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Research Article

Error Analysis Using Three and Four Stage Eighth Order Embedded Runge-Kutta Method for Sixth Order Ordinary Differential Equation $v^{vi}(u) = f(u, v, v', v'', v''', v^{iv})$

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Abstract: The present paper aims at providing an insight to embedded Runge-Kutta sixth order (RKSD) ordinary differential equation method for solving the initial value problem of order six of type $v^{vi}(u) = f(u, v, v', v'', v''', v^{iv})$. The concept of order conditions for the three and four stages up to the eighth and ninth orders, respectively, is designed and evaluated; furthermore, the zero-stability of the proposed method is proved. Comparisons are made between these orders with the help of a mathematical example, and global and local truncated error norms are evaluated.

Keywords: ordinary differential equations, embedded Runge-Kutta methods, initial value problem, local and global truncation error, zero stability

MSC: 34A45, 65L70, 65L07

1. Introduction

This paper focuses on solving the equations of the form

$$v^{vi}(u) = f(u, v, v', v'', v''', v^{iv}) \tag{1}$$

with initial conditions as

$$v(u_0) = \alpha_0, v'(u_0) = \alpha'_0, v''(u_0) = \alpha''_0, v'''(u_0) = \alpha'''_0, v^{iv}(u_0) = \alpha_0^{iv}, v^{v}(u_0) = \alpha_0^{v}. \tag{2}$$

The solution and optimization of ordinary differential equations (ODEs) have been of great interest to scientists for a very long time [1, 2], as they occupy a special place in many executions in the field of science and engineering [3-6]. For instance, third-order ODEs are used in thin film flow problems [7-10], and fifth-order differential equations are used in fiber preservation transformations [11]. Beccar et al. [12] and Abdulsalam et al. [13] contributed towards analyzing various orders of ODEs. Malhotra et al. [14] optimized the real-life problem, and Kaur et al. [15] used an improvised concept for solving the differential equations. Hussain et al. [16] have worked on numerical integration of third-

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order differential equations using the four-stage fifth-order Runge-Kutta method. The method proposed is considered to require a smaller number of stages than existing Runge-Kutta methods [17]. Hatun et al. [18] have developed a simulator containing 38 different Runge-Kutta-based methods for solving differential equations. The solution to the problem can be presented numerically as well as graphically, and a performance analysis can also be obtained through this. Mechee et al. [19] in their paper solve special fourth-order ODEs by using direct explicit integrators of the Runge-Kutta method. By using this technique, the authors have saved computational time, further increasing efficiency [20]. However, some authors, like Pandey et al. [21] have worked on sixth-order differential equations, but their work is limited to only boundary value problems, and not much work has been reported on solving sixth-order ordinary initial value problems. Moreover, the calculation of the errors associated with the methods is still an issue.

The current research paper brings forth the Runge-Kutta technique for evaluating global and local truncated errors of the initial value problem (1-2).

2. Runge-Kutta type sixth order ODE

The sixth order ODE evaluated in this paper be represented as:

$$v^{vi}\left(u\right)=f\left(u,v,v',v'',v''',v^{iv}\right) \tag{3}$$

with initial conditions as

$$v\left(u_0\right)=\alpha_0, v'\left(u_0\right)=\alpha_0', v''\left(u_0\right)=\alpha_0'', v'''\left(u_0\right)=\alpha_0''', v^{iv}\left(u_0\right)=\alpha_0^{iv}, v^{v}\left(u_0\right)=\alpha_0^{v}. \tag{4}$$

2.1 *Derivation of Runge-Kutta sixth order (RKSD) method*

$$v_{n+1}=v_n+hv_n'+\frac{h^2}{2}v_n''+\frac{h^3}{3!}v_n'''+\frac{h^4}{4!}v_n^{iv}+\frac{h^5}{5!}v_n^{v}+h^6\sum_{i=1}^{s}b_ik_i \tag{5}$$

$$v_{n+1}'=v_n'+hv_n''+\frac{h^2}{2}v_n'''+\frac{h^3}{3!}v_n^{iv}+\frac{h^4}{4!}v_n^{v}+h^5\sum_{i=1}^{s}b_i'k_i \tag{6}$$

$$v_{n+1}''=v_n''+hv_n'''+\frac{h^2}{2}v_n^{iv}+\frac{h^3}{3!}v_n^{v}+h^4\sum_{i=1}^{s}b_i''k_i \tag{7}$$

$$v_{+1}'''=v_n'''+hv_n^{iv}+\frac{h^2}{2}v_n^{v}+h^3\sum_{i=1}^{s}b_i'''k_i \tag{8}$$

$$v_{n+1}^{iv}=v_n^{iv}+hv_n^{iv}+h^2\sum_{i=1}^{s}b_i^{iv}k_i \tag{9}$$

$$v_{n+1}^{v}=v_n^{v}+h\sum_{i=1}^{s}b_i^{v}k_{i}, \tag{10}$$

where

$$k_1 = f\left(u_n, v_n, v'_n, v''_n, v'''_n, v_n^{iv}\right)$$

$$k_i = f\Bigg(u_n + c_i h, v_n + hc_i v'_n + \frac{h^2c_i^2}{2}v''_n + \frac{h^3c_i^3}{3!}v'''_n + \frac{h^4c_i^4}{4!}v_n^{iv} + \frac{h^5c_i^5}{5!}v_n^{v} + h^6\sum_{j=1}^{s} a_{ij}k_j, v'_n + hc_i v''_n + \frac{h^2c_i^2}{2}v'''_n + \frac{h^3c_i^3}{3!}v_n^{iv}$$
$$+ \frac{h^4c_i^4}{4!}v_n^{v} + h^5\sum_{j=1}^{s}\overline{a_{ij}}k_j, v''_n + hc_i v'''_n + \frac{h^2c_i^2}{2}v_n^{iv} + \frac{h^3c_i^3}{3!}v_n^{v} + h^4\sum_{j=1}^{s}\overline{\overline{a_{ij}}}k_j, v'''_n + hc_i v_n^{iv} + \frac{h^2c_i^2}{2}vn_n^{v}v$$
$$+ h^3\sum_{j=1}^{s}\overline{\overline{\overline{a_{ij}}}}k_j, v_n^{iv} + hc_i v_n^{v} + h^2\sum_{j=1}^{s}\overline{\overline{\overline{\overline{a_{ij}}}}}k_j\Bigg) \text{ for } i = 1, 2, 3, ..., s.$$

The parameters a_{ij}, b_i^p where p is a derivative, i.e., $p = 0 \ldots iv$ and $c_i \in R\,\forall i,\ j = 1, 2, 3, \ldots, s$.

- Explicit method: $a_{ij} = 0$ at $i > j$ and
- Explicit method: $a_{ij} \neq 0$ at $i < j$.

The Taylor's series concept been applied in evaluating the expression for different variables linked with Runge-Kutta method in equation (5-10).

c	A
	b_i^T
	$b_i^{'T}$
	$b_i^{''T}$
	$b_i^{'''T}$
	b_i^{ivT}

Mathematica is used to deal with complex calculations. The main purpose of the construction of embedded explicit RKSD methods is to lower local truncated error. The method computes v_{n+1}^p to obtain an approximate value to $v^p\left(u_{n+1}\right)$ where p is a derivative, i.e., $p = 0 \ldots v$ and v_{n+1} is the calculated solution and $v\left(u_{n+1}\right)$ is the exact solution. Equation (5-10) can also be represented as:

$$v_{n+1} = v_n + h\psi, \tag{11}$$

$$v'_{n+1} = v'_n + h\psi', \tag{12}$$

$$v''_{n+1} = v''_n + h\psi'', \tag{13}$$

$$v'''_{n+1} = v'''_n + h\psi''', \tag{14}$$

$$v_{n+1}^{iv} = v_n^{iv} + h\psi^{iv}, \tag{15}$$

$$v_{n+1}^{v} = v_n^{v} + h\psi^{v} \tag{16}$$

The differentials for the equation are as under:

$$F_1^{(6)} = v^{(vi)} = f\left(u, v, v_n', v_n'', v_n''', v_n^{iv}\right), \tag{17}$$

$$F_1^{(7)} = g\left(u, v, v_n', v_n'', v_n''', v_n^{iv}\right) = f_u + f_v v' + f_{v'} v_{uu} + f_{v''} v_{uuu} + f_{v'''} v_{uuuu} + f_{v^{iv}} v_{uuuuu}, \tag{18}$$

$$F_1^{(8)} = g_u + g_v v' + g_{v'} v_{uu} + g_{v''} v_{uuu} + g_{v'''} v_{uuuu} + g_{v^{iv}} v_{uuuuu} \tag{19}$$

the local truncation errors [24-26] of $v^p(u)$ where p is a derivative, i.e., $p = 0 \ldots v$ are calculated after substitution of the exact solution of (1) into (11-16).

$$\tau_{n+1}^p = h\left[\psi^p - \Delta^{\mathrm{p}}\right], \text{ where } p = (0), (i), (ii), \ldots\ldots, (v) \tag{20}$$

Further, using equations (17-19) in equations (11-16), the increment functions $\psi, \psi', \psi'', \psi'''$ are obtained as:

$$\sum_{i=1}^s b_i k_i = \sum_{i=1}^s b_i F_1^{(6)} + \sum_{i=1}^s b_i c_i h F_1^{(7)} + \frac{1}{2}\sum_{i=1}^s b_i c_i^2 h^2 F_1^{(8)} + \frac{1}{3!}\sum_{i=1}^s b_i c_i^3 h^3 F_1^{(9)} + + O\left(h^6\right) \tag{21}$$

Similarly,

$$\sum_{i=1}^s b_i' k_i = \sum_{i=1}^s b_i' F_1^{(6)} + \sum_{i=1}^s b_i' c_i h F_1^{(7)} + \frac{1}{2}\sum_{i=1}^s b_i' c_i^2 h^2 F_1^{(8)} + \frac{1}{3!}\sum_{i=1}^s b_i' c_i^3 h^3 F_1^{(9)} + + O\left(h^6\right) \tag{22}$$

$$\sum_{i=1}^s b_i''' k_i = \sum_{i=1}^s b_i''' F_1^{(6)} + \sum_{i=1}^s b_i''' c_i h F_1^{(7)} + \frac{1}{2}\sum_{i=1}^s b_i''' c_i^2 h^2 F_1^{(8)} + \frac{1}{3!}\sum_{i=1}^s b_i''' c_i^3 h^3 F_1^{(9)} + + O\left(h^6\right) \tag{23}$$

$$\sum_{i=1}^s b_i^{iv} k_i = \sum_{i=1}^s b_i^{iv} F_1^{(6)} + \sum_{i=1}^s b_i^{iv} c_i h F_1^{(7)} + \frac{1}{2}\sum_{i=1}^s b_i^{iv} c_i^2 h^2 F_1^{(8)} + \frac{1}{3!}\sum_{i=1}^s b_i^{iv} c_i^3 h^3 F_1^{(9)} + + O\left(h^6\right) \tag{24}$$

$$\sum_{i=1}^s b_i^{v} k_i = \sum_{i=1}^s b_i^{v} F_1^{(6)} + \sum_{i=1}^s b_i^{v} c_i h F_1^{(7)} + \frac{1}{2}\sum_{i=1}^s b_i^{v} c_i^2 h^2 F_1^{(8)} + \frac{1}{3!}\sum_{i=1}^s b_i^{v} c_i^3 h^3 F_1^{(9)} + + O\left(h^6\right) \tag{25}$$

The local truncation errors (20) are as follows:

$$\tau_{n+1} = h^6\left[\sum b_i k_i - \left(\frac{1}{6!}F_1^{(6)} + \frac{1}{7!}hF_1^{(7)} + \frac{1}{8!}h^2 F_1^{(8)} + \frac{1}{9!}h^3 F_1^{(9)} \ldots\ldots\ldots\right)\right] \tag{26}$$

$$\tau_{n+1}' = h^5\left[\sum b_i' k_i - \left(\frac{1}{5!}F_1^{(6)} + \frac{1}{6!}hF_1^{(7)} + \frac{1}{7!}h^2 F_1^{(8)} + \frac{1}{8!}h^3 F_1^{(9)} \ldots\ldots\ldots\right)\right] \tag{27}$$

$$\tau_{n+1}'' = h^4\left[\sum b_i'' k_i - \left(\frac{1}{4!}F_1^{(6)} + \frac{1}{5!}hF_1^{(7)} + \frac{1}{6!}h^2 F_1^{(8)} + \frac{1}{7!}h^3 F_1^{(9)} \ldots\ldots\ldots\right)\right] \tag{28}$$

$$\tau_{n+1}''' = h^3\left[\sum b_i''' k_i - \left(\frac{1}{3!}F_1^{(6)} + \frac{1}{4!}hF_1^{(7)} + \frac{1}{5!}h^2 F_1^{(8)} + \frac{1}{6!}h^3 F_1^{(9)} \ldots\ldots\ldots\right)\right] \tag{29}$$

$$\tau_{n+1}^{iv} = h^2\left[\sum b_i^{iv} k_i - \left(\frac{1}{2!}F_1^{(6)} + \frac{1}{3!}hF_1^{(7)} + \frac{1}{4!}h^2 F_1^{(8)} + \frac{1}{5!}h^3 F_1^{(9)} \ldots\ldots\ldots\right)\right] \tag{30}$$

$$\tau_{n+1}^{v} = h\left[\sum b_i^{v} k_i - \left(F_1^{(6)} + \frac{1}{2!}hF_1^{(7)} + \frac{1}{3!}h^2F_1^{(8)} + \frac{1}{4!}h^3F_1^{(9)}.........\right)\right] \tag{31}$$

3. Order conditions of third/fourth stages eighth-/ninth-order RKSD method (RKSD8/RKSD9)

As per the findings in Section 2, the required third-stage eighth-order RKSD (RKSD8) and fourth-stage ninth-order RKSD (RKSD9) order conditions are framed as follows:

The order terms for v :

Sixth order: $\sum b_i = \frac{1}{6!} = \frac{1}{720}$ (32)

Seventh order: $\sum b_i c_i = \frac{1}{7!} = \frac{1}{5040}$ (33)

Eighth order: $\sum b_i c_i^2 = \frac{1}{20160}, \sum b_i \overline{\overline{\overline{a_{ij}}}} = \frac{1}{40320}$ (34)

The order terms for v' :

Fifth order: $\sum b_i^{'} = \frac{1}{5!} = \frac{1}{120}$ (35)

Sixth order: $\sum b_i^{'} c_i = \frac{1}{6!} = \frac{1}{720}$ (36)

Seventh order: $\sum b_i^{'} c_i^2 = \frac{1}{2520}, \sum b_i^{'} \overline{\overline{\overline{a_{ij}}}} = \frac{1}{5040},$ (37)

Eighth order: $\sum b_i^{'} c_i^3 = \frac{1}{6720}, \sum b_i^{'} c_j \overline{\overline{\overline{a_{ij}}}} = \frac{1}{40320}, \sum b_i^{'} c_i \overline{\overline{\overline{a_{ij}}}} = \frac{1}{13440}, \sum b_i^{'} \overline{\overline{\overline{a_{ij}}}} = \frac{1}{40320}$ (38)

The order terms for v'' :

Fourth order: $\sum b_i^{''} = \frac{1}{4!} = \frac{1}{24}$ (39)

Fifth order: $\sum b_i^{''} c_i = \frac{1}{5!} = \frac{1}{120},$ (40)

Sixth order: $\sum b_i^{''} c_i^2 = \frac{1}{360}, \sum b_i^{'} \overline{\overline{\overline{a_{ij}}}} = \frac{1}{720}$ (41)

Seventh order: $\sum b_i^{''} c_i^3 = \frac{1}{840}, \sum b_i^{''} c_j \overline{\overline{\overline{a_{ij}}}} = \frac{1}{5040}, \sum b_i^{''} c_i \overline{\overline{\overline{a_{ij}}}} = \frac{1}{1680}, \sum b_i^{''} \overline{\overline{\overline{a_{ij}}}} = \frac{1}{5040}$ (42)

Eighth order: $\sum b_i^{''} c_i^4 = \frac{1}{1680}, \sum b_i^{''} c_j^2 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{20160}, \sum b_i^{''} c_i^2 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{3360}, \sum b_i^{''} c_i c_j \overline{\overline{\overline{a_{ij}}}} = \frac{1}{10080}, \sum b_i^{''} c_j \overline{\overline{\overline{a_{ij}}}} = \frac{1}{40320},$

$\sum b_i^{''} c_i \overline{\overline{\overline{a_{ij}}}} = \frac{1}{10080}, \sum b_i^{''} \overline{\overline{a_{ij}}} = \frac{1}{40320}$ (43-49)

The order terms for v''' :

Third order: $\sum b_i^{'''} = \frac{1}{3!} = \frac{1}{6}$ (50)

Fourth order: $\sum b_i^{'''} c_i = \frac{1}{4!} = \frac{1}{24}$ (51)

Fifth order: $\sum b_i^{'''} c_i^2 = \frac{1}{60}, \sum b_i^{'''} \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{120}$ (52-53)

Sixth order: $\sum b_i^{'''} c_i^3 = \frac{1}{120}, \sum b_i^{'''} c_j \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{720}, \sum b_i^{'''} c_i \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{240}, \sum b_i^{'''} \overline{\overline{\overline{a}}}_{ij} = \frac{1}{720}$ (54-57)

Seventh order: $\sum b_i^{'''} c_i^4 = \frac{1}{210}, \sum b_i^{'''} c_j^2 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{2520}, \sum b_i^{'''} c_i^2 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{420,} \sum b_i^{'''} c_i c_j \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{a}{1260}, \sum b_i^{'''} c_j \overline{\overline{\overline{a}}}_{ij} = \frac{1}{5040},$

$\sum b_i^{'''} c_i \overline{\overline{\overline{a}}}_{ij} = \frac{1}{1260}, \sum b_i^{'''} \overline{\overline{a}}_{ij} = \frac{1}{5040}$ (58-64)

Eighth order: $\sum b_i^{'''} c_i^5 = \frac{1}{336}, \sum b''{}'_i \; c_j^3 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{6720}, \sum b''{}'_i \; c_i^3 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{672}, \sum b_i^{'''} c_i^2 c_j \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{2016}, \sum b_i^{'''} c_i c_j^2 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{8064}$

$\sum b_i^{'''} c_j^2 \overline{\overline{\overline{a}}}_{ij} = \frac{1}{20160}, \sum b_i^{'''} c_i^2 \overline{\overline{\overline{a}}}_{ij} = \frac{1}{2016}, \sum b_i^{'''} c_i c_j \overline{\overline{\overline{a}}}_{ij} = \frac{1}{8064}, \sum b_i^{'''} c_i \overline{\overline{a}}_{ij}, = \frac{1}{8064}, \sum b_i^{'''} c_j \overline{\overline{a}}_{ij} = \frac{1}{40320},$

$\sum b_i^{'''} \overline{a}_{ij} = \frac{1}{40320}$ (65)

The order terms for v^{iv} :

First order: $\sum b_i^{iv} = \frac{1}{2}$ (66)

Third order: $\sum b_i^{iv} c_i = \frac{1}{3!} = \frac{1}{6}$ (67)

Fourth order: $\sum b_i^{iv} c_i^2 = \frac{1}{12}, \sum b_i^{iv} \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{24}$ (68-69)

Fifth order: $\sum b_i^{iv} c_i^3 = \frac{1}{20}, \sum b_i^{iv} c_j \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{120}, \sum b_i^{iv} c_i \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{40}, \sum b_i^{iv} \overline{\overline{\overline{a}}}_{ij} = \frac{1}{120}$ (70-73)

Sixth order: $\sum b_i^{iv} c_i^4 = \frac{1}{30}, \sum b_i^{iv} \overline{\overline{a}}_{ij} = \frac{1}{720}, \sum b_i^{iv} c_j^2 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{360}, \sum b_i^{iv} c_i^2 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{60}, \sum b_i^{iv} c_i c_j \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{180}$

$\sum b_i^{iv} c_i \overline{\overline{\overline{a}}}_{ij} = \frac{1}{180}, \sum b_i^{iv} c_j \overline{\overline{\overline{a}}}_{ij} = \frac{1}{720}, \sum b_i^{iv} \overline{\overline{a}}_{ij} = \frac{1}{720}$ (74-81)

Seventh order: $\sum b_i^{iv} c_i^5 = \frac{1}{42}, \sum b_i^{iv} c_j^3 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{840}, \sum b_i^{iv} c_i^3 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{84}, \sum b_i^{iv} \overline{a}_{ij} = \frac{1}{5040}, \sum b_i^{iv} c_i^2 c_j \overline{\overline{\overline{\overline{a}}}}_{ij}$
$= \frac{1}{252} \sum b_i^{iv} c_j^2 c_i \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{1008}, \sum b_i^{iv} c_j^2 \overline{\overline{\overline{a}}}_{ij} = \frac{1}{2520}, \sum b_i^{iv} c_i^2 \overline{\overline{\overline{a}}}_{ij} = \frac{1}{252}, \sum b_i^{iv} c_i c_j \overline{\overline{\overline{a}}}_{ij} = \frac{1}{1008},$
$\sum b_i^{iv} c_j \overline{\overline{a}}_{ij} = \frac{1}{5040}, \sum b_i^{iv} c_i \overline{\overline{a}}_{ij} = \frac{1}{1008}, \sum b_i^{iv} \overline{a}_{ij} = \frac{1}{5040}$ (82-93)

Eighth order: $\sum b_i^{iv} c_i^6 = \frac{1}{56}, \sum b_i^{iv} c_j^4 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{1680}, \sum b_i^{iv} c_i^4 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{112}, \sum b_i^{iv} c_i^3 c_j \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{336},$
$\sum b_i^{iv} c_i^2 c_j^2 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{1344}, \sum b_i^{iv} c_i c_j^3 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{6720}, \sum b_i^{iv} c_j^3 \overline{\overline{\overline{a}}}_{ij} = \frac{1}{6720}, \sum b_i^{iv} c_i^3 \overline{\overline{\overline{a}}}_{ij} = \frac{1}{336},$
$\sum b_i^{iv} c_i^2 c_j \overline{\overline{\overline{a}}}_{ij} = \frac{1}{1344}, \sum b_i^{iv} c_i c_j^2 \overline{\overline{\overline{a}}}_{ij} = \frac{1}{6720}, \sum b_i^{iv} c_j \overline{a}_{ij} = \frac{1}{40320}, \sum b_i^{iv} c_i \overline{a}_{ij} = \frac{1}{6720},$
$\sum b_i^{iv} a_{ij} = \frac{1}{40320}, \sum b_i^{iv} c_j^2 \overline{\overline{a}}_{ij} = \frac{1}{20160}, \sum b_i^{iv} c_i^2 \overline{\overline{a}}_{ij} = \frac{1}{1344}, \sum b_i^{iv} c_i c_j \overline{\overline{a}}_{ij} = \frac{1}{6720}.$ (94)

The order terms for v^v:

First order: $\sum b_i^v = 1$ (95)

Second order: $\sum b_i^v c_i = \frac{1}{2},$ (96)

Third order: $\sum b_i^v c_i^2 = \frac{1}{3}, \sum b_i^v \overline{\overline{\overline{a}}}_{ij} = \frac{1}{6}$ (97-98)

Fourth order: $\sum b_i^v c_i^3 = \frac{1}{4}, \sum b_i^v c_j \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{24}, \sum b_i^v c_i \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{8}, \sum b_i^v \overline{\overline{\overline{a}}}_{ij} = \frac{1}{24}$ (99-102)

Fifth order: $\sum b_i^v c_i^4 = \frac{1}{5}, \sum b_i^v \overline{\overline{a}}_{ij} = \frac{1}{120}, \sum b_i^v c_j^2 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{60}, \sum b_i^v c_i^2 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{10}, \sum b_i^v c_i c_j \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{30}, \sum b_i^v c_i \overline{\overline{\overline{a}}}_{ij} = \frac{1}{30},$
$\sum b_i^v c_j \overline{\overline{\overline{a}}}_{ij} = \frac{1}{120}$ (103-109)

Sixth order: $\sum b_i^v c_i^5 = \frac{1}{6}, \sum b_i^v c_j^3 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{120}, \sum b_i^v c_i^3 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{12}, \sum b_i^v c_i^3 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{12}, \sum b_i^v c_i c_j^2 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{144}, \sum b_i^v c_j^2 \overline{\overline{\overline{\overline{a}}}}_{ij}$
$= \frac{1}{360}, \sum b_i^v c_i^2 \overline{\overline{\overline{a}}}_{ij} = \frac{1}{36}, \sum b_i^v c_i c_j \overline{\overline{\overline{a}}}_{ij} = \frac{1}{144}, \sum b_i^v c_j \overline{\overline{a}}_{ij} = \frac{1}{720}, \sum b_i^v c_i \overline{\overline{a}}_{ij} = \frac{1}{144}, \sum b_i^v \overline{a}_{ij} = \frac{1}{720}$ (110-120)

Seventh order: $\sum b_i^v c_i^6 = \frac{1}{7}, \sum b_i^v c_j^4 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{210}, \sum b_i^v c_i^4 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{14}, \sum b_i^v c_i^3 c_j \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{42}, \sum b_i^v c_i^2 c_j^2 \overline{\overline{\overline{\overline{a}}}}_{ij} = \frac{1}{168}, \sum b_i^v c_i c_j^3 \overline{\overline{\overline{\overline{a}}}}_{ij}$
$= \frac{1}{840}, \sum b_i^v c_j^{32400} \overline{\overline{\overline{a}}}_{ij} = \frac{1}{840}, \sum b_i^v c_i^{32400} \overline{\overline{\overline{a}}}_{ij} = \frac{1}{42}, \sum b_i^v c_i^2 c_j \overline{\overline{\overline{a}}}_{ij} = \frac{1}{168}, \sum b_i^v c_i^{268} c_j^2 \overline{\overline{\overline{a}}}_{ij} = \frac{1}{840},$ (121-130)

$$\sum b_i^v c_j^2 \overline{\overline{a_{ij}}} = \frac{1}{2520}, \sum b_i^v c_i^2 \overline{\overline{a_{ij}}} = \frac{1}{168}, \sum b_i^v c_i^{268} c_j \overline{\overline{a_{ij}}} = \frac{1}{840}\ \sum b_i^v c_j^{268} \overline{a_{ij}} = \frac{1}{5040}$$

$$\sum b_i^v c_i^{268} \overline{a_{ij}} = \frac{1}{840}, \sum b_i^v a_{ij} = \frac{1}{5040}. \tag{131-136}$$

Eighth order: $\sum b_i^v c_i^7 = \frac{1}{8}, \sum b_i^v c_j^5 \overline{\overline{\overline{\overline{a_{ij}}}}} = \frac{1}{336}, \sum b_i^v c_i^5 \overline{\overline{\overline{\overline{a_{ij}}}}} = \frac{1}{16}, \sum b_i^v c_i^4 c_j \overline{\overline{\overline{\overline{a_{ij}}}}} = \frac{1}{48}, \sum b_i^v c_i^3 c_j^2 \overline{\overline{\overline{\overline{a_{ij}}}}} = \frac{1}{192},$

$$\sum b_i^v c_i^2 c_j^3 \overline{\overline{\overline{\overline{a_{ij}}}}} = \frac{1}{960}, \sum b_i^v c_i c_j^4 \overline{\overline{\overline{\overline{a_{ij}}}}} = \frac{1}{5760}, \sum b_i^v c_j^{42400} \overline{\overline{\overline{a_{ij}}}} = \frac{1}{1680}, \sum b_i^v c_i^{42400} \overline{\overline{\overline{a_{ij}}}} = \frac{1}{48}, \sum b_i^v c_i^3 c_j \overline{\overline{\overline{a_{ij}}}}$$

$$= \frac{1}{192}, \sum b_i^v c_i^2 c_j^2 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{960}, \sum b_i^v c_i c_j^3 \overline{\overline{\overline{a_{ij}}}} = \frac{1}{5760}, \sum b_i^v c_j^3 \overline{\overline{a_{ij}}} = \frac{1}{6720}, \sum b_i^v c_i^3 \overline{\overline{a_{ij}}} = \frac{1}{192},$$

$$\sum b_i^v c_i^2 c_j \overline{\overline{a_{ij}}} = \frac{1}{960}, \sum b_i^v c_i c_j^2 \overline{\overline{a_{ij}}} = \frac{1}{5760}, \sum b_i^v c_j^2 \overline{a_{ij}} = \frac{1}{20160}, \sum b_i^v c_i^2 \overline{a_{ij}} = \frac{1}{960},$$

$$\sum b_i^v c_i c_j \overline{a_{ij}} = \frac{1}{5760}, \sum b_i^v c_j a_{ij} = \frac{1}{40320}, \sum b_i^v c_i a_{ij} = \frac{1}{5760}. \tag{137-157}$$

4. Zero-stability of RKSD8 and RKSD9 method

The most important precondition for obtaining the convergence [21, 23] of numerical problem is evaluating zero-stability of the system, as explained by Dormand et al. [10]. The array form of the findings is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{n+1} \\ hv'_{n+1} \\ h^2 v''_{n+1} \\ h^3 v'''_{n+1} \\ h^4 v_{n+1}^{iv} \\ h^5 v_{n+1}^{v} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_n \\ hv'_n \\ h^2 v''_n \\ h^3 v'''_n \\ h^4 v_n^{iv} \\ h^5 v_n^{v} \end{bmatrix}$$

The characteristic equation will be expressed as follows:

$$\rho(\varepsilon) = |I\varepsilon - A|$$

Hence, $\rho(\varepsilon) = (\varepsilon - 1)^6$ we get the roots to be $\varepsilon = 1$ with multiplicity 6. In other words, the finding proves its authenticity to the desired zero stability of the proposed method as no root is found to be greater than 1 and also the multiplicity of the roots is at most 6.

5. Construction of third stage RKSD8 method

The present section focused towards the development of third stage RKSD8 method based on results (32-157) for evaluating values of c_3, b_i^p, where p is a derivative, i.e., $p = 0 \ldots v$ as follows:

$$b_1 = \frac{28c_2c_3 - 4(c_2 + c_3) + 1}{20160c_2c_3}, \quad b_2 = \frac{4c_3 - 1}{20160c_2(c_3 - c_2)}, \quad b_3 = \frac{1 - 4c_2}{20160c_3(c_3 - c_2)},$$

$$b_1' = \frac{378c_2c_3 - 63(c_2 + c_3) + 18}{45360c_2c_3}, \quad b_2' = \frac{63c_3 - 18}{45360c_2(c_3 - c_2)}, \quad b_3' = \frac{18 - 63c_2}{45360c_3(c_3 - c_2)},$$

$$b_1''' = \frac{20c_2c_3 - 5(c_2 + c_3) + 2}{120c_2c_3}, \quad b_2''' = \frac{5c_3 - 2}{120c_2(c_3 - c_2)}, \quad b_3''' = \frac{-5c_2 + 2}{120c_3(c_3 - c_2)},$$

$$b_1^{iv} = \frac{6c_2c_3 - 2(c_2 + c_3) + 1}{12c_2c_3}, \quad b_2^{iv} = \frac{2c_3 - 1}{12c_2(c_3 - c_2)}, \quad b_3^{iv} = \frac{1 - 2c_2}{12c_3(c_3 - c_2)},$$

$$b_1^{v} = \frac{6c_2c_3 - 3(c_3 + c_2) + 2}{6c_2c_3}, \quad b_2^{v} = \frac{3c_3 - 2}{6c_2(c_3 - c_2)}, \quad b_3^{v} = \frac{2 - 3c_2}{6c_3(c_3 - c_2)},$$

$$c_2 = \frac{-3 + 4c_3}{-4 + 6c_3}$$

Table 1. The Butcher tableau for third stage RKSD8 method

0	0			0														
$\frac{3}{5}$	$\frac{1}{487}$	0		$\frac{-3}{800}$	0													
$\frac{2}{7}$	$\frac{26}{735}$	$\frac{-11}{588}$	0	$\frac{13}{170}$	$\frac{-55}{504}$	0												
	$\frac{1}{1531}$	$\frac{1}{1330}$	$\frac{1}{1293}$	$\frac{1}{288}$	0	$\frac{4}{823}$	$\frac{2}{135}$	$\frac{1}{475}$	$\frac{22}{889}$	$\frac{7}{144}$	$\frac{5}{198}$	$\frac{49}{528}$	$\frac{1}{8}$	$\frac{25}{132}$	$\frac{49}{264}$	$\frac{13}{36}$	$\frac{100}{99}$	$\frac{-49}{132}$

Hence, the findings of variables reflected in Table 1 helps in evaluating the the result values of error norms as: $\left\|\tau^{(8)}\right\|_2 = -1.43561 \times 10^{-16}$, $\left\|\tau'^{(8)}\right\|_2 = -139462 \times 10^{-9}$, $\left\|\tau'^{(8)}\right\|_2 = -139462 \times 10^{-9}$, $\left\|\tau''^{(8)}\right\|_2 = -3177332 \times 10^{-9}$, $\left\|\tau'''^{(8)}\right\|_2 = -166233 \times 10^{-9}$, $\left\|\tau^{iv(8)}\right\|_2 = 1363695 \times 10^{-9}$ and $\left\|\tau^{v(8)}\right\|_2 = 2310676 \times 10^{-9}$ and $\left\|\tau_g^{(8)}\right\|_2 = 2310676 \times 10^{-9}$

6. Construction of fourth stage RKSD9 method

The present section focused towards the development of third stage RKSD8 method based on results (32-157) for evaluating values of c_3, b_i^p, where p is a derivative, i.e., $p = 0 \ldots v$ as follows:

$$b_1 = \frac{1}{720} - b_2 - b_3 - b_4, \quad b_2 = \frac{\left(\frac{c_3}{4} + \frac{c_4}{4} - c_3c_4 - \frac{1}{12}\right)}{5040c_2(c_2 - c_3)(c_4 - c_2)}, \quad b_3 = \frac{3(c_4 + c_2) - 1 - 12c_4c_2}{60480c_3(c_2 - c_3)(c_3 - c_4)},$$

$$b_4 = \frac{-(1 - 3(c_2 + c_3) + 12c_2c_3)}{60480c_4(c_3 - c_4)(c_4 - c_2)}, \quad b_1' = \frac{1}{120} - b_2' - b_3' - b_4', \quad b_2' = -\frac{[28c_3c_4 - 8(c_4 + c_3) + 3}{20160c_2(c_2 - c_3)(c_4 - c_2)},$$

$$b_3' = \frac{\left[-28c_3c_4 + 8(c_4 + c_3) - 3\right]}{20160c_3(c_2 - c_3)(c_3 - c_4)}, \quad b_4' = \frac{-\left[28c_3c_4 - 8(c_2 + c_3) - 3\right]}{20160c_4(c_3 - c_4)(c_4 - c_2)}, \quad b_1'' = \frac{1}{24} - b_2'' - b_3'' - b_4'',$$

$$b_2'' = \frac{-\left[21c_4c_3 - 7(c_4 + c_3) + 3\right]}{2520c_2(c_2 - c_3)(c_4 - c_2)}, \quad b_3''' = \frac{-\left[5c_2c_4 + 2(c_4 + c_2) - 1\right]}{120c_3(c_2 - c_3)(c_3 - c_4)}, \quad b_4''' = \frac{-\left[5c_2c_3 - 2(c_3 + c_2) + 3\right]}{120c_3(c_2 - c_3)(c_3 - c_4)},$$

$$b_1^{iv} = \frac{1}{2} - b_2^{iv} - b_3^{iv} - b_4^{iv}, \quad b_3^{v} = \frac{\left[-6c_2c_4 + 4(c_4 + c_2) - 3\right]}{60c_3(c_2 - c_3)(c_3 - c_4)}, \quad b_4^{v} = \frac{\left[6c_2c_3 - 4(c_3 + c_2) + 3\right]}{12c_4(c_3 - c_4)(c_4 - c_2)},$$

$$c_2 = \frac{5 - 2c_3}{24 - 90c_3}$$

Table 2. The Butcher tableau for fourth stage RKSD9 method

c	A				$\bar{A}$			
0	0				0			
$\frac{6}{7}$	$\frac{351}{490}$	0			$\frac{-11}{8}$	0		
$\frac{109}{372}$	$\frac{-98}{643}$	$\frac{14}{605}$	0		$\frac{110}{403}$	$\frac{14}{605}$	0	
1	$\frac{125}{36}$	$\frac{80}{491}$	$\frac{-53}{133}$	0	$\frac{-310}{39}$	$\frac{-10}{17}$	$\frac{861}{667}$	0
	$\frac{1}{156}$	$\frac{1}{655}$	$\frac{1}{123}$	$\frac{1}{108}$	$\frac{1}{295}$	$\frac{1}{161}$	$\frac{1}{135}$	$\frac{1}{194}$

$\frac{1}{2450}$	$\frac{1}{877}$	$\frac{1}{170}$	$\frac{1}{589}$	$\frac{1}{114}$	$\frac{1}{206}$	$\frac{1}{137}$	$\frac{1}{705}$	$\frac{1}{272}$	$\frac{1}{912}$	$\frac{1}{860}$	$\frac{1}{408}$	$\frac{1}{76}$	$\frac{1}{919}$
$\frac{1}{853}$	$\frac{1}{214}$	$\frac{-69}{789}$											

Hence, the findings of variables reflected in Table 2 helps in evaluating the result values of error norms as:

$\left\|\tau^{(8)}\right\|_2 = 3.33726644 \times 10^{-2}$, $\left\|\tau'^{(8)}\right\|_2 = -1.53149 \times 10^{-19}$, $\left\|\tau''^{(8)}\right\|_2 = -7.65743 \times 10^{-18}$, $\left\|\tau'''^{(8)}\right\|_2 = -2.37144 \times 10^{-4}$, $\left\|\tau^{iv(8)}\right\|_2 =$ -2.611267×10^{-3}, $\left\|\tau^{v(8)}\right\|_2 = -1.135281 \times 10^{-12}$, and $\tau_g^{(8)}{}_2 = 1.327371 \times 10^{-3}$.

7. Numerical results

The present section is focused towards verification of results in above sections with the help of an example of order six.

Problem 1: Consider the initial value problem given as: $v^{vi}(u) + 2v^{iv}(u) + v''(u) = 0$ with initial conditions $v(0) = 0, v'(0) = 0, v''(0) = 1, v'''(0) = 1,\ v^{iv}(0) = 1, v^{v}(0) = 0.$

Solution 1: The exact solution interval [0, 5] in this:

$$v(x) = 3 + 2x - \frac{5}{2}\sin x - 3\cos x - x\sin x + \frac{1}{2}x\cos x$$

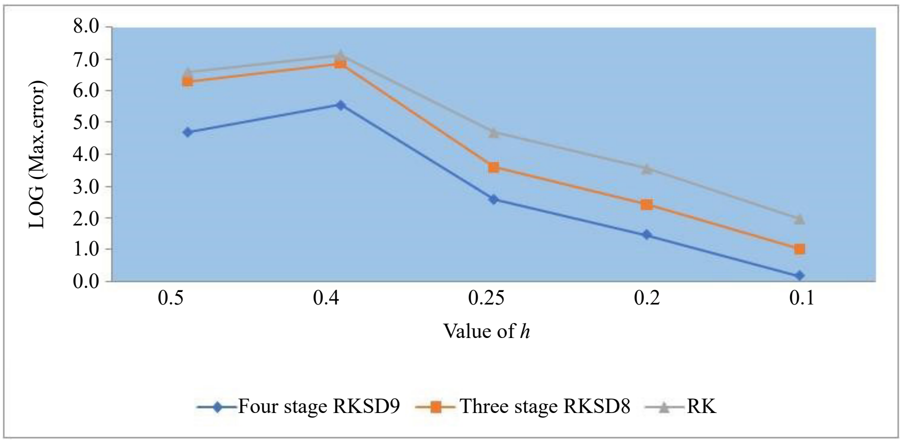

Figure 1. The efficiency of RKSD8, RKSD9 in comparison to Runge-Kutta method with $h = 0.1$, 0.2, 0.25, 0.4, and 0.5

Problem 2: Consider the initial value problem given as: $v^{vi}(u) - 8v^{iv}(u) + 16v''(u) = 0$ with initial conditions $v(0) = 1, v'(0) = 1, v''(0) = 0, v'''(0) = 0,\ v^{iv}(0) = 0, v^{v}(0) = 2$.

Solution 2: The exact solution is in the interval [0, 3]:

$$v(x) = \frac{9}{8} - \frac{1}{16}e^{-2x} - \frac{1}{16}e^{-2x}x - \frac{1}{16}e^{2x} + \frac{1}{16}e^{2x}x + x$$

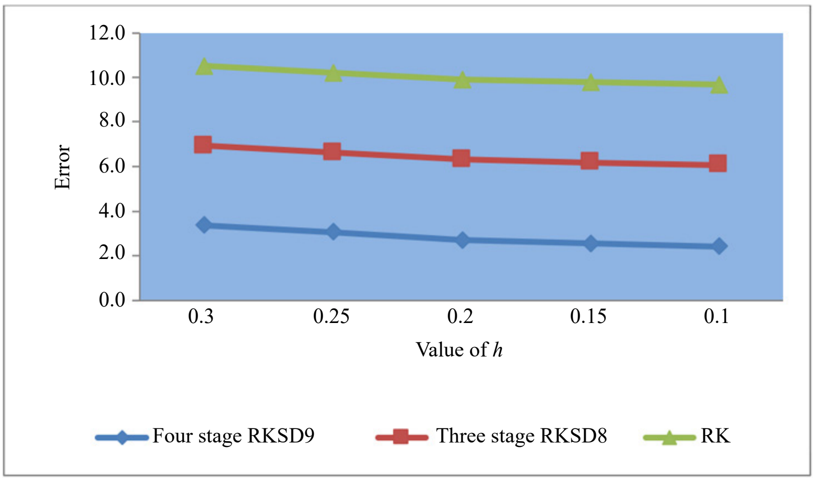

Figure 2. The efficiency of RKSD8 and RKSD9 in comparison to Runge-Kutta method with $h = 0.1$, 0.15, 0.20, 0.25, and 0.3

Problem 3: Consider the initial value problem given as: $v^{vi}(u) - 9v^{iv}(u) = 0$ with initial conditions $v(0) = 1, v'(0) = 1, v''(0) = 0, v'''(0) = 0,\ v^{iv}(0) = 0, v^{v}(0) = 2$.

Solution 2: The exact solution is in the interval [0, 4]:

$$v(x) = -\frac{1}{81} + \frac{26}{27}x + \frac{4}{9}x^2 + \frac{5}{18}x^3 + \frac{1}{81}e^{3x}$$

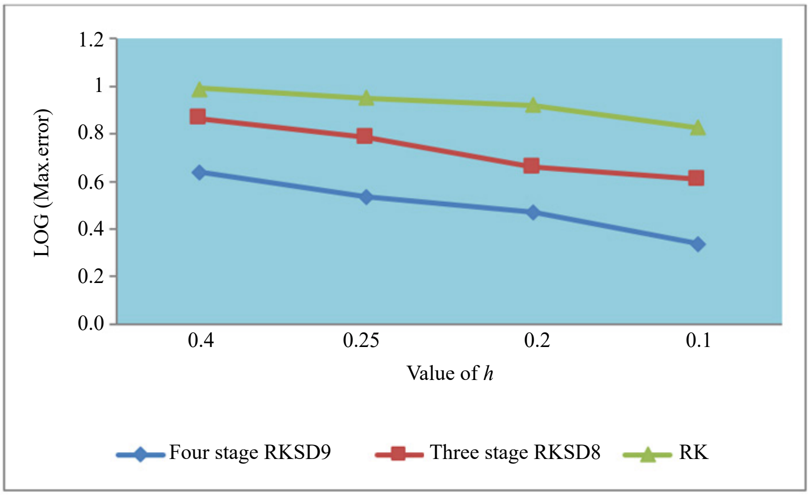

Figure 3. The efficiency of RKSD8 and RKSD9 in comparison to Runge-Kutta method with $h = 0.1, 0.2, 0.25$, and 0.4

8. Conclusion

The present research paper accomplished its objective of finding the solution of sixth order IVP using the embedded Runge-Kutta method. The basic motive for introducing the method at the third and fourth stages is that it has proven beneficial for the evaluation of parameters under RKSD8 and RKSD9. The results obtained in problems 1-3 proved the benefits of current findings with the traditional Runge-Kutta method by decreasing the errors values of the required function w.r.t. different values of step sizes. Hence, the observed results will be useful for introspecting many real-life problems related to engineering, science, and medical areas with more accuracy by minimizing the errors and achieving zero stability in comparison to other existing Runge-Kutta methods.

Conflict of interest

The authors declare no conflict of interest financial or otherwise.

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