get

$$\varrho(\mathfrak{m}, \mathbb{S}\mathfrak{m}) = \mathbb{S}\mathfrak{m} = 1 \le \mathfrak{m} = \mathfrak{m} - 0 = \varphi(\mathfrak{m}) - \varphi(1) = \varphi(\mathfrak{m}) - \varphi(\mathbb{S}\mathfrak{m}).$$

This shows that $\varrho(\mathfrak{m}, \mathbb{S}\mathfrak{m}) \le \varphi(\mathfrak{m}) - \varphi(\mathbb{S}\mathfrak{m})$.

On the other hand, inequality (7) doesn't hold for metric d because for $\frac{1}{2} < \mathfrak{m} \le 1$, we have

$$d(\mathfrak{m}, \mathbb{S}\mathfrak{m}) = \|\mathfrak{m} - \mathbb{S}\mathfrak{m}\| = \|\mathfrak{m} - 0\| = \mathfrak{m} > 1-\mathfrak{m} = \varphi(\mathfrak{m}) - \varphi(\mathbb{S}\mathfrak{m}).$$

In other words, it follows

$$d(\mathfrak{m}, \mathbb{S}\mathfrak{m}) \not\le \varphi(\mathfrak{m}) - \varphi(\mathbb{S}\mathfrak{m}) \qquad \forall \mathfrak{m} < 0.$$

Therefore, this shows that relation (7) holds for w-distance ϱ but metric d doesn't satisfy this inequality.

So, Theorem 1.1 and Theorem 2.10 in [6] do not work.

This yields that $\mathbb{S}$ satisfies all the conditions of Theorem 3.1. Thus, $\mathbb{S}$ has a fixed-point $\mathfrak{m} = 0$.

The following theorem is an extended version of Bollenbacher and Hicks' [9] theorem in setting w-distance.

Theorem 3.2. Suppose that $(\mathcal{M}, d)$ is a complete metric space endowed with a w-distance ϱ on it. Let $\mathbb{S}: \mathcal{M} \to \mathcal{M}$ be a self-map and let $\varphi: \mathcal{M} \to [0, \infty)$ be a function such that for each $\mathfrak{x}$ in $\mathcal{M}$, we have

$$\varrho(\mathfrak{y}, \mathbb{S}\mathfrak{y}) \le \varphi(\mathfrak{y}) - \varphi(\mathbb{S}\mathfrak{y}), \tag{6}$$

for each $\mathfrak{y} \in O(\mathfrak{x}, \infty)$ and each Cauchy sequence in $O(\mathfrak{x}, \infty)$ converges to some point in $\mathcal{M}$. Assume that either of the conditions (a)-(d) in Theorem 3.1 holds. Then, $\mathbb{S}$ has a unique fixed point. Moreover, for this fixed-point $\mathfrak{u}$, we have $\varrho(\mathfrak{u}, \mathfrak{u}) = 0$.

Proof. The proof is similar to the proof of Theorem 3.1, and so we deleted it.

The following theorem, improve and generalize Hicks and Rhoades' theorem [10] in the setting of w-distances.

Theorem 3.3. Suppose that $(\mathcal{M}, d)$ is a complete metric space endowed with a w-distance ϱ on it and $0 < k < 1$. Let for the self-map $\mathbb{S}: \mathcal{M} \to \mathcal{M}$ there exists an $\mathfrak{x}$ in $\mathcal{M}$ such that

$$\varrho(\mathbb{S}\mathfrak{y}, \mathbb{S}^2\mathfrak{y}) \le k\varrho(\mathfrak{y}, \mathbb{S}\mathfrak{y}), \tag{7}$$

for each $\mathfrak{y} \in O(\mathfrak{x}, \infty)$. Assume that either of the conditions (a)-(d) in Theorem 3.1 holds. Then, $\mathbb{S}$ has a unique fixed point. Moreover, for this fixed-point $\mathfrak{u}$, we have $\varrho(\mathfrak{u}, \mathfrak{u}) = 0$.

Proof. Putting $\varphi(\mathfrak{y}) = (1-k)^{-1}\varrho(\mathfrak{y}, \mathbb{S}\mathfrak{y})$, for all $\mathfrak{y} \in O(\mathfrak{x}, \infty)$, and using Theorem 3.2, one can conclude the statement.

Remark 3.1. Respectively, taking $\varrho = d$ in Theorems 3.1, 3.2, and 3.3, we can obtain Caristi's [1] theorems, Theorem 2.10 in [6], Bollenbacher and Hicks' [9] theorem, and also Hicks and Rhoades' [10] theorem as special cases.

4. Some new versions of Caristi's theorem

In this section, we will prove some results which is a generalization of Caristi's theorem for the lower semi-continuous function $\psi : \mathcal{M} \times \mathcal{M} \to [0,\infty)$. We start with the following theorem, which is a new version of Caristi's theorem in the setting of w-distances:

Theorem 4.1. Suppose that $(\mathcal{M}, d)$ is a complete metric space endowed with a w-distance ϱ on it. Let $\mathbb{S}$ be a self-mapping on $\mathcal{M}$, and let $\psi : \mathcal{M} \times \mathcal{M} \to [0,\infty)$ be a lower semi-continuous function with respect to the first variable such that for every $\mathfrak{m}, \mathfrak{n} \in \mathcal{M}$,

$$\varrho(\mathfrak{m},\mathfrak{n}) \le \psi(\mathfrak{m},\mathfrak{n}) - \psi(\mathbb{S}\mathfrak{m},\mathbb{S}\mathfrak{n}). \tag{8}$$

Assume that either of the conditions (a)-(d) in Theorem 3.1 holds. Then, $\mathbb{S}$ has a unique fixed point. Moreover, for this fixed-point $\mathfrak{u}$, we have $\varrho(\mathfrak{u}, \mathfrak{u}) = 0$.

Proof. Let $\mathfrak{m} \in \mathcal{M}$ be an arbitrary point. Define $\mathfrak{n} = \mathbb{S}\mathfrak{m}$ and $\varphi(\mathfrak{m}) = \psi(\mathfrak{m},\mathbb{S}\mathfrak{m})$. Then, from (8) for all $\mathfrak{m} \in \mathcal{M}$, we obtain

$$\varrho(\mathfrak{m},\mathbb{S}\mathfrak{m}) \le \varphi(\mathfrak{m}) - \varphi(\mathbb{S}\mathfrak{m}).$$

Also, by the hypothesis for ψ, this implies φ is a lower semi-continuous mapping. Hence, $\mathbb{S}$ can satisfy the conditions of Theorem 1.1. So, it follows the desired result. Now, we will show that $\mathbb{S}$ has a unique fixed point. Let r, t be two distinct fixed points of $\mathbb{S}$. From (8), we have

$$\varrho(r,t) \le \psi(r,t) - \psi(\mathbb{S}r,\mathbb{S}t) = \psi(r,t) - \psi(r,t) = 0.$$

Hence, we get $\varrho(r, t) = 0$, which yields $r = t$. The uniqueness concludes the proof of the theorem.

In the following corollary, we generalize and improve Theorem 2 in [15].

Corollary 1. Suppose that $(\mathcal{M}, d)$ is a complete metric space endowed with a w-distance ϱ on it. Let $\mathbb{S}$ be a self-mapping on $\mathcal{M}$. If, for every $\mathfrak{m}, \mathfrak{n} \in \mathcal{M}$, there exists $\eta \in [0,1)$ such that

$$\varrho(\mathbb{S}\mathfrak{m},\mathbb{S}\mathfrak{n}) \le \eta\varrho(\mathfrak{m},\mathfrak{n}). \tag{9}$$

Assume that either of the conditions (a)-(d) in Theorem 3.1 holds. Then, $\mathbb{S}$ has a unique fixed point. Moreover, for this fixed point $\mathfrak{u}$, we have $\varrho(\mathfrak{u}, \mathfrak{u}) = 0$.

Proof. Let $\mathfrak{m}$, $\mathfrak{n}$ be arbitrary points on $\mathcal{M}$. We define $\psi(\mathfrak{m},\mathfrak{n}) = \dfrac{\varrho(\mathfrak{m},\mathfrak{n})}{1-\eta}$. In (9), we get

$$(1-\eta)\varrho(\mathfrak{m},\mathfrak{n}) \le \varrho(\mathfrak{m},\mathfrak{n}) - \varrho(\mathbb{S}\mathfrak{m},\mathbb{S}\mathfrak{n}).$$

It follows from the above inequality that

$$\varrho(\mathfrak{m},\mathfrak{n}) \le \frac{\varrho(\mathfrak{m},\mathfrak{n})}{1-\eta} - \frac{\varrho(\mathbb{S}\mathfrak{m},\mathbb{S}\mathfrak{n})}{1-\eta} = \psi(\mathfrak{m},\mathfrak{n}) - \psi(\mathbb{S}\mathfrak{m},\mathbb{S}\mathfrak{n}).$$

Thus, by applying the relation (8) in Theorem 4.1, one can conclude that $\mathbb{S}$ has a unique fixed point.

Corollary 2. Suppose that ϱ be a w-distance on a complete metric space $(\mathcal{M}, d)$ such that $\varrho(\mathfrak{m}, \mathfrak{m}) = 0$ for all $\mathfrak{m} \in \mathcal{M}$, and $\mathbb{S}$ be a self-mapping on $\mathcal{M}$. If there exists a lower semi-continuous mapping $\alpha : [0,+\infty) \to [0,+\infty)$ with $\alpha(t) < t$, for all $t > 0$ and $\dfrac{\alpha(t)}{t}$ is a non-decreasing map and that

$$\varrho(\mathbb{S}\mathfrak{m},\mathbb{S}\mathfrak{n}) \le \alpha(\varrho(\mathfrak{m},\mathfrak{n})), \tag{10}$$

for every $\mathfrak{m},\mathfrak{n}\in\mathcal{M}$. Assume that either of the conditions (a)-(d) in Theorem 3.1 holds. Then, $\mathbb{S}$ has a unique fixed point. Moreover, for this fixed-point $\mathfrak{u}$, we have $\varrho(\mathfrak{u},\mathfrak{u})=0$.

Proof. Let $\mathfrak{m},\mathfrak{n}$ be arbitrary points on $\mathcal{M}$. We define $\psi(\mathfrak{m},\mathfrak{n})=\dfrac{\varrho(\mathfrak{m},\mathfrak{n})}{1-\dfrac{\alpha(\varrho(\mathfrak{m},\mathfrak{n}))}{\varrho(\mathfrak{m},\mathfrak{n})}}$, if $\mathfrak{m}\neq\mathfrak{n}$; and $\psi(\mathfrak{m},\mathfrak{m})=0$.

Using the inequality (10), we have

$$\left(1-\frac{\alpha(\varrho(\mathfrak{m},\mathfrak{n}))}{\varrho(\mathfrak{m},\mathfrak{n})}\right)\varrho(\mathfrak{m},\mathfrak{n})\leq\varrho(\mathfrak{m},\mathfrak{n})-\varrho(\mathbb{S}\mathfrak{m},\mathbb{S}\mathfrak{n}).$$

It follows from the above inequality that

$$\varrho(\mathfrak{m},\mathfrak{n})\leq\frac{\varrho(\mathfrak{m},\mathfrak{n})}{1-\dfrac{\alpha(\varrho(\mathfrak{m},\mathfrak{n}))}{\varrho(\mathfrak{m},\mathfrak{n})}}-\frac{\varrho(\mathbb{S}\mathfrak{m},\mathbb{S}\mathfrak{n})}{1-\dfrac{\alpha(\varrho(\mathfrak{m},\mathfrak{n}))}{\varrho(\mathfrak{m},\mathfrak{n})}}.$$

We know that $\alpha(t)/t$ is non-decreasing and $\varrho(\mathbb{S}\mathfrak{m},\mathbb{S}\mathfrak{n})<\varrho(\mathfrak{m},\mathfrak{n})$; thus

$$\varrho(\mathfrak{m},\mathfrak{n})\leq\frac{\varrho(\mathfrak{m},\mathfrak{n})}{1-\dfrac{\alpha(\varrho(\mathfrak{m},\mathfrak{n}))}{\varrho(\mathfrak{m},\mathfrak{n})}}-\frac{\varrho(\mathbb{S}\mathfrak{m},\mathbb{S}\mathfrak{n})}{1-\dfrac{\alpha(\varrho(\mathbb{S}\mathfrak{m},\mathbb{S}\mathfrak{n}))}{\varrho(\mathbb{S}\mathfrak{m},\mathbb{S}\mathfrak{n})}}=\psi(\mathfrak{m},\mathfrak{n})-\psi(\mathbb{S}\mathfrak{m},\mathbb{S}\mathfrak{n}).$$

Using Corollary 1, one can obtain that $\mathbb{S}$ has a unique fixed point.

Remark 4.1. Respectively, putting $\varrho=d$ in Theorem 4.1, Corollaries 1 and 2, we can obtain theorem Caristi [1], Corollaries 2.1-2.3 [16], and also Banach principle theorem [17] as a special case.

5. Another approach of Caristi's theorem using Zorn's lemma

In Theorem 2, Kada et al. [8] generalized the original Caristi's theorem in the setting of w-distances without using Zorn's lemma. Recently, Kirk and Shahzad [18] gave an easy and nice proof for the original Caristi's theorem in complete metric spaces using Zorn's lemma. In the following theorem, we will improve and generalize Theorem 3 [18] and Theorem 2 [8] in the setting of w-distances using Zorn's lemma.

Theorem 5.1. Suppose that $(\mathcal{M},d)$ is a complete metric space endowed with a w-distance ϱ on it. Let $S:\mathcal{M}\to\mathcal{M}$ be a self-map, and let $\varphi:\mathcal{M}\to[0,\infty)$ be a lower semi-continuous function such that

$$\varrho(\mathfrak{m},\mathbb{S}\mathfrak{m})\leq\varphi(\mathfrak{m})-\varphi(\mathbb{S}\mathfrak{m})\qquad\forall\mathfrak{m}\in\mathcal{M}.\tag{11}$$

Then, $\mathbb{S}$ has a fixed point. Moreover, for this fixed point $\overline{\mathfrak{m}}$, we have $\varrho(\overline{\mathfrak{m}},\overline{\mathfrak{m}})=0$.

Proof. Define the Brøndsted partial order on $\mathcal{M}$ as follows:

$$\mathfrak{m}\preceq\mathfrak{n}\Leftrightarrow\varrho(\mathfrak{m},\mathfrak{n})\leq\varphi(\mathfrak{m})-\varphi(\mathfrak{n}).$$

Suppose that J is an ordered set, and $\{\mathfrak{m}_\delta\}_{\delta\in J}$ is a chain in $(\mathcal{M},\preceq)$. So, we have

$$\left(\mu\leq\eta\Rightarrow\mathfrak{m}_\mu\preceq\mathfrak{m}_\eta\right)\;\Leftrightarrow\;\varrho(\mathfrak{m}_\mu,\mathfrak{m}_\eta)\leq\varphi(\mathfrak{m}_\mu)-\varphi(\mathfrak{m}_\eta).$$

Thus, it implies $\{\varphi(\mathfrak{m}_\delta)\}_{\delta\in J}$ is decreasing. Moreover, φ is bounded below. Therefore, $\lim_{\delta}\varphi(\mathfrak{m}_\delta)=t$. Hence, we get $\lim_{\mu,\eta}\varrho(\mathfrak{m}_\mu,\mathfrak{m}_\eta)=0$, and Lemma 1 implies that the sequence $\{\mathfrak{m}_\delta\}_{\delta\in J}$ is a Cauchy net. From the completeness of the space $\mathcal{M}$, there exists $\mathfrak{m}\in\mathcal{M}$ such that $\lim_\delta(\mathfrak{m}_\delta)=\mathfrak{m}$. Thus, for each $\mu\in J$, we conclude that

$$\varrho(\mathfrak{m}_\mu, \mathfrak{m}) = \lim_\delta \varrho(\mathfrak{m}_\mu, \mathfrak{m}_\delta) \le \lim_\delta(\varphi(\mathfrak{m}_\mu) - \varphi(\mathfrak{m}_\delta)) = \varphi(\mathfrak{m}_\mu) - t \le \varphi(\mathfrak{m}_\mu) - \varphi(\mathfrak{m}).$$

So, for every $\mu \in J$, we get $\mathfrak{m}_\mu \preceq \mathfrak{m}$. Therefore, $\mathfrak{m}$ is an upper bound for the chain $\{\varphi(\mathfrak{m}_\delta)\}_{\delta\in J}$. Now, from Zorn's lemma, $(\mathcal{M}, \preceq)$ has a maximal element $\overline{\mathfrak{m}}$. On the other hand, the relation (11) implies that $\overline{m} \preceq S\overline{m}$. Thus, $\overline{\mathfrak{m}} = \mathbb{S}\overline{\mathfrak{m}}$. The rest of proof is similar to the proof of Theorem 3.1.

6. Conclusion

In this paper, we generalized and proved many famous theorems in fixed point theory, such as Caristi's [1] theorem, Bollenbacher and Hicks' [9] theorem, and also Hicks and Rhoades' [10] theorem, by substituting the continuity assumption $\mathbb{S}$ with relatively weaker conditions of k-continuity or when $\mathbb{S}$ obeys the $(C; k)$ condition in a complete metric space with a w-distance. Then, we gave some new versions of Caristi's theorem. In the end, we obtained another proof of Caristi's theorem using Zorn's lemma.

Conflict of interest

There is no conflict of interest in this study.

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