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Research Article

Analytical Solution of One-Dimensional Keller-Segel Equations via New Homotopy Perturbation Method

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Abstract: For solving a system of nonlinear partial differential equations (PDE) emerging in an attractor one-dimensional chemotaxis model, we used a relatively new analytical method called the new modified homotopy perturbation method (NMHPM). We use NMHPM for solving one-dimensional Keller-Segel models for different types. Some properties show biologically acceptable dependency on parameter values, and numerical solutions are provided. NMHPM's stability and reduced computing time provide it with a broader range of applications. The algorithm provides analytical approximations for different types of Keller-Segel equations. Some numerical illustrations are given to show the efficiency of the algorithm.

***Keywords*:** HPM, Keller-Segel model, systems of PDE chemotaxis, numerical solution

MSC: 92C17, 35K58, 82C22

1. Introduction

Many phenomena in fluid mechanics, viscoelasticity, biology, physics, engineering, and other areas of science can be successfully modeled with the use of PDE. There are many numerical methods proposed for solving PDE phenomena, for example, the finite volume method (FVM) [1], the finite different method (FDM) [2-4], the variational iteration method (VIM) [5-8]. There are also many works in numerical methods for solving PDE phenomena of all kinds; we mention some of them [9-14], and the homotopy perturbation method (HPM) [15-26]. This is the last one (HPM), which was established by He [5, 27]. The HPM has been used by many authors to solve many problems in mathematics. This method, which does not require a small parameter in an equation, has a significant advantage in that it provides an approximate analytical solution to a wide range of linear and nonlinear problems in applied sciences.

The chemotaxis model is an important model analysis for many researchers, and it is regarded as one of the most important phenomenon studies, with Patlak [28] and Keller-Segel [29] being the first to discover it. Chemotaxis is represented by a model to explain the chemotaxis phenomenon, a chemical attraction that exists between organisms. Among the most important works accomplished for this phenomenon, we mention [30-32].

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2. Alternative framework

The algorithm of the HPM was introduced in [33, 34] and a modification algorithm of the HPM was introduced by Momani and Odibat [24]. The basic concepts of the HPM for the following nonlinear differential equation are as follows:

$$L(u) + N(u) = f(r), \; r \in \Omega, \tag{1}$$

with the boundary condition of

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \; r \in \Gamma, \tag{2}$$

where L is a linear differential operator and N is a nonlinear differential operator, $f(r)$ is a known analytic function. B is a boundary operator, n is the unit outward normal, and Γ is the boundary of the domain $\mathbf{\Omega}$. This HPM defined homotopy as

$$v(r, p): \Omega \times [0,1] \to \mathbb{R}, \tag{3}$$

which corresponds to

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0, \tag{4}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{5}$$

where $r \in \Omega$ and $p \in [0,1]$ are the attached parameters, and u_0 is the initial approach value that fulfills the initial condition. From equations (4) and (5), it is obtained

$$H(v, 0) = L(v) - L(u_0) = 0, \tag{6}$$

and

$$H(v, 1) = L(v) + N(v) - f(r) = 0. \tag{7}$$

He [5] assumes that the solutions of (4) and (5), can be expressed as the power series of p:

$$v = \sum_{i=0} p^i v_i = v_0 + pv_1 + p^2 v^2 + \ldots \tag{8}$$

the approach solution of (1) is

$$u = \lim_{p \to 1} \sum_{i=0} v_i = v_0 + v_1 + v_2 + \ldots \tag{9}$$

Furthermore, the HPM method is modified [35] into the modified HPM (MHPM) by including u^m on both sides of the homotopy equation (10). In this paper, we study a new modification of HPM (NMHPM) for solving a one-dimensional Keller-Segel model of different types. Now, we consider the system of nonlinear coupled equations given below:

$$\begin{cases} u_t + L_1(u) + N_1(u,c) = f_u(r),\ r \in \Omega, \\ c_t + L_2(c) + N_2(v,c) = f_c(r),\ r \in \Omega, \end{cases} \tag{10}$$

where L_1 and L_2 are linear operators, N_1 and N_2 are nonbilinear operators, and $f_u(r)$ and $f_c(r)$ are analytical functions. The initial conditions are

$$\begin{cases} u(x,0) = u_0(x), \\ c(x,0) = c_0(x), \end{cases} \tag{11}$$

From equation (10), we obtained the homotopy equation (12)

$$\begin{cases} H(v,p) = (1-p)[L(v) - L_1(u_0)] + p[v_t - L_1(v) - N_1(v,u) - f_u(r)] = 0, \\ H(\overline{v},p) = (1-p)[L(\overline{v}) - L_1(c_0)] + p[\overline{v}_t - L_2(\overline{v}) - N_2(\overline{v},c) - f_c(r)] = 0. \end{cases} \tag{12}$$

Hence, the solution of (4) and (5) in the form of p-power series is

$$\begin{cases} v = \sum_{i=0} p^i v_i = v_0 + pv_1 + p^2 v_2 + \ldots \\ \overline{v} = \sum_{i=0} p^i \overline{v}_i = \overline{v}_0 + p\overline{v}_1 + p^2 \overline{v}_2 + \ldots \end{cases} \tag{13}$$

by substituting (12) to (13) and taking $p = 1$, the solution function of (9) will be

$$u(t) = \sum_{n=0} v_n(t), \tag{14}$$

and

$$c(t) = \sum_{n=0} \overline{v}_n(t), \tag{15}$$

or

$$S(t) = (u(t), c(t)) = \sum_{n=0} (v_n(t), \overline{v}_n(t)).$$

3. Convergence of HPM

We have the equation

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{16}$$

According to equation (16), we obtain

$$L(v) = L(u_0) + pL(u_0) + p[N(v) - f(r)], \tag{17}$$

Applying the inverse operator, L^{-1}, to both sides of equation (17), we obtain

$$v = u_0 + p[L^{-1} f(r) - L^{-1} N(v) - u_0], \tag{18}$$

Suppose that $v = \sum_{i=0} p^i v_i$, substituting (3) into the right-hand side of equation (18), we have equation (18) in the following form:

$$v = u_0 + p[L^{-1} f(r) - L^{-1} N(\sum_{i=0} p^i v_i) - u_0], \tag{19}$$

if $p \to 1$, the exact solution may be obtained by using

$$U = \lim_{p \to 1} V \tag{20}$$

$$= L^{-1} f(r) - L^{-1} N\left(\sum_{i=0} p^i v_i\right) \tag{21}$$

$$= L^{-1} f(r) - \left(\sum_{i=0} p^i (L^{-1} N) v_i\right) \tag{22}$$

Theorem 1. (Sufficient condition of convergence.) Suppose that X and Y are Banach spaces and N: $X \to Y$ is a contractive nonlinear mapping, that is:

$$\forall W, Z \in X : \| N(W) - N(Z) \| \leq \varepsilon \| W - Z \|, \leq \varepsilon \leq 1.$$

Then, according to Banach's fixed point theorem, N has a unique fixed point u, that is, $N(u) = u$. Assume that the sequence generated by the HPM can be written as

$$W_n = N(W_n - 1), N(W_n - 1) = \sum_{i=0}^{n-1} W_i,$$

and suppose that $W_0 = W_0 \in Br(W)$, where

$$Br(W) = Z \in X \| Z - W \| \leq r,$$

Then, we have

i. $W_n \in Br(W)0$.

ii. $\lim_{n \to \infty} W_n = W$.

The proof of convergence of the series (14) and (15) has been proved in [26, 36, 37]. Hence, the convergence of the series $u(t)$, $v(t)$ is proved, and as $S(t) = (u(t), v(t))$, so $S(t)$ is convergent.

4. Application

We consider the Keller-Segel model as follows:

$$\begin{cases} u_t - u_{xx} + (uc_x)_x = 0, \text{ in } (x,t) \in \mathbb{R} \times [0,1], \\ c_t - c_{xx} - u + c = 0, \text{ in } (x,t) \in \mathbb{R} \times [0,1], \end{cases} \tag{23}$$

with subject to the initial conditions of

$$\begin{cases} u(x,0) = u_0(x), x \in \mathbb{R}, \\ c(x,0) = c_0(x), x \in \mathbb{R}, \end{cases} \tag{24}$$

where $u = u(x, t)$ denotes the population density of biological individuals and $c = c(t, x)$ denotes the concentration of chemical substance. Now that we have solved the Keller-Segel model by NMHPM, we take into account the homotopy defined as follows:

$$\begin{cases} H(v,p) = (1-p)[v_t - \frac{\partial u_0}{\partial t}] + p(v_t - v_{xx} + (v\overline{v}_x)_x) = 0, \\ H(\overline{v},p) = (1-p)[\overline{v}_t - \frac{\partial c_0}{\partial t}] + p(\overline{v}_t - \overline{v}_{xx} - v\overline{v}) = 0, \end{cases} \tag{25}$$

or

$$H((v,\overline{v}),p) = (1-p)\begin{pmatrix} v_t - \frac{\partial u_0}{\partial t} \\ \\ \overline{v}_t - \frac{\partial c_0}{\partial t} \end{pmatrix} + p\begin{pmatrix} v_t - v_{xx} + (v\overline{v}_x)_x \\ \\ \overline{v}_t - \overline{v}_{xx} - v\overline{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}. \tag{26}$$

Substituting v and $\overline{v}$ from (13) into (25) or (37) terms, we can obtain:

$$p^0 : \begin{cases} \frac{\partial v_0}{\partial t} = \frac{\partial u_0}{\partial t}, \\ \\ \frac{\partial \overline{v}_0}{\partial t} = \frac{\partial c_0}{\partial t}, \end{cases} \tag{27}$$

$$p^1 : \begin{cases} \frac{\partial v_1}{\partial t} - \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial v_0}{\partial x}\frac{\partial \overline{v}_0}{\partial x} + v_0 \frac{\partial^2 \overline{v}_0}{\partial x^2} = 0, \\ \frac{\partial \overline{v}_1}{\partial t} - \frac{\partial^2 \overline{v}_0}{\partial x^2} - v_0 + \overline{v}_0 = 0, \end{cases} \tag{28}$$

$$p^2 : \begin{cases} \frac{\partial v_2}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial v_0}{\partial x}\frac{\partial \overline{v}_1}{\partial x} + \frac{\partial v_1}{\partial x}\frac{\partial \overline{v}_0}{\partial x} + v_1 \frac{\partial^2 \overline{v}_0}{\partial x^2} + v_0 \frac{\partial^2 \overline{v}_1}{\partial x^2} = 0, \\ \frac{\partial \overline{v}_2}{\partial t} - \frac{\partial^2 \overline{v}_1}{\partial x^2} - v_0 + \overline{v}_0 - v_1 + \overline{v}_1 = 0. \end{cases} \tag{29}$$

We can write (29) as follows:

$$\begin{cases} \partial v_i \partial t = \frac{\partial^2 v_{i-1}}{\partial x^2} - \sum_{\substack{j=0 \\ i \neq j}}^{i-1} \left(\frac{\partial v_j}{\partial x}\frac{\partial \overline{v}_{i-j-1}}{\partial x} + v_j \frac{\partial^2 \overline{v}_{i-j-1}}{\partial x^2} \right), i = 1:n, \\ \frac{\partial \overline{v}_i}{\partial t} = \frac{\partial^2 v_{i-1}}{\partial x^2} + \sum_{\substack{j=0 \\ i \neq j}}^{i-1} v_j - \overline{v}_j, i = 1:n, \end{cases} \tag{30}$$

for solving (29) or (30), we integrate with t, we obtain

$$v_i = \int\left[\frac{\partial^2 v_{i-1}}{\partial x^2} - \sum_{\substack{j=0 \\ i\neq j}}^{i-1}\left(\frac{\partial v_j}{\partial x}\frac{\partial \overline{v}_{i-j-1}}{\partial x} + v_j\frac{\partial^2 \overline{v}_{i-j-1}}{\partial x^2}\right)\right]dt,\ i = 1:n,$$

$$\overline{v}_i = \int\left[\frac{\partial^2 \overline{v}_{i-1}}{\partial x^2} + \sum_{\substack{j=0 \\ i\neq j}}^{i-1} v_j - \overline{v}_j\right]dt, i = 1:n,$$

with initial conditions $u_0(x)$ and $c_0(x)$ is known.

5. Numerical solutions test for Keller-Segel model

Now, we solve this problem with the initial condition defined as

$$\begin{cases} v_0(x) = u_0(x) = \cos kx,\ (x) \in \mathbb{R}, \\ \overline{v}_0(x) = c_0(x) = \cos kx,\ (x) \in \mathbb{R}, \end{cases} \tag{31}$$

we have for $i = 1, 2, ...,$

$$\begin{cases} v_1 = -t(k^2\sin(kx)^2 - k^2\cos(kx)^2 + k^2\cos(kx)), \\ \overline{v}_1 = -k^2 t\cos(kx), \end{cases}$$

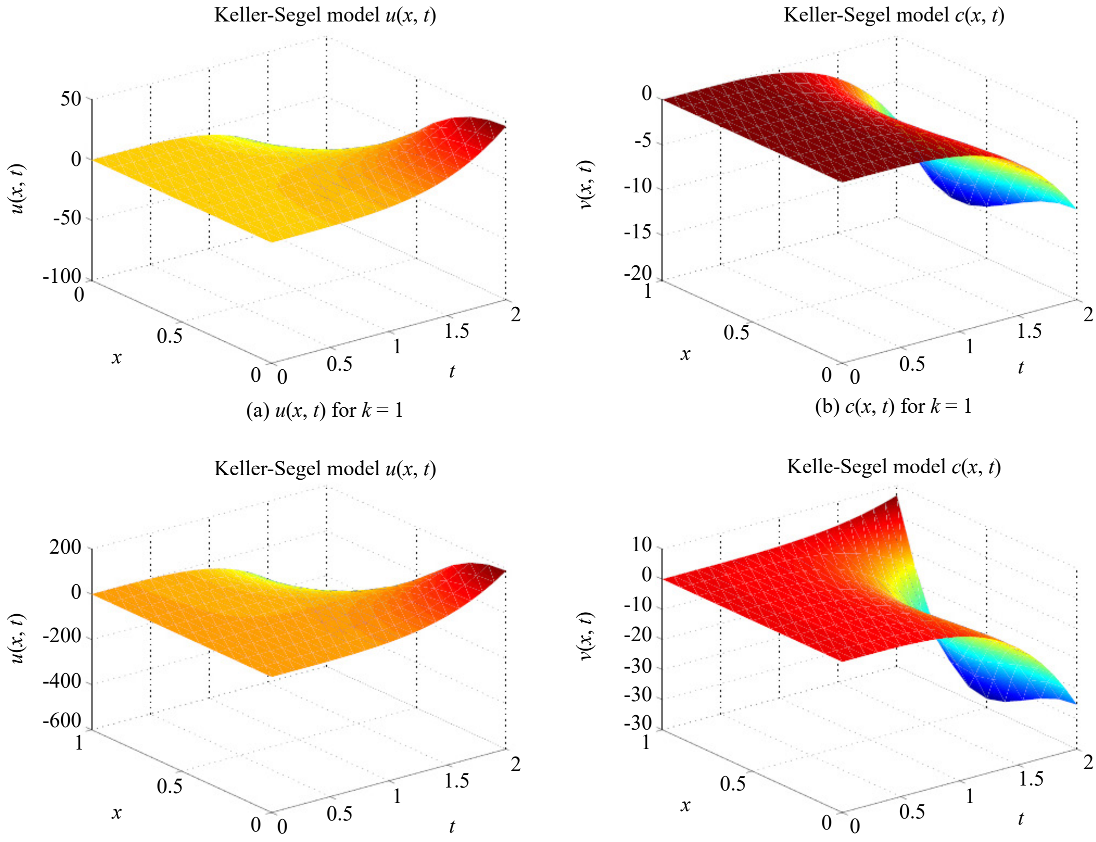
and

$$\begin{cases} v_2 = \frac{(k^4t^2\cos(kx)^3)}{2} - \frac{(5k^4t^2\cos(kx)^2)}{2} + \frac{(5k^4t^2\sin(kx)^2)}{2} + \frac{(k^4t^2\cos(kx))}{2} - \frac{(5k^4t^2\cos(kx)\sin(kx)^2)}{2}, \\ \overline{v}_2 = \frac{(k^2t^2)}{2} - \frac{k^2t^2\cos(kx)^2 + (k^4t^2\cos(kx))}{2}. \end{cases}$$

The remaining terms can be obtained using the iterative formula. However, we only consider a few terms of the series of solutions, and the asymptotic solution is given as:

$$\begin{cases} u(x,t) = v_1(x,t) + v_2(x,t) + v_3(x,t) + v_4(x,t) + ... \\ c(x,t) = \overline{v}_1(x,t) + \overline{v}_2(x,t) + \overline{v}_3(x,t) + \overline{v}_4(x,t) + ... \end{cases}$$

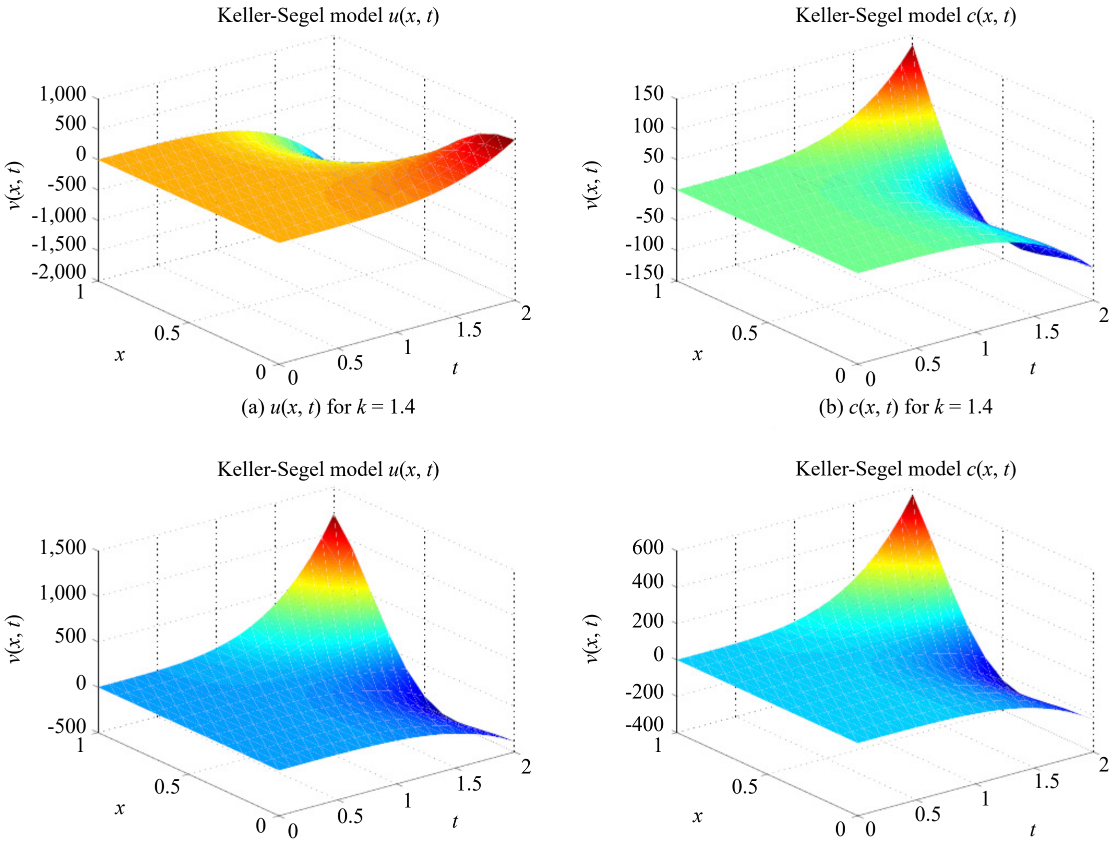
In the below, we show some results of the approximation solution of one-dimensional Keller-Segel with different parameter types. The following Figures 1 to 3 illustrate the biological behavior of the coupled solution for the following set of constants: k = 1, 1.2, 1.4, 1.6, 1.7, and 1.8.



(a) $u(x, t)$ for $k = 1$ (b) $c(x, t)$ for $k = 1$

(c) $u(x, t)$ for $k = 1.2$ (d) $c(x, t)$ for $k = 1.2$

Figure 1. Coupled solution for $k = 1$ and 1.2



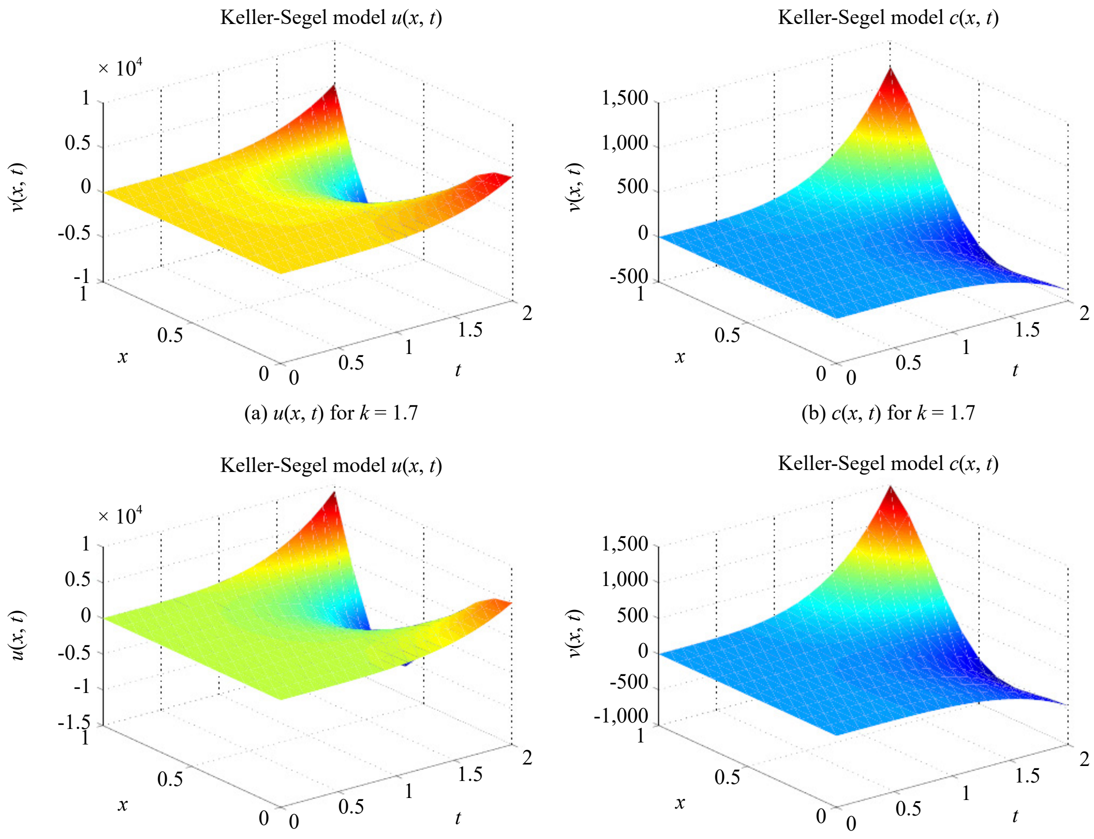
(a) $u(x, t)$ for $k = 1.4$

(b) $c(x, t)$ for $k = 1.4$

(c) $u(x, t)$ for $k = 1.6$

(d) $c(x, t)$ for $k = 1.6$

Figure 2. Coupled solution for $k = 1.4$ and 1.6



(a) $u(x, t)$ for $k = 1.7$ (b) $c(x, t)$ for $k = 1.7$

(c) $u(x, t)$ for $k = 1.8$ (d) $c(x, t)$ for $k = 1.8$

Figure 3. Coupled solution for k =1.7 and 1.8

6. Test method for classical Keller-Segal model

Where we study the numerical solution for the classical version of the new HPM (NHPM) with initial conditions, we present the results of the approximate coupled solution of different types of parameters and source terms and functions. We define the new version of the Keller-Segel model as follows:

$$\begin{cases} u_t(x,t) - \delta_u u_{xx} + (\chi u c_x)_x = 0, \text{ in } (x,t) \in \mathbb{R} \times [0,1], \\ \\ c_t(x,t) - \delta_c c_{xx} + \rho u - \tau c = 0, \text{ in } (x,t) \in \mathbb{R} \times [0,1], \end{cases} \tag{32}$$

where $u = u(x, t)$ denotes the density of the cells in position $x \in [0]$, at time $t \in \mathbb{R}, c = c(x,t)$ is the concentration of chemical attractant in position $x \in \mathbb{R}$, at time $t \in [0,1], \tau, \rho$, and χ are positive constants, where diffusion coefficients δ_u and δ_c, respectively, are assumed as constants. With zero Dirichlet boundary conditions and initial conditions, we define

$$\begin{cases} u(x,0) = u_0(x), & x \in \mathbb{R}, \\ c(x,0) = c_0(x), & x \in \mathbb{R}. \end{cases}$$

Now, we study numerical coupled solutions of the problem (32) by NHPM with initial conditions as follows:

$$\begin{cases} u_0(x) = (x^2+1)e^x, \\ c_0(x) = (x^2+1)e^{-x}, \end{cases} \tag{33}$$

and parameters $\tau = \rho = \chi = d_1 = d_2 = \delta_u = \delta_c = 1$.

Now, we compute the approximate solution of the problem associated with the initial condition and the parameters defined in (33) and (6) (respectively) by NHPM, and we find

$$\begin{cases} v_0(x) = (x^2+1)e^x, \\ \overline{v}_0(x) = (x^2+1)e^{-x}, \end{cases} \tag{34}$$

we have for $i = 1, 2, \ldots$

$$\begin{cases} v_1 = t(2e^x + e^x(x+1) - (e^{-x} - e^{-x}(x+1))(e^x + e^x(x+1)) + e^x(2e^{-x} - e^x(x+1))(x+1)), \\ \overline{v}_1 = -t(2e^{-x} + e^x(x+1) - 2e^{-x}(x+1)), \end{cases}$$

and

$$\begin{cases} v_2 = \frac{(5t^2e^x)}{2} + \frac{(t^2e^{-x}))}{2} + t^2x + \frac{(3t^2)}{2} - t^2x^2e^{-x} + \frac{(t^2xe^x)}{2} + \frac{(3t^2xe^{-x})}{2}(3t^2xe^{-x}), \\ \overline{v}_2 = 2t^2xe^{-x} - 2t^2e^{-x} - t^2x - \frac{t^2}{2} - \frac{(3t^2xe^x)}{2} - \frac{(7t^2e^x)}{2}. \end{cases}$$

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Now, we show the approximate solution in Figures 4 to 8 with different parameters.

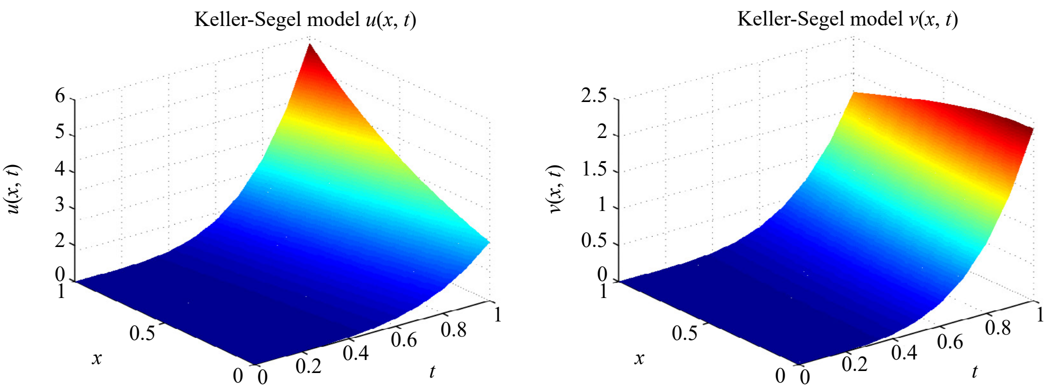

Figure 4. The approximate solution of $u(x, t)$ and $v(x, t)$ with the parameters $\tau = 2.75$, $\rho = 10^{-3}$, $\chi = 0.05$, $\delta_u = 1.5$, $\delta_c = 10^{-3}$

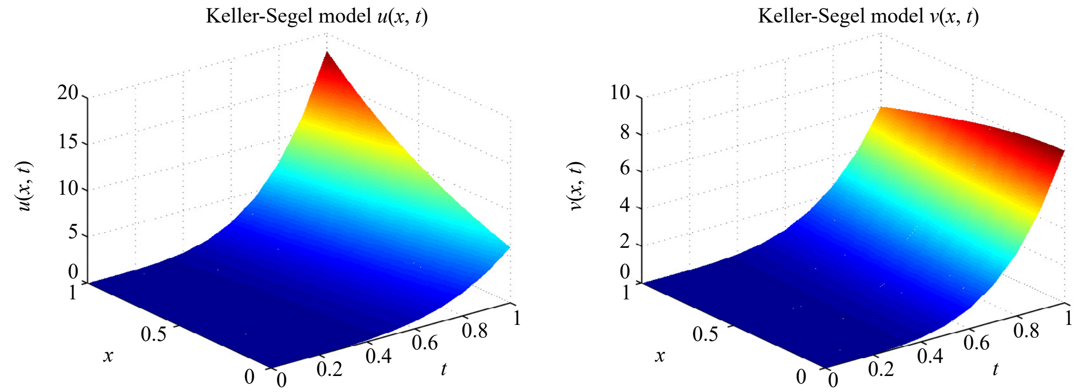

Figure 5. The approximate solution of $u(x, t)$ and $v(x, t)$ with the parameters $\tau = 3.75$, $\rho = 5 \times 10^{-5}$, $\chi = 5 \times 10^{-4}$, $\delta_u = 2$, $\delta_c = 10^{-5}$

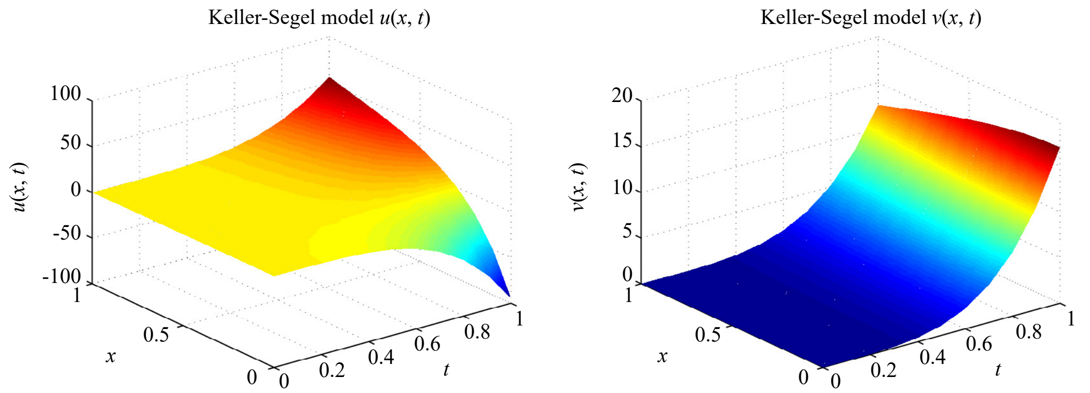

Figure 6. The approximate solution of $u(x, t)$ and $v(x, t)$ with the parameters $\tau = 4.5$, $\rho = 10^{-9}$, $\chi = 2.5$, $\delta_u = 2.5$, $\delta_c = 10^{-7}$

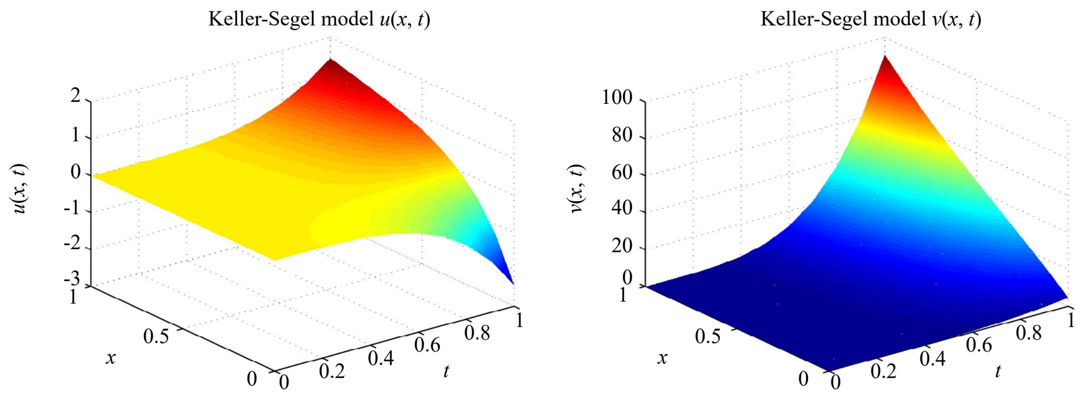

Figure 7. The approximate solution of $u(x, t)$ and $v(x, t)$ with the parameters $\tau = 1$, $\rho = 1$, $\chi = 1$, $\delta_u = 1$, $\delta_v = 1$

In this case, we present a solution to the problem (8) with the source term functions $f_u = \cos(x)e^x + 1, f_v = \cos(x)$ $e^{-x} - 1$, and $\tau = 1, \rho = 1, \chi = 1, \delta_u = 1, \delta_c = 1$.

The behavior of the solution of the system of equations (8) with initial conditions in equations (33) and various types of parameters (6) is shown in the above Figures 4 to 7. These solutions describe the biological cell density and chemical substance concentrations in places $x \in R$ and $t \in [0, 1]$ for a particular set of theoretical parameters chosen from the literature. While Figure 8 shows the behavior of a coupled solution for as source terms functions f_u and f_c of space, we can deduce that the cell density biological increases in space as the concentration of the chemical substance decreases. When source terms functions f_u and f_c are used, the cell density biological and chemical substance concentrations increase in space. The approximate solutions obtained, as seen in the graphical depiction, mimic the behavior of the real-world situation.

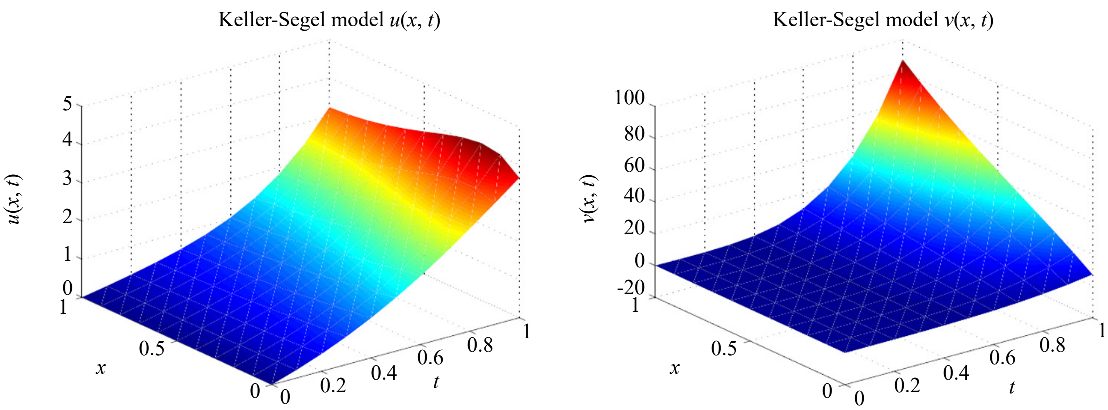

Figure 8. Coupled solution for source term functions f_u and f_c

7. Test method for the new version of the Keller-Segal model

We consider the new version of the Keller-Segel model defined as:

$$\begin{cases} u_t(x,t) - u_{xx} + (uc_x)_x = 0, \text{ in } (x,t) \in \mathbb{R} \times [0,1], \\ c_t(x,t) - c_{xx} + uc - u_x c_x = 0, \text{ in } (x,t) \in \mathbb{R} \times [0,1], \end{cases} \tag{35}$$

with the initial condition as follows:

$$\begin{cases} u(x,0) = u_0(x) = me^x, \text{ in } (x) \in \mathbb{R}, \\ c(x,0) = c_0(x) = ke^{-x}, \text{ in } (x) \in \mathbb{R}. \end{cases} \tag{36}$$

Now, we use homotopy (12) to solve the new version of the Keller-Segel model by NMHPM, so we have

$$H((v,\bar{v}),p) = (1-p)\begin{pmatrix} v_t - \dfrac{\partial u_0}{\partial t} \\ \bar{v}_t - \dfrac{\partial c_0}{\partial t} \end{pmatrix} + p\begin{pmatrix} v_t - v_{xx} + (v\bar{v}_x)_x \\ \bar{v}_t - \bar{v}_{xx} - v\bar{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{37}$$

The NMHPM method gives the solution of the new version of the Keller-Segel model as follows:

$$\begin{cases} \dfrac{\partial v_i}{\partial t} = \dfrac{\partial^2 v_{i-1}}{\partial x^2} - \sum\limits_{\substack{j=0 \\ i \neq j}}^{i-1} \left(\dfrac{\partial v_j}{\partial x} \dfrac{\partial \overline{v}_{i-j-1}}{\partial x} + v_j \dfrac{\partial^2 \overline{v}_{i-j-1}}{\partial x^2} \right), \ \ i = 1 : n, \\ \dfrac{\partial \overline{v}}{\partial t} = \dfrac{\partial^2 \overline{v}_{i-1}}{\partial x^2} + \sum\limits_{\substack{j=0 \\ i \neq j}}^{i-1} \left(v_{i-j-1} \overline{v}_j + \dfrac{\partial v_{i-j-1}}{\partial x} \dfrac{\partial \overline{v}_j}{\partial x} \right), \ \ i = 1 : n, \end{cases} \tag{38}$$

$$v_i = \int \left[\frac{\partial^2 v_{i-1}}{\partial x^2} - \sum_{\substack{j=0 \\ i \neq j}}^{i-1} \left(\frac{\partial v_j}{\partial x} \frac{\partial \overline{v}_{i-j-1}}{\partial x} + v_j \frac{\partial^2 \overline{v}_{i-j-1}}{\partial x^2} \right) \right] dt, \ i = 1 : n,$$

$$\overline{v}_i = \int \left[\frac{\partial^2 \overline{v}_{i-1}}{\partial x^2} - \sum_{\substack{j=0 \\ i \neq j}}^{i-1} v_{i-j-1} \overline{v}_j + \frac{\partial v_{i-j-1}}{\partial x} \frac{\partial \overline{v}_j}{\partial x} \right] dt, \ i = 1 : n,$$

So, the solution reads

$$\begin{cases} v_0(x) = me^x, \\ \overline{v}_0 = ke^{-x}, \end{cases} \tag{39}$$

$$\begin{cases} v_1(x, t) = me^x t, \\ \overline{v}_1(x, t) = ke^{-x} t, \end{cases} \tag{40}$$

$$\begin{cases} v_2(x, t) = \dfrac{1}{2} me^x t^2, \\ \overline{v}_2(x, t) = \dfrac{1}{2} ke^{-x} t^2, \end{cases} \tag{41}$$

$$\begin{cases} v_3(x, t) = \dfrac{1}{6} me^x t^3, \\ \overline{v}_3(x, t) = \dfrac{1}{6} ke^{-x} t^3, \end{cases} \tag{42}$$

$$\begin{cases} v_4(x, t) = \dfrac{1}{24} me^x t^4, \\ \overline{v}_4(x, t) = \dfrac{1}{24} ke^{-x} t^4, \end{cases} \tag{43}$$

$$\begin{matrix} . \\ . \\ . \end{matrix} \tag{44}$$

The iteration formula can be used to acquire the remaining terms. Only a few terms of the solutions are considered, and the asymptotic solution is as follows with $n = 0, 1, \ldots$:

$$\begin{cases} v_n(x,t) = \dfrac{1}{\prod_{h=2}^{n} h} m e^{x} t^{n} \\ \overline{v}_n(x,t) = \dfrac{1}{\prod_{h=2}^{n} h} k e^{-x} t^{n}, \end{cases} \tag{45}$$

Thus, we obtain the approximation solution of the problem as follows:

$$\begin{cases} u_N(x,t) = \sum_{n=0}^{N} v_N = \sum_{n=0}^{N} \dfrac{1}{\prod_{n=2}^{N} n} m e^{x} t^{N}, \\ c_N = \sum_{n=0}^{N} \overline{v}_N(x,t) = \sum_{n=0}^{N} \dfrac{1}{\prod_{n=2}^{N} n} k e^{-x} t^{N}. \end{cases} \tag{46}$$

The biological behavior of the coupled solution for the following and fixed time $t = 0.5$ and $t = 1$ sets is shown in Figures 9 and 10:

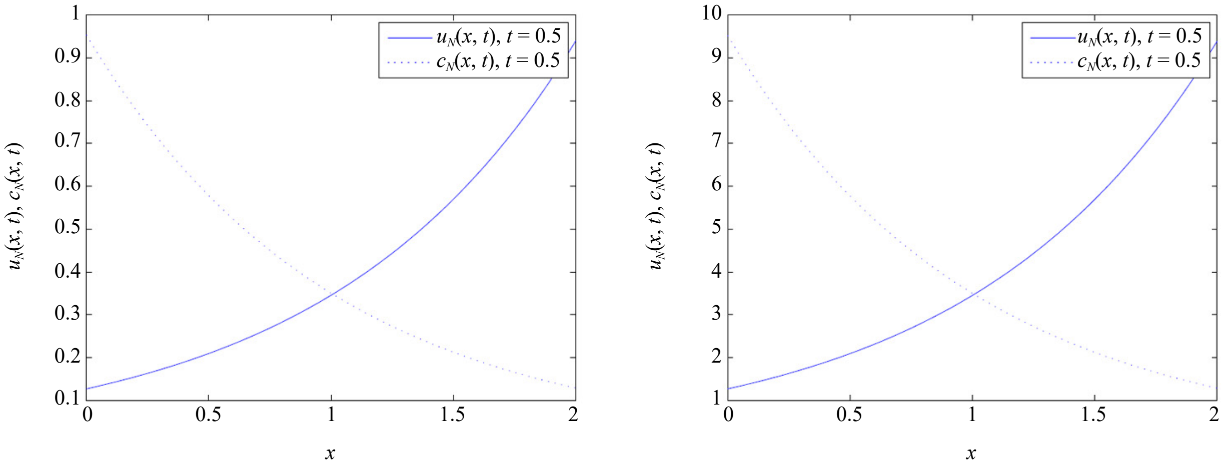

Figure 9. Coupled solutions as function of *x* for a fixed time

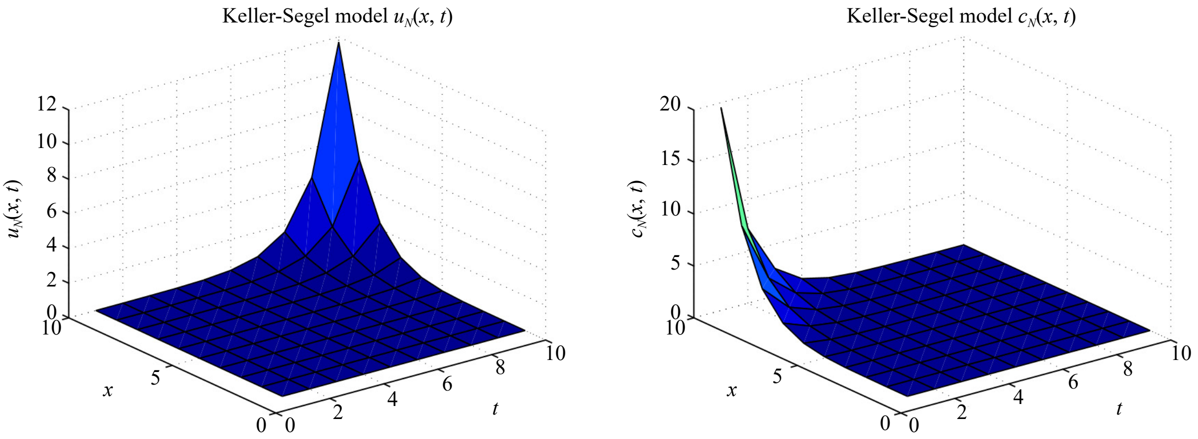

Figure 10. Coupled solutions of $u_N(x, t)$ and $c_N(x, t)$

8. Conclusion

In this research, we demonstrate how to solve nonlinear coupled partial differential equations emerging in an attractor one-dimensional Keller-Segel dynamics system using a relatively new analytical technique, the NMHPM. The model mimics the regular biological diffusion behavior seen in the field, according to the analysis and conclusions of the nonlinear system of attractors in the one-dimensional Keller-Segel equation.

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Conflict of interest

The authors have no conflicts to disclose.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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