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Research Article

On r-dynamic Coloring of Ladder Graph and Tadpole Graph Using m-shadow Operation

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Abstract: An r-dynamic coloring is a proper k-coloring of a graph $G = \{V, E\}$ such that the neighbors of every vertex $v \in V(G)$ are colored using ς: $V(G) \rightarrow S(c)$ where $S(c)$ is a set of colors. The coloring is made in such a way that it satisfies the conditions: (i) For any edge $uv \in E(G)$ the color of u and color of v are distinct and (ii) the cardinality of coloring the neighbors of any vertex v should be greater than or equal to $\min\{r, d(v_G)\}$, where $d(v_G)$ is the degree of the vertex v. In this paper, the lower bounds for the r-dynamic coloring of the m-shadow graph of the ladder graph $D_m(L_n)$ and the tadpole graph $D_m(T_{n,p})$ are attained. Using the lower bounds, the exact solution of the r-dynamic chromatic number of the ladder graph L_n and tadpole graph $T_{n,p}$ by the m-shadow operation is obtained.

Keywords: r-dynamic coloring, m-shadow graph, ladder graph, tadpole graph

MSC: 05C15

1. Introduction

The graphs used in this paper are simple and finite. Let G be a simple graph that is connected and undirected. The other typical notations used here are $V(G)$ and (G), which are the vertices and edges of G, respectively. The minimum degree of G is $\delta(G)$, and the maximum degree is $\Delta(G)$. For any $v \in V$, $N(v)$ denotes the neighborhood vertex of v that is adjacent to v. The concept of dynamic chromatic number was first introduced by Montgomery [1], and the study of r-dynamic coloring is an extension of dynamic coloring, so one of the obvious results that holds is $\chi(G) \leq \chi_r(G) \leq \chi_{r+1}(G)$. An r-dynamic coloring of G is a mapping of ς from $V(G)$ to the set of colors $S(c)$ such that the following conditions hold:

1. For any $uv \in E(G), \varsigma(u) \neq \varsigma(v)$.
2. $|\varsigma(N(v))| \geq \min\{r, d(v_G)\}$, where $d(v_G)$ is the degree of v and r is a positive integer.

When $r = 1$, the 1-dynamic chromatic number of G is equal to its chromatic number. When $r = 2$, the 2-dynamic chromatic number of G is the result of the dynamic chromatic number. The r values are extended up to the maximum degree $\Delta(G)$. The r-dynamic chromatic number remains the same even after r values exceed $\Delta(G)$. Some of the

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following observations were proposed by Montgomery [1] on r-dynamic chromatic number, and some of the bounds are studied from [2-8]. Nandini et al. [9] have studied the r-dynamic coloring of para-line graph of some standard graphs. In [10], there are five theorems studied, including the graph of a flower graph $C(F_n)$, the line graph of a flower graph $L(F_n)$, the subdivision graph of a flower graph $S(F_n)$, the para-line graph of a flower graph $L[S(F_n)]$, and the splitting graph of a flower graph $S[F_n]$. In [11], there are six theorems studied, including the central vertex join of path graph P_m with cycle graph C_n, the central vertex join of cycle graph C_m with path graph P_n, the central vertex join of cycle graph C_3 with path graph in P_n, the central vertex join of cycle graph C_m with complete graph K_n, the central vertex join of cycle graph C_3 with complete graph K_n, and the central vertex join of cycle graph C_m with cycle graph C_n. In this paper, we determined the r-dynamic chromatic number of the ladder graph and the tadpole graph using the m-shadow operation.

2. Preliminaries

In this section, the basic definitions and preliminary lemmas that are used in the next sections are given. A graph G is a pair $(V(G), E(G))$, where $V(G)$ denotes the vertex set and $E(G)$ denotes the edge set. If G has the same end vertices, it is called a loop, and an undirected, loopless graph is said to be a simple graph. A graph G is finite if its order and size are finite. In a graph G, the minimum degree $\delta(G)$ is the minimum number of edges that are incident from any vertex $v \in V$, and the maximum degree $\Delta(G)$ is the maximum number of edges that are incident from any vertex $v \in V$.

Definition 2.1. The shadow graph $D_2(G)$ of a simple connected graph G is obtained by taking two copies of G, i.e., G' and G'', and joining each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.

Definition 2.2. [12, 13] A m-shadow graph of G denoted by $D_m(G)$ is a graph obtained by taking m-copies of G, i.e., G', G'', G''', ..., $G^{(m)}$ and then joining each vertex $u^i \in G^i$, $i \in [1, m-1]$ to all the neighbors of the corresponding vertex $v^j \in G^{i+1}, G^{i+2}, ..., G^{(m)}$, $i < j \leq m$.

Definition 2.3. The ladder graph is a planar undirected graph that is the Cartesian product of two path graphs and is denoted by $L_n = P_n \times P_2$. In other words, a ladder graph is obtained by taking two copies of a path graph of the same order whose corresponding vertices are connected by an edge.

Definition 2.4. The tadpole graph is a special type of graph consisting of a cycle C_n of at least $n \geq 3$ vertices and a path P_p with p vertices connected by a bridge. It is denoted by $T_{n,p}$.

Lemma 2.1. Let G be a finite, connected graph, then the following condition holds:

1. $\chi_r(G) \leq \chi_{r+1}(G)$
2. $\chi_r(G) \geq \min\{r, \Delta(G)\} + 1$
3. $\chi_r(G) = \chi_1(G) \leq \chi_2(G) \leq \ldots \leq \chi_{\Delta(G)}(G)$
4. At $r \geq \Delta(G)$, then $\chi_r(G) = \chi_{\Delta(G)}(G)$

3. Results

Lemma 3.1. For a ladder graph L_n, the lower bound for the r-dynamic chromatic number of the m-shadow graph of the ladder graph $D_m(L_n)$ is

$$\chi_r(D_m(L_n)) \geq 2r, \text{ for } 1 \leq r \leq \Delta(D_m(L_n)), \forall m, n.$$

Proof. Let $V(D_m(L_n)) = \{v'_j, v''_j, ..., v^m_j : 1 \leq j \leq 2n\}$ be the vertex set and $E(D_m(L_n)) = \{\{v^a_{2j-1}v^x_{2j+1}, v^a_{2j}v^x_{2j+2} : 1 \leq a \leq m; a \leq x \leq m;\ 1 \leq j \leq n-1\} \cup \{v^a_j v^x_{j+1} : 1 \leq a \leq m; a \leq x \leq m; j = 1, 3, 5, \ldots, 2n-1\}\}$ be the edge set whose corresponding cardinalities are $|V(D_m(L_n))| = 2mn$ and $|E(D_m(L_n))| = m^2(3n-2)$, respectively. The vertex v'_j is adjacent to $v''_k, v'''_k, ..., v^{(m)}_k$ only where $v''_j, v'''_j, ..., v^m_j$ is adjacent.

$$\text{The minimum degree is } \delta(D_m(L_n)) = \begin{cases} mn & \text{for } n = 1,2 \\ 2m & \text{for } n \geq 3 \end{cases}$$

and

$$\text{the maximum degree is } \Delta(D_m(L_n)) = \begin{cases} mn & \text{for } n = 1, 2 \\ 3m & \text{for } n \geq 3. \end{cases}$$

For $n = 1, 2$, the value of r varies from $1 \leq r \leq mn$, and for $n \geq 3$, the value of r varies from $1 \leq r \leq 3m$, and hence the result remains same. Let L be a simple, connected graph. By the definition of a m-shadow graph, every $v_j^i (1 \leq j \leq 2n)$ vertex in the ith copy of L is adjacent to $v_l^{i+1}, v_l^{i+2}, \cdots, v_l^{(m)}$ of all $i+1$, $i+2, \cdots$, mth copies of L, wherever v_j^i are adjacent. In order to prove the lemma, we consider two cases.

Case 1. $1 \leq r \leq mn, \forall n = 1, 2.$

First, consider $r = 1$. Assign the colors 1, 2 to all the vertices of m-copies of L_n. For instance, assign the color class 1 to $v_j', v_j'', \cdots, v_j^{(m)}$ and color class 2 to $v_j', v_j'', \cdots, v_j^{(m)}$ for j = even. The 1-dynamic coloring of $D_m(L_n)$ results as the same as the chromatic number of L_n.

Next, consider $r = 2$. For a vertex v_j', where $1 \leq j \leq 2n$, the maximum degree is four. Assign a color (say, ς_1) to v_j', so that two adjacent vertices of v_j', are colored with ς_2, and the other two adjacent vertices are colored with ς_3. For instance, if v_{2k-1}' where $1 \leq k \leq n$ are colored with ς_1, whose two adjacent vertices v_{2k}' and v_{2k}'' are colored with ς_2 and ς_3. Now, to satisfy r-adjacency v_{2k-1}'', it requires a new color, ς_4. Similarly, when $3 \leq r \leq mn - 1$, each odd copy of $D_m(L_n)$ receives $r - 1$ colors, and each even copy of $D_m(L_n)$ receives $r + 1$ new colors. Therefore, it shows a total of $2r$ colors are required.

Finally, when $r = mn$, we assign a new color to each vertex in order to achieve r-adjacency. Since there are $2mn$ vertices in $D_m(L_n)$ and $r = mn$, it is clearly seen that $2r$ colors are required to satisfy the r-dynamic coloring.

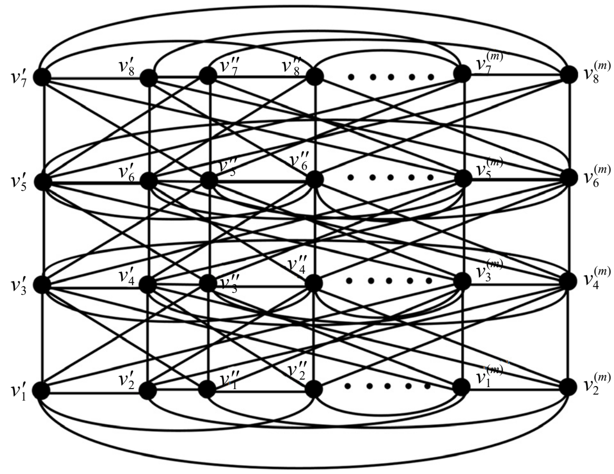

Figure 1. $(D_m(L_4))$-m-shadow graph of ladder graph L_4

Case 2. $1 \leq r \leq 3m, \forall n \geq 3.$

In the case of $r = 1$, the 1-dynamic coloring of $D_m(L_n)$ is the same as the chromatic number of L_n, and thus the proof holds. When $r = 2$, odd and even copies of $D_m(L_n)$ each receive r new colors, and for $3 \leq r \leq 3m - 1$, each odd copy of $D_m(L_n)$ receives $r - 1$ colors, and each even copy of $D_m(L_n)$ receives $r + 1$ new colors. Therefore, it shows a total of $2r$ colors are required.

Further, consider the maximum degree, $r = 3m$, for which we assign the colors $1, 2, \cdots, 3m$ to the $(2j - 1)$th copies of $D_m(L_n), 1 \leq j \leq \left\lceil \frac{m}{2} \right\rceil$ and the colors $3m + 1, 3m + 2, \cdots, 6m$ to the $(2j)$th copies of $D_m(L_n), 1 \leq j \leq \left\lceil \frac{m}{2} \right\rceil$, showing that $2r$ colors are required to satisfy r-adjacency.

Theorem 3.1. Let r, $n \geq 1$ and $m \geq 2$ be any positive integers, then the r-dynamic chromatic number of m-shadow graph of ladder graph $D_m(L_n)$ is

$$\chi_r(D_m(L_n)) = 2r \text{ for } 1 \le r \le \Delta(D_m(L_n)).$$

Proof. To ascertain the r-dynamic chromatic number of $D_m(L_n)$, we have to prove that, $\chi_r(D_m(L_n)) \ge 2r$ and $\chi_r(D_m(L_n)) \le 2r$. In accordance with Lemma 3.1, we have $\chi_r(D_m(L_n)) \ge 2r$. So, it completes the proof of lower bound. Then, we have to prove the upper bound. To prove $\chi r(D_m(L_n)) \le 2r$, we divide into some cases and consider a function $\varsigma : V(D_m(L_n)) \to S(\varsigma)$, where $S(\varsigma) = \{1, 2, 3, \ldots, 2r\}$.

1. Consider $m = 2$.

When $r = 1$ and $\forall n$, the r-dynamic chromatic number is given by,

$$\varsigma(v'_j) = \varsigma(v''_j) = \{1, 2\} : \forall 1 \le j \le 2n.$$

Therefore, the minimum number of colors required is 2. When $r = \Delta(D_2(L_n))$, the r-dynamic chromatic number is given by, for $n = 1$, $r = \Delta(D_2(L_1)) = 2$

$$\varsigma : V(D_2(L_1)) = \begin{cases} \{1,2\} & \text{for } v'_j, \forall 1 \le j \le 2n \\ \{3,4\} & \text{for } v''_j, \forall 1 \le j \le 2n \end{cases}$$

Therefore, the minimum number of colors required is 4. For $n = 2$, $r = \Delta(D_2(L_2)) = 4$

$$\varsigma : V(D_2(L_2)) = \begin{cases} \{1,2,3,4\} & \text{for } v'_j, \forall 1 \le j \le 2n \\ \{5,6,7,8\} & \text{for } v''_j, \forall 1 \le j \le 2n \end{cases}$$

Therefore, the minimum number of colors required is 8. For $n \ge 3$, $r = \Delta(D_2(L_n)) = 6$

$$\varsigma : V(D_2(L_n)) = \begin{cases} \{1,2,3,4,5,6\} & \text{for } v'_j, \forall 1 \le j \le 2n \\ \{7,8,9,10,11,12\} & \text{for } v''_j, \forall 1 \le j \le 2n \end{cases}$$

Therefore, the minimum number of colors required is 12.

2. Consider $m = 3$.

When $r = 1$ and $\forall n$, the r-dynamic chromatic number is given by,

$$\varsigma(v'_j) = \varsigma(v''_j) = \varsigma(v'''_j) = \{1, 2\} : \forall\ 1 \le j \le 2n$$

Therefore, the minimum number of colors required is 2.

When $r = \Delta(D_3(L_n))$, the r-dynamic chromatic number is given by, for $n = 1$, $r = \Delta(D_3(L_1)) = 3$

$$\varsigma : V(D_3(L_1)) = \begin{cases} \{1,2\} & \text{for } v'_j, \forall 1 \le j \le 2n \\ \{3,4\} & \text{for } v''_j, \forall 1 \le j \le 2n \\ \{5,6\} & \text{for } v'''_j, \forall 1 \le j \le 2n \end{cases}$$

Therefore, the minimum number of colors required is 6. For $n = 2$, $r = \Delta(D_3(L_2)) = 6$

$$\varsigma : V(D_3(L_2)) = \begin{cases} \{1,2,3,4\} & \text{for } v'_j, \forall 1 \le j \le 2n \\ \{5,6,7,8\} & \text{for } v''_j, \forall 1 \le j \le 2n \\ \{9,10,11,12\} & \text{for } v'''_j, \forall 1 \le j \le 2n \end{cases}$$

Therefore, the minimum number of colors required is 12. For $n \ge 3$, $r = \Delta(D_3(L_n)) = 9$

$$\varsigma : V(D_3(L_n)) = \begin{cases} \{1,2,3,4,5,6\} & \text{for } v'_j, \forall 1 \le j \le 2n \\ \{7,8,9,10,11,12\} & \text{for } v''_j, \forall 1 \le j \le 2n \\ \{13,14,15,16,17,18\} & \text{for } v'''_j, \forall 1 \le j \le 2n \end{cases}$$

Therefore, the minimum number of colors required is 18.

3. For m-shadow graph, when $r = 1$ and $\forall n$, the r-dynamic chromatic number is given by,

$$\varsigma(v'_j) = \varsigma(v''_j) = \cdots = \varsigma(v_j^{(m)}) = \{1,2\} : \forall 1 \le j \le 2n.$$

Therefore, the minimum number of colors required is 2.

When $r = \Delta(D_m(L_n))$, the r-dynamic chromatic number is given by, for $n = 1$, $r = \Delta(D_m(L_1)) = m$

$$\varsigma : V(D_m(L_n)) = \begin{cases} \{1,2\} & \text{for } (v'_j), \forall 1 \le j \le 2n \\ \{3,4\} & \text{for } (v''_j), \forall 1 \le j \le 2n \\ \vdots & \\ \{2m-1, 2m\} & \text{for } v_j^{(m)}, \forall 1 \le j \le 2n \end{cases}$$

Since $r = m$, the minimum number of colors required is $2r$. For $n = 2$, $r = \Delta(D_m(L_2)) = 2m$

$$\varsigma : V(D_m(L_2)) = \begin{cases} \{1,2,3,4\} & \text{for } (v'_j), \forall 1 \le j \le 2n \\ \{5,6,7,8\} & \text{for } (v''_j), \forall 1 \le j \le 2n \\ \vdots & \\ \{4m-3, 4m-2, 4m-1, 4m\} & \text{for } v_j^{(m)}, \forall 1 \le j \le 2n \end{cases}$$

Since $r = 2m$, the minimum number of colors required is $2r$. For $n \ge 3$, $r = \Delta(D_m(L_n)) = 3m$

$$\varsigma : V(D_m(L_n)) = \begin{cases} \{1,2,3,4,5,6\} & \text{for } (v'_j), \forall 1 \le j \le 2n \\ \{7,8,9,10,11,12\} & \text{for } (v''_j), \forall 1 \le j \le 2n \\ \vdots & \\ \{6m-5, 6m-4, 6m-3, 6m-2, 6m-1, 6m\} & \text{for } v_j^{(m)}, \forall 1 \le j \le 2n \end{cases}$$

Since $r = 3m$, the minimum number of colors required is $2r$.

Thus, $\chi_r(D_m(L_n)) \le 2r$. In accordance with Lemma 3.1, we have $\chi_r(D_m(L_n)) \ge 2r$.

Hence, $\chi_r(D_m(L_n)) = 2r$ for $1 \le r \le \Delta(D_m(L_n))$.

Lemma 3.2. For a tadpole graph $T_{n,p}$, the lower bound for the r-dynamic chromatic number of the m-shadow graph of the tadpole graph $D_m(T_{n,p})$ is

$$\chi_r(D_m(T_{n,p})) \ge \min\{r+2, r+m\}$$

Proof. Let $V(D_m(T_{n,p})) = \{v'_{nj}, v''_{nj}, \cdots, v_{n_j}^{(m)}, v'_{p_k}, v''_{p_k}, \cdots, v_{p_k}^{(m)} : 1 \le j \le n$ and $1 \le k \le p\}$ be the vertex set and $E(D_m(T_{n,p}))$ $= \{\{v_{n_j}^a v_{n_{j+1}}^x, v_{p_k}^a v_{p_{k+1}}^x : 1 \le a \le m; a \le x \le m; 1 \le j \le n-1; 1 \le k \le p-1\} \cup \{v_{n_1}^a v_{n_j}^x, v_{n_1}^a v_{p_k}^x : 1 \le a \le m; a \le x \le m; j = n; k = p\}\}$ be the edge set whose corresponding cardinalities are $|V(D_m(T_{n,p}))| = m(n+p)$ and $|E(D_m(T_{n,p}))| = m^2(n+p)$, respectively. The vertex v'_j are adjacent to $v''_k, v'''_k, \ldots, v_k^{(m)}$ only where $v''_j, v'''_j, \ldots, v_j^m$ are adjacent. The minimum degree is $\delta(D_m(T_{n,p})) = m$ and the maximum degree is $\Delta(D_m(T_{n,p})) = 3m$.

Let T be an undirected simple connected graph. By the definition of m-shadow graph, every $v_{n_j}^i, v_{p_k}^i$ (for $1 \le j \le n$ and $1 \le k \le p$) vertex in the ith copy of T is adjacent to $v_{n_l}^{i+1}, v_{n_l}^{i+2}, \cdots, v_{n_l}^{(m)}, v_{p_q}^{i+1}, v_{p_q}^{i+2}, \cdots, v_{p_q}^{(m)}$ of all $i+1, i+2, \cdots, m$th copies of T, wherever $v^i_{n_j}, v^i_{p_k}$ are adjacent.

For all m-copies of $T_{n,p}$, when r lies in the range, $1 \le r \le m - 1$, the minimum number of colors required to satisfy r-adjacency are $r + 2$, whereas for $m \le r \le \Delta((D_m(T_{n,p})))$, minimum of $r + m$ colors are required. Considering the r-dynamic coloring condition, we take $\min\{r+2, r+m\}$ to be the lower bound for $D_m(T_{n,p})$.

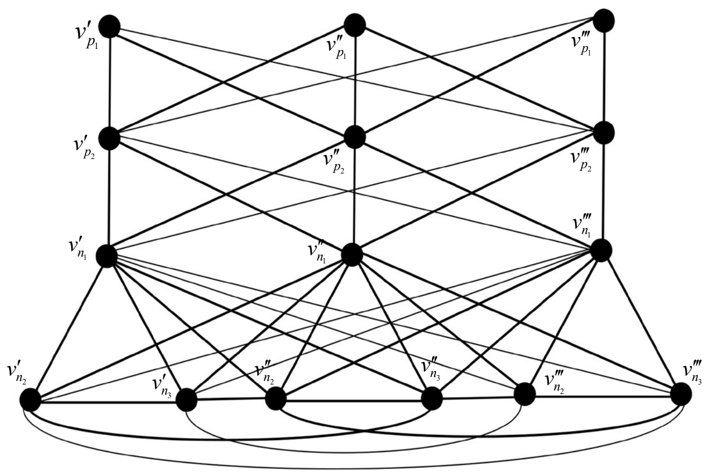

Figure 2. $(D_3(T_{3,2}))$-m-shadow graph of tadpole graph $T_{3,2}$

For example, when $m \ge 2$ and $r = 2$, we have $\min\{4, 2 + m\}$ to be 4.

(i) When n is odd and for all p, assign the colors (say) ς_1, ς_2, ς_3 to the vertices $v'_{nj}, v'''_{nj}, \cdots, v^{(m-1)}_{n_j}$ and $v'_{p_k}, v'''_{p_k}, \cdots, v^{(m-1)}_{p_k}$. To achieve 2-dynamic coloring, a new color, ς_4, is required along with the colors ς_2 and ς_3, which are assigned to the vertices $v''_{n_j}, v^{iv}_{n_j}, \cdots, v^{(m)}_{n_j}, v'_{p_k}, v'''_{p_k}, \cdots, v^{(m-1)}_{p_k}$, $(1 \le j \le n)$ and $(1 \le k \le p)$.

(ii) When n is even and for all p, to achieve proper coloring and satisfy 2-dynamic coloring, assign the colors ς_1 and ς_2 to the vertices $v'_{n_j}, v'''_{n_j}, \cdots, v^{(m-1)}_{n_j}$, $v'_{p_k}, v'''_{p_k}, \cdots, v^{(m-1)}_{p_k}$, and colors ς_3 and ς_4 to the vertices $v''_{n_j}, v^{iv}_{n_j}, \cdots, v^{(m)}_{n_j}$, $v''_{p_k}, v^{iv}_{p_k}, \cdots, v^{(m)}_{p_k}$, $(1 \le j \le n)$, and $(1 \le k \le p)$.

Theorem 3.2. Let r, $m \ge 2$, $n \ge 3$, and $p \ge 1$ be any positive integers. Then, the r-dynamic chromatic number of the m-shadow graph of the tadpole graph $D_m(T_{n,p})$ is

$$
\chi_r(D_m(T_{n,p})) = \begin{cases}
3 & r=1, n=1(\text{mod}2), \forall p, m \\
7 & r=4, n=1(\text{mod}3), \forall p, m \\
 & r=5, n=1(\text{mod}3), \forall n \geq 7, p \text{ and } m=2 \\
8 & r=\Delta(D_m(T_{n,p})), n=1(\text{mod}3), \forall n \geq 8, p \text{ and } m=2 \\
 & r=\Delta(D_m(T_{n,p})), n=2(\text{mod}3), \forall n \geq 8, p \text{ and } m=2 \\
10 & r=\Delta(D_m(T_{n,p})), n=5, \forall p, m=2 \\
2r & r=1, n=0(\text{mod}2), \forall p, m \\
 & r=2, \forall n, p \text{ and } m \\
 & r=4, n=4 \text{ and } n=2(\text{mod}3), \forall p, m \\
r+m & r=3, n=1(\text{mod}3), \forall m, n \geq 7 \text{ and } p=1,2 \text{ and } p=1,2(\text{mod}3) \\
 & r=3, n=2(\text{mod}3), \forall p, m \\
 & 3 \leq r \leq \Delta(D_m(T_{n,p})), n=0(\text{mod}3), \forall p, m=2 \\
 & 3 \leq r \leq \Delta(D_m(T_{n,p}))-4, n=0(\text{mod}3), \forall p, m \geq 3 \\
 & \Delta(D_m(T_{n,p}))-3 \leq r \leq \Delta(D_m(T_{n,p})), n=3, \forall p, m \geq 3 \\
r+m+1 & r=3, n=4, \forall p, m \\
 & r=3, n=1(\text{mod}3), \forall n \geq 7, p=0(\text{mod}3), m \\
r+m+2 & r=5, n=5, \forall p, m \\
4m & \Delta(D_m(T_{n,p}))-3 \leq r \leq \Delta(D_m(T_{n,p})), \forall n > 3, p, m \geq 3 \\
2(m+1) & r=5, n=1(\text{mod}3), \forall n \geq 7, p, m \geq 3 \\
\left\lceil \frac{8(m+3)}{5} \right\rceil & r=5, n=4, \forall p, m
\end{cases}
$$

Proof. To ascertain the r-dynamic chromatic number of $D_m(T_{n,p})$, we have to prove the theorem and divide it into some cases.

Case 1. $r=1, n \equiv 1(\text{mod}2), \forall p, m.$

To ascertain the r-dynamic chromatic number of $D_m(T_{n,p})$, we have to prove that $\chi_r(D_m(T_{n,p})) \geq 3$ and $\chi_r(D_m(T_{n,p})) \leq 3$. In accordance with Lemma 3.2, we have $\chi_r(D_m(T_{n,p})) \geq 3$. So, it completes the proof of lower bound. Then, we have to prove the upper bound. To prove $\chi_r(D_m(T_{n,p})) \leq 3$, let us define a function $\varsigma : V(D_m(T_{n,p})) \rightarrow S(\varsigma)$, where $S(\varsigma) = \{1, 2, 3\}$.

For this case, we divide into two subcases, namely Subcase 1 and Subcase 2.

Subcase 1. $r = 1$, $n \equiv 1(\text{mod}2)$, $p \equiv 1(\text{mod}2)$

$$
\varsigma : V(D_m(T_{n,p})) = \begin{cases}
\varsigma_1 & v'_{n_{j\equiv 2(mod\,3)}}, v''_{n_{j\equiv 2(mod\,3)}}, \cdots, v^{(m)}_{n_{j\equiv 2(mod\,3)}}, \forall 1 \leq j \leq n \\
 & v'_{pk=odd}, v''_{pk=odd}, \cdots, v^{(m)}_{pk=odd}, \forall k \text{ is odd} \\
\varsigma_2 & v'_{n_{j\equiv 1(mod\,3)}}, v''_{n_{j\equiv 1(mod\,3)}}, \cdots, v^{(m)}_{n_{j\equiv 1(mod\,3)}}, \forall 1 \leq j \leq n \\
 & v'_{pk=even}, v''_{pk=even}, \cdots, v^{(m)}_{pk=even}, \forall k \text{ is even} \\
\varsigma_3 & v'_{n_{j\equiv 0(mod\,3)}}, v''_{n_{j\equiv 0(mod\,3)}}, \cdots, v^{(m)}_{n_{j\equiv 0(mod\,3)}}, \forall 1 \leq j \leq n
\end{cases}
$$

Subcase 2. $r = 1$, $n \equiv 1(\text{mod}2)$, $p \equiv 0(\text{mod}2)$

$$
\varsigma : V(D_m(T_{n,p})) = \begin{cases}
\varsigma_1 & v'_{n_{j\equiv 1(mod\,3)}}, v''_{n_{j\equiv 1(mod\,3)}}, \cdots, v^{(m)}_{n_{j\equiv 1(mod\,3)}}, \forall 1 \leq j \leq n \\
 & v'_{pk=odd}, v''_{pk=odd}, \cdots, v^{(m)}_{pk=odd}, \forall k \text{ is odd} \\
\varsigma_2 & v'_{n_{j\equiv 2(mod\,3)}}, v''_{n_{j\equiv 2(mod\,3)}}, \cdots, v^{(m)}_{n_{j\equiv 2(mod\,3)}}, \forall 1 \leq j \leq n \\
 & v'_{pk=even}, v''_{pk=even}, \cdots, v^{(m)}_{pk=even}, \forall k \text{ is even} \\
\varsigma_3 & v'_{n_{j\equiv 0(mod\,3)}}, v''_{n_{j\equiv 0(mod\,3)}}, \cdots, v^{(m)}_{n_{j\equiv 0(mod\,3)}}, \forall 1 \leq j \leq n
\end{cases}
$$

Based on the lower bound and the upper bound, we have $3 \le (D_m(T_{n,p})) \le 3$. Now, it is easy to establish the r-adjacency, hence $\chi_r(D_m(T_{n,p}))$ for $r = 1$, $n \equiv 1(\text{mod}2)$, $\forall p, m$.

Case 2. For $4 \le r \le 5$, $n \equiv 1(\text{mod}3)$, $\forall p, m$.

To ascertain the r-dynamic chromatic number of $(D_m(T_{n,p}))$, we have to prove that $\chi_r(D_m(T_{n,p})) \ge 7$ and $\chi_r(D_m(T_{n,p})) \le 7$. In accordance with Lemma 3.2, we have $\chi_r(D_m(T_{n,p})) \ge 7$. So, it completes the proof of lower bound. Then, we have to prove the upper bound. To prove $\chi_r(D_m(T_{n,p})) \le 7$, let us define a function ς: $V(D_m(T_{n,p})) \to S(\varsigma)$, where $S(\varsigma) = \{1, 2, 3, \ldots, 7\}$. For this case, we divide into two subcases, namely Subcase 3 and Subcase 4.

Subcase 3. $r = 4$, $n \equiv 1(\text{mod}3)$, $\forall n \ge 7, p, m$.

Consider $m = 2$ when $r = 4$, $n = 7, 10, 13, \ldots$ and $\forall p$

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } v'_{n_j}, v'_{p_k}, \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{4,5,6,7\} & \text{for } v''_{n_j}, v''_{p_k}, \forall 1 \le j \le n \text{ and } 1 \le k \le p. \end{cases}$$

Consider m-shadow graph when $r = 4$, $n = 7, 10, 13, \ldots$ and $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \left(v'''_{n_j}, v'''_{p_k}\right), \cdots, \left(v^{(m-1)}_{n_j}\right), \forall \le j \le n \\ & \text{and } 1 \le k \le p \\ \{4,5,6,7\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \left(v^{iv}_{n_j}, v^{iv}_{p_k}\right), \cdots, \left(v^{(m)}_{n_j}, v^{(m)}_{p_k}\right), \forall \le j \le n \\ & \text{and } 1 \le k \le p. \end{cases}$$

Subcase 4. $r = 5$, $n \equiv 1(\text{mod}3)$, $\forall n \ge 7, p, m$.

Consider $m = 2$ when $r = 5$, $n = 7, 10, 13, \ldots$ and $\forall p$

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{4,5,6,7\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider m-shadow graph when $r = 4$, $n = 7, 10, 13, \ldots$ and $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \left(v'''_{n_j}, v'''_{p_k}\right), \cdots, \left(v^{(m-1)}_{n_j}\right), \forall \le j \le n \\ & \text{and } 1 \le k \le p \\ \{4,5,6,7\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \left(v^{iv}_{n_j}, v^{iv}_{p_k}\right), \cdots, \left(v^{(m)}_{n_j}, v^{(m)}_{p_k}\right), \forall \le j \le n \\ & \text{and } 1 \le k \le p. \end{cases}$$

Based on Subcases 3 and 4, a minimum of seven colors is required to satisfy r-adjacency, $\chi_r(D_m(T_{n,p})) \le 7$. In accordance with the lower bound and the upper bound, we have $7 \le \chi_r(D_m(T_{n,p})) \le 7$. Now, it is easy to establish the r-adjacency, hence $\chi_r(D_m(T_{n,p})) = 7$ for $4 \le r \le 5$, $n \equiv 1(\text{mod}3)$, $\forall p, m$.

Case 3. $r = \Delta(D_m(T_{n,p}))$, $n \equiv 1(\text{mod}3)$, $\forall p, m = 2$ and $r = \Delta(D_m(T_{n,p}))$, $n \equiv 2(\text{mod}3)$, $\forall n \ge 8$, $\forall p$, $m = 2$.

In accordance with Lemma 3.2, we have $\chi_r(D_m(T_{n,p})) \ge 8$. So, it completes the proof of lower bound. Then, we have to prove the upper bound. To prove $\chi_r(D_m(T_{n,p})) \le 8$, let us define a function ς: $V(D_m(T_{n,p})) \to S(\varsigma)$, where $S(\varsigma) = \{1, 2, 3, \ldots, 8\}$. For this case, we divide into two subcases, namely Subcase 5 and Subcase 6.

Subcase 5. $m = 2$, $r = \Delta(D_m(T_{n,p}))$, $n \equiv 1(\text{mod}3)$, $\forall p$.

When $m = 2$, $r = 6$, $n = 4, 7, 10, \ldots$, and $\forall p$.

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Subcase 6. $m = 2$, $r = \Delta(D_2(T_{n,p}))$, $n \equiv 2(\text{mod}3)$, $\forall n \geq 8$, p.

When $m = 2$, $r = 6$, $n = 8, 11, 14, \ldots$ and $\forall p$.

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \\ \{5,6,7,8\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \end{cases}$$

Based on Subcases 5 and 6, a minimum of eight colors is required to satisfy r-adjacency, $\chi_r(D_m(T_{n,p})) \leq 8$. In accordance with the lower bound and the upper bound, we have $8 \leq \chi_r(D_m(T_{n,p})) \leq 8$. Now, it is easy to establish the r-adjacency, hence $\chi_r(D_m(T_{n,p})) = 8$ for $r = \Delta(D_m(T_{n,p}))$, $n \equiv 1(\text{mod}3)$, $\forall p$, $m = 2$ and $r = \Delta(D_m(T_{n,p}))$, $n \equiv 2(\text{mod}3)$, $\forall n \geq 8$ $\forall p$, $m = 2$.

Case 4. $m = 2$, $r = \Delta(D_2(T_{5,p}))$, $n = 5$, $\forall p$.

In accordance with Lemma 3.2, we have $\chi_r(D_m(T_{n,p})) \geq 10$. So, it completes the proof of lower bound. Then, we have to prove the upper bound. To prove $\chi_r(D_m(T_{n,p})) \leq 10$, let us define a function ς: $V(D_m(T_{n,p})) \rightarrow S(\varsigma)$ where $(\varsigma) = \{1, 2, 3, \ldots, 10\}$.

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3,4,5\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \\ \{5,6,7,8,9,10\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \end{cases}$$

Thus, a minimum of 10 colors is required to satisfy r-adjacency, $\chi_r(D_m(T_{n,p})) \leq 10$. Based on the lower bound and the upper bound, we have $10 \leq \chi_r(D_m(T_{n,p})) \leq 10$. So, we can conclude that $\chi_r(D_m(T_{n,p})) = 10$ for $m = 2$, $r = \Delta(D_2(T_{5,p}))$, $n = 5$, $\forall p$.

Case 5. $r = 1$, $n \equiv 0(\text{mod}2)$, $\forall p$, m.

To ascertain the r-dynamic chromatic number of $D_m(T_{n,p})$, we have to prove that $\chi_r(D_m(T_{n,p})) \leq 2r$ and $\chi_r(D_m(T_{n,p})) \geq 2r$. In accordance with Lemma 3.2, we have $\chi_r(D_m(T_{n,p})) \geq 2r$. To prove $\chi_r(D_m(T_{n,p})) \leq 2r$, let us define a function ς: $V(D_m(T_{n,p})) \rightarrow S(\varsigma)$ where $(\varsigma) = \{1,2,3, \ldots, 2r\}$.

Subcase 7. $r = 1$, $n \equiv 0(\text{mod}2)$, $p \equiv 1(\text{mod}2)$.

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \varsigma_1 & \text{for } v'_{p_{k=odd}}, v''_{p_{k=odd}}, \forall 1 \leq k \leq p \\ & \text{for } v'_{n_{j=even}}, v''_{n_{j=even}}, \forall 1 \leq j \leq n \\ \varsigma_2 & \text{for } v'_{p_{k=even}}, v''_{p_{k=even}}, \forall 1 \leq k \leq p \\ & \text{for } v'_{n_{j=odd}}, v''_{n_{j=odd}}, \forall 1 \leq j \leq n \end{cases}$$

Subcase 8. $r = 1$, $n \equiv 1(\text{mod}2)$, $p \equiv 0(\text{mod}2)$.

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \varsigma_1 & \text{for } v'_{p_{k=odd}}, v''_{p_{k=odd}}, \forall 1 \leq k \leq p \\ & \text{for } v'_{n_{j=odd}}, v''_{n_{j=odd}}, \forall 1 \leq j \leq n \\ \varsigma_2 & \text{for } v'_{p_{k=even}}, v''_{p_{k=even}}, \forall 1 \leq k \leq p \\ & \text{for } v'_{n_{j=even}}, v''_{n_{j=even}}, \forall 1 \leq j \leq n \end{cases}$$

Subcase 9. $r = 2$, $\forall n$, p, m.

1. Consider $m = 2$.

When $r = 2$, $n \equiv 0(\text{mod}2)$, $\forall p$.

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \\ \{3,4\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \end{cases}$$

When $r = 2$, $n \equiv 1(\text{mod}2)$, $\forall p$

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{2,3,4\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

2. Consider m-shadow graph.
When $r = 2$, $n \equiv 0(\text{mod}2)$, $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2\} & \text{for } v'_{n_j}, v'_{p_k} = v'''_{n_j}, v'''_{p_k} = \cdots = v^{(m-1)}_{n_j}, v^{(m-1)}_{p_k}, \forall \le j \le n \text{ and } 1 \le k \le p \\ \{3,4\} & \text{for } v''_{n_j}, v''_{p_k}, v^{iv}_{n_j}, v^{iv}_{p_k} = \cdots = v^{(m)}_{n_j}, v^{(m)}_{p_k}, \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

When $r = 2$, $n \equiv 1(\text{mod}2)$, $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } v'_{n_j}, v'_{p_k} = v'''_{n_j}, v'''_{p_k} = \cdots = v^{(m-1)}_{n_j}, v^{(m-1)}_{p_k} \forall \le j \le n \text{ and } 1 \le k \le p \\ \{2,3,4\} & \text{for } v''_{n_j}, v''_{p_k}, v^{iv}_{n_j}, v^{iv}_{p_k} = \cdots = v^{(m)}_{n_j}, v^{(m)}_{p_k}, \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Subcase 10. $r = 4$, $n = 4$ and $n \equiv 2(\text{mod}3)$, $\forall p,m$.
Consider $m = 2$, when $r = 4$, $n = 4$, $\forall p$

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } v'_{n_j}, v'_{p_k}, \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8\} & \text{for } v''_{n_j}, v''_{p_k}, \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider m-shadow graph, when $r = 4$, $n = 4$, $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } v'_{n_j}, v'_{p_k} = v'''_{n_j}, v'''_{p_k} = \cdots = v^{(m-1)}_{n_j}, v^{(m-1)}_{p_k} \forall \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8\} & \text{for } v''_{n_j}, v''_{p_k}, v^{iv}_{n_j}, v^{iv}_{p_k} = \cdots = v^{(m)}_{n_j}, v^{(m)}_{p_k}, \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider $m = 2$, when $r = 4$, $n = 5, 8, 11, \ldots$, $\forall p$

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } v'_{n_j}, v'_{p_k}, \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8\} & \text{for } v''_{n_j}, v''_{p_k}, \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider m-shadow graph, when $r = 4$, $n = 5, 8, 11, \ldots$, $\forall p$

$$\varsigma : V(D_3(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{3,4,5\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{4,5,6\} & \text{for } \left(v'''_{n_j}, v'''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Thus, a minimum of $2r$ colors is required to satisfy r-adjacency, $\chi_r(D_m(T_{n,p})) \le 2$. In accordance with the lower bound and the upper bound, we have $2 \le \chi_r(D_m(T_{n,p})) \le 2$, hence $\chi_r(D_m(T_{n,p})) = 2$ for $r = 4$, $n = 4$ and $n \equiv 2(\text{mod}3)$, $\forall p, m$.

Case 6. $r = 3$, $n \equiv 1(\text{mod}3)$, $\forall$ m, $n \ge 7$ and $p = 1, 2$ and $p \equiv 1, 2(\text{mod}3)$; $r = 3$, $n \equiv 2(\text{mod}3)$ $\forall$ p, m; $3 \le r \le \Delta(D_m(T_{n,p}))$, $n \equiv 0(\text{mod}3)$, $\forall$ $p,m = 2$; $3 \le r \le \Delta((D_m(T_{n,p})) - 4$, $n \equiv 0(\text{mod}3)$, $\forall$ p, $m \ge 3$; $\Delta(D_m(T_{n,p})) - 3 \le r \le \Delta(D_m(T_{n,p}))$, $n = 3$, $\forall$ p, $m \ge 3$.

To ascertain the r-dynamic chromatic number of $D_m(T_{n,p})$, we have to prove that, $\chi_r(D_m(T_{n,p})) \le r + m$ and $\chi_r(D_m(T_{n,p})) \ge r + m$. In accordance with Lemma 3.2, we have, $\chi_r(D_m(T_{n,p})) \ge r + m$. It completes the proof of lower bound. Then, we have to prove the upper bound. To prove, $\chi_r(D_m(T_{n,p})) \leq r + m$, let us define a function, $\varsigma : V(D_m(T_{n,p})) \rightarrow S(\varsigma)$, where $S(\varsigma) = \{1, 2, 3, \dots, r + m\}$. For this case, we divide into five subcases.

Subcase 11. $r = 3$, $n \equiv 1(\text{mod}3)$, $\forall m$, $n \geq 7$, $p = 1, 2$ and $p \equiv 1, 2(\text{mod}3)$.

Consider $m = 2$, when $r = 3$, $n = 7, 10, 13, \dots$ and $p = 1, 2, 4, 5, 7, 8, \dots$

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } (v'_{n_j}, v'_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \\ \{3,4,5\} & \text{for } (v''_{n_j}, v''_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \end{cases}$$

Consider $m = 3$, when $r = 3$, $n = 7, 10, 13, \dots$ and $p = 1, 2, 4, 5, 7, 8, \dots$

$$\varsigma : V(D_3(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } (v'_{n_j}, v'_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \\ \{3,4,5\} & \text{for } (v''_{n_j}, v''_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \\ \{4,5,6\} & \text{for } (v'''_{n_j}, v'''_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \end{cases}$$

Consider m-shadow graph, when $r = 3$, $n = 7, 10, 13, \dots$ and $p = 1, 2, 4, 5, 7, 8, \dots$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } (v'_{n_j}, v'_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \\ \{3,4,5\} & \text{for } (v''_{n_j}, v''_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \\ \vdots & \\ \{r+m-2, r+m-1, r+m\} & \text{for } (v^{(m)}_{n_j}, v^{(m)}_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \end{cases}$$

Subcase 12. $r = 3$, $n \equiv 2(\text{mod}3)$, $\forall p, m$.

Consider $m = 2$, when $r = 3$, $n = 5, 8, 11, \dots$ and $\forall p$

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } (v'_{n_j}, v'_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \\ \{3,4,5\} & \text{for } (v''_{n_j}, v''_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \end{cases}$$

Consider m-shadow graph, when $r = 3$, $n = 5, 8, 11, \dots$ and $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } (v'_{n_j}, v'_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \\ \{3,4,5\} & \text{for } (v''_{n_j}, v''_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \\ \vdots & \\ \{r+m-2, r+m-1, r+m\} & \text{for } (v^{(m)}_{n_j}, v^{(m)}_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \end{cases}$$

Subcase 13. $m = 2$, $3 \leq r \leq \Delta(D_2(T_{n,p}))$, $n \equiv 0(\text{mod}3)$, $\forall p$.

When $r = 3$, $n = 3, 6, 9, \dots$ and $\forall p$

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } (v'_{n_j}, v'_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \\ \{3,4,5\} & \text{for } (v''_{n_j}, v''_{p_k}), \forall 1 \leq j \leq n \text{ and } 1 \leq k \leq p \end{cases}$$

When $r = \Delta(D_2(T_{n,p})) = 6$, $n = 3, 6, 9, \dots$ and $\forall p$

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Subcase 14. $m \ge 3$, $3 \le r \le \Delta(D_m(T_{n,p})) - 4$, $n \equiv 0(\text{mod}3)$, $\forall p$.
Consider $m = 3$, when $r = 3$, $n = 3, 6, 9, \ldots$ and $\forall p$

$$\varsigma : V(D_3(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{2,3,4\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{4,5,6\} & \text{for } \left(v'''_{n_j}, v'''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider $m = 3$, when $r = \Delta(D_3(T_{n,p})) - 4 = 5$, $n = 3, 6, 9, \ldots$ and $\forall p$

$$\varsigma : V(D_3(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{3,4,5,6\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8\} & \text{for } \left(v'''_{n_j}, v'''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider m-shadow graph, when $r = 3$, $n = 3, 6, 9, \ldots$ and $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{2,3,4\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \vdots & \\ \{r+m-2, r+m-1, r+m\} & \text{for } \left(v^{(m)}_{n_j}, v^{(m)}_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider m-shadow graph, when $r = \Delta(D_m(T_{n,p})) - 4$, $n = 3, 6, 9, \ldots$ and $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{3,4,5,6\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \vdots & \\ \{r+m-3, r+m-2, r+m-1, r+m\} & \text{for } \left(v^{(m)}_{n_j}, v^{(m)}_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Subcase 15. $m \ge 3$, $\Delta(D_m(T_{n,p})) - 3 \le r \le \Delta(D_m(T_{n,p}))$, $n = 3$, $\forall p$.
Consider $m = 3$, when $r = \Delta(D_3(T_{n,p})) - 3 = 6$, $n = 3$ and $\forall p$

$$\varsigma : V(D_3(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{4,5,6,7\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{7,8,9\} & \text{for } \left(v'''_{n_j}, v'''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider $m = 3$, when $r = \Delta(D_3(T_{n,p})) = 9$, $n = 3$ and $\forall p$

$$\varsigma : V(D_3(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{9,10,11,12\} & \text{for } \left(v'''_{n_j}, v'''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider m-shadow graph, when $r = \Delta(D_m(T_{n,p})) - 3$, $n = 3$ and $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{4,5,6,7\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{7,8,9,10\} & \text{for } \left(v'''_{n_j}, v'''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \vdots & \\ \{r+m-2, r+m-1, r+m\} & \text{for } \left(v^{(m)}_{n_j}, v^{(m)}_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider m-shadow graph, when $r = \Delta(D_m(T_{n,p}))$, $n = 3$ and $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \vdots & \\ \{r+m-3, r+m-2, r+m-1, r+m\} & \text{for } \left(v^{(m)}_{n_j}, v^{(m)}_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Based on Subcase 11 until Subcase 15, a minimum of $r + m$ colors is required to satisfy r-adjacency, $\chi_r(D_m(T_{n,p})) \le r + m$. In accordance with the lower bound and the upper bound, we have $r + m\ \chi_r(D_m(T_{n,p})) \le r + m$, hence $\chi_r(D_m(T_{n,p})) = r + m$ for $m \ge 3\ \Delta(D_m(T_{n,p})) - 3 \le r \le \Delta(D_m(T_{n,p}))$, $n = 3$ and $\forall p$.

Case 7. For $r = 3$, $n = 4$, $\forall p$ and for $r = 3$, $n \equiv 1(\text{mod}3)$, $\forall\ n \ge 7$, $p \equiv 0(\text{mod}3)$.

To ascertain the r-dynamic chromatic number of $D_m(T_{n,p})$, we have to prove that $\chi_r(D_m(T_{n,p})) \ge r + m + 1$ and $\chi_r(D_m(T_{n,p})) \le r + m + 1$. In accordance with Lemma 3.2, we have $\chi_r(D_m(T_{n,p})) \ge r + m + 1$. It completes the proof of lower bound. Then, we have to prove the upper bound. To prove $\chi_r(D_m(T_{n,p})) \le r + m + 1$, let us define a function $\varsigma : V(D_m(T_{n,p})) \to S(\varsigma)$, where $S(\varsigma) = \{1, 2, 3, \dots , r + m +1\}$.

Subcase 16. $r = 3$, $n = 4$, $\forall p$.

Consider $m = 2$, when $r = 3$, $n = 4$ and $\forall p$, m

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{4,5,6\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider m-shadow graph, when $r = 3$, $n = 4$ and $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3,\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{4,5,6\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \vdots & \\ \{r+m-1, r+m, r+m+1\} & \text{for } \left(v^{(m)}_{n_j}, v^{(m)}_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Subcase 17. $r = 3$, $n \equiv 1(\text{mod}3)$, $\forall\ n \ge 7$, $p \equiv 0(\text{mod}3)$.

Consider $m = 2$, when $r = 3$, $n = 7, 10, 13, \dots$ and $p = 3, 6, 9, \dots$

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{4,5,6\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider m-shadow graph, when $r = 3$, $n = 7, 10, 13, \dots$ and $p = 3, 6, 9, \dots$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{4,5,6\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \vdots & \\ \{r+m-1, r+m, r+m+1\} & \text{for } \left(v^{(m)}_{n_j}, v^{(m)}_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Based on Subcases 16 and 17, a minimum of $r + m + 1$ colors is required to satisfy r-adjacency, $\chi_r(D_m(T_{n,p})) \le r + m + 1$. In accordance with the upper bound and the lower bound, we have $r + m + 1 \le \chi_r(D_m(T_{n,p})) \le r + m + 1$, hence $\chi_r(D_m(T_{n,p})) = r + m + 1$ for $r = 3$, $n \equiv 1(\text{mod}3)$, $\forall\, n \ge 7$, $p \equiv 0(\text{mod}3)$.

Case 8. $r = 5$, $n = 5$, $\forall p, m$.

To ascertain the r-dynamic chromatic number of $D_m(T_{n,p})$, we have to prove that $\chi_r(D_m(T_{n,p})) \ge r + m + 2$ and $\chi_r(D_m(T_{n,p})) \le r + m + 2$. In accordance with Lemma 3.2, we have $\chi_r(D_m(T_{n,p})) \ge r + m + 2$. It completes the proof of lower bound. Then, we have to prove the upper bound. To prove $\chi_r(D_m(T_{n,p})) \le r + m + 2$, let us define a function $\varsigma : V(D_m(T_{n,p})) \rightarrow S(\varsigma)$, where $S(\varsigma) = \{1, 2, 3, \dots, r + m + 2\}$.

Consider $m = 2$, when $r = 5$, $n = 5$ and $\forall p$

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3,4,5\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{6,7,8,9\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider m-shadow graph, when $r = 5$, $n = 5$ and $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3,4,5\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{2,3,4,5,6\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{3,4,5,6,7\} & \text{for } \left(v'''_{n_j}, v'''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \vdots & \\ \{r+m-1, r+m, r+m+1, r+m+2\} & \text{for } \left(v^{(m)}_{n_j}, v^{(m)}_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Thus, a minimum of $r + m + 2$ colors is required to satisfy r-adjacency, $\chi_r(D_m(T_{n,p})) \le r + m + 2$. In accordance with the upper bound and the lower bound, we have $r + m + 2 \le \chi_r(D_m(T_{n,p})) \le r + m + 2$, hence $\chi_r(D_m(T_{n,p})) = r + m + 2$ for $r = 5$, $n = 5$, $\forall p, m$.

Case 9. $m \ge 3$, $\Delta(D_m(T_{n,p})) - 3 \le r \le \Delta(D_m(T_{n,p}))$, $\forall n > 3, p$.

To ascertain the r-dynamic chromatic number of $D_m(T_{n,p})$, we have to prove that $\chi_r(D_m(T_{n,p})) \ge 4m$ and $\chi_r(D_m(T_{n,p})) \le 4m$. In accordance with Lemma 3.2, we have $\chi_r(D_m(T_{n,p})) \ge 4m$. It completes the proof of lower bound. Then, we have to prove the upper bound. To prove $\chi_r(D_m(T_{n,p})) \le 4m$, let us define a function $\varsigma : V(D_m(T_{n,p})) \rightarrow S(\varsigma)$, where $S(\varsigma) = \{1, 2, 3, \dots, 4m\}$.

Consider $m = 3$, when $r = \Delta(D_m(T_{n,p})) - 3 = 6$, $n = 4, 5, 6, \dots$ and $\forall p$

$$\varsigma : V(D_3(T_{n,p})) = \begin{cases} \{1,2,3,4,5\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8,9\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{9,10,11,12\} & \text{for } \left(v'''_{n_j}, v'''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider $m = 3$, when $r = \Delta(D_m(T_{n,p})) = 9$, $n = 4, 5, 6, \cdots$ and $\forall p$

$$\varsigma : V(D_3(T_{n,p})) = \begin{cases} \{1,2,3,4,\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8,\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{9,10,11,12\} & \text{for } \left(v'''_{n_j}, v'''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider m-shadow graph, when $r = \Delta(D_m(T_{n,p})) - 3$, $n = 4, 5, 6, \ldots$ and $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3,4,5\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8,9\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{9,10,11,12,13\} & \text{for } \left(v'''_{n_j}, v'''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \vdots & \\ \{4m-3,4m-2,4m-1,4m\} & \text{for } \left(v^{(m)}_{n_j}, v^{(m)}_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Consider m-shadow graph, when $r = \Delta(D_m(T_{n,p}))$, $n = 4, 5, 6, \ldots$ and $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{9,10,11,12\} & \text{for } \left(v'''_{n_j}, v'''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \vdots & \\ \{4m-3,4m-2,4m-1,4m\} & \text{for } \left(v^{(m)}_{n_j}, v^{(m)}_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Thus, a minimum of $4m$ colors is required to satisfy r-adjacency, $\chi_r(D_m(T_{n,p})) \le 4m$. In accordance with Lemma 3.2, we have $\chi_r(D_m(T_{n,p})) \ge 4m$, hence $\chi_r(D_m(T_{n,p})) = 4m$ for $m \ge 3$, $\Delta(D_m(T_{n,p})) - 3 \le r \le \Delta(D_m(T_{n,p}))$, $\forall n > 3, p$.

Case 10. $m \ge 3$, $r = 5$, $n \equiv 1(\text{mod}3)$, $\forall n \ge 7, p$.

To ascertain the r-dynamic chromatic number of $D_m(T_{n,p})$, we have to prove that $\chi_r(D_m(T_{n,p})) \ge 2m + 2$ and $\chi_r(D_m(T_{n,p})) \le 2m + 2$. To prove $\chi_r(D_m(T_{n,p})) \le 2m + 2$, let us define a function $\varsigma : V(D_m(T_{n,p})) \to S(\varsigma)$, where $S(\varsigma) = \{1, 2, 3, \ldots, r + 2m + 2\}$.

1. Consider $m = 3$, when $r = 5$, $n = 7, 10, 13, \ldots$ and $\forall p$

$$\varsigma : V(D_3(T_{n,p})) = \begin{cases} \{1,2,3,4,\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{3,4,5,6\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8\} & \text{for } \left(v'''_{n_j}, v'''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

2. Consider m-shadow graph, when $r = 5$, $n = 7, 10, 13, \ldots$ and $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{3,4,5,6\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \vdots & \\ \{2m-1, 2m, 2m+1, 2m+2\} & \text{for } \left(v^{(m)}_{n_j}, v^{(m)}_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Thus, a minimum of $2m + 2$ colors is required to satisfy r-adjacency, $\chi_r(D_m(T_{n,p})) \le 2m + 2$. In accordance with Lemma 3.2, we have $\chi_r(D_m(T_{n,p})) \ge 2m + 2$, hence $\chi_r(D_m(T_{n,p})) = 2m + 2$ for $m \ge 3$, $r = 5$, $n \equiv 1(\text{mod}3)$, $\forall n \ge 7$, p.

Case 11. $r = 4$, $n = 5$, $\forall p,m$.

To ascertain the r-dynamic chromatic number of $D_m(T_{n,p})$, we have to prove that $\chi_r(D_m(T_{n,p})) \ge \left\lceil \frac{8(m+3)}{5} \right\rceil$ and $\chi_r(D_m(T_{n,p})) \le \left\lceil \frac{8(m+3)}{5} \right\rceil$. To prove $\chi_r(D_m(T_{n,p})) \le \left\lceil \frac{8(m+3)}{5} \right\rceil$, let us define a function $\varsigma : V(D_m(T_{n,p})) \to S(\varsigma)$, where $S(\varsigma) = \left\{1, 2, 3, \ldots, \left\lceil \frac{8(m+3)}{5} \right\rceil \right\}$.

1. Consider $m = 2$, when $r = 4$, $n = 5$, $\forall p$

$$\varsigma : V(D_2(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

2. Consider m-shadow graph, when $r = 4$, $n = 5$, $\forall p$

$$\varsigma : V(D_m(T_{n,p})) = \begin{cases} \{1,2,3,4\} & \text{for } \left(v'_{n_j}, v'_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \{5,6,7,8\} & \text{for } \left(v''_{n_j}, v''_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \\ \vdots & \\ \left\{\frac{8m+9}{5}, \frac{8m+14}{5}, \frac{8m+19}{5}, \frac{8m+24}{5}\right\} & \text{for } \left(v^{(m)}_{n_j}, v^{(m)}_{p_k}\right), \forall 1 \le j \le n \text{ and } 1 \le k \le p \end{cases}$$

Thus, on generalizing, a minimum of $\left\lceil \frac{8(m+3)}{5} \right\rceil$ colors is required to satisfy r-adjacency, $\chi_r(D_m(T_{n,p})) \le \left\lceil \frac{8(m+3)}{5} \right\rceil$. In accordance with Lemma 3.2, we have $\chi_r(D_m(T_{n,p})) \ge \left\lceil \frac{8(m+3)}{5} \right\rceil$, hence $\chi_r(D_m(T_{n,p})) = \left\lceil \frac{8(m+3)}{5} \right\rceil$ for $r = 4$, $n = 5$, $\forall p$, m.

4. Concluding remarks

We have studied the r-dynamic chromatic number of the ladder graph and the tadpole graph using the m-shadow operation of graphs. Further, we are working on the r-dynamic coloring of various graphs in the ladder graph family using the block circulant matrix approach. Since obtaining the exact value of the r-dynamic chromatic number is considered a nondeterministic polynomial time-complete problem, solving this problem is still widely open. Therefore, we propose the following open problem:

- Determine the r-dynamic chromatic number of other special graph operations.
- Characterize the existence of r-dynamic coloring of any graph.

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Conflict of interest

There is no conflict of interest in this study.

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