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Research Article

A Harmonic-Type Method for Nonlinear Equations in Banach Space

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Abstract: In this work, we investigate the local and semi-local convergence of a harmonic mean Newton-type fourth-order technique for estimating the locally unique solutions of nonlinear systems in Banach spaces. The local analysis is established in previous works under assumptions reaching the fifth derivative of the involved operator. Therefore, the applicability of the method is restricted to solving nonlinear equations containing operators that are at least five times differentiable. However, this method may converge even if these assumptions are not satisfied. Other limitations include the lack of a priori error estimates and the isolation of the solution results. The local analysis in this work is shown using only the first derivative of the method. Moreover, a priori estimates on the error distances and uniqueness results are provided based on generated continuity assumptions on the Fréchet derivative of the operator. Furthermore, the more interesting semi-local case not studied previously is developed by means of majorizing sequences. The analysis in both cases is given not in the finite-dimensional Euclidean but in the more general setting of Banach spaces. Some numerical tests are performed to validate the theoretical results further.

***Keywords*:** harmonic-type iterative method, Banach space, convergence order, Fréchet-derivative

MSC: 49M15, 65E99, 47H99, 65S15, 41A25

1. Introduction

Let S_1 and S_2 be Banach spaces, and let S denote a nonempty and open subset of the space S_1. Moreover, the notation $\mathfrak{L}(S_1, S_2)$ denotes the space of continuous operators from S_1 into S_2, which are linear. Furthermore, H is an operator between S and S_2, which is differentiable according to Fréchet.

A multitude of applications from diverse fields is governed by mathematical modeling [1-4]. In particular, the applications are reduced to

$$H(x) = 0. \tag{1}$$

The determination of a solution $x^* \in S$ of (1) is a very challenging undertaking in general. The analytical form

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of x^* is most desirable. However, that is fulfilled only on rare occasions. Thus, iterative solution methods have been developed, when starting from a certain point $x_0 \in S$, a sequence is generated that approximates x^* (see, for example, [4-7]).

The most popular method is Newton's method (NM) defined for each $m = 0, 1, 2, \ldots$ by

$$x_{m+1} = x_m - H'(x_m)^{-1} H(x_m). \tag{2}$$

The convergence order (CO) for NM is quadratic [5]. In order to increase convergence order numerous methods have been developed based on geometrical or algebraic considerations (see [8-14] and references cited therein).

Among those, we select the two-step Harmonic mean method (TSHM) given as

$$y_m = x_m - \frac{2}{3} H'(x_m)^{-1} H(x_m),$$

$$s_m = H'(x_m)^{-1} H'(y_m),$$

$$T_m = I - \frac{1}{4}(s_m - I) + \frac{1}{2}(s_m - I)^2,$$

$$B_m = H'(x_m) + H'(y_m), \quad A_m = \frac{1}{2} B_m^{-1},$$

and

$$x_{m+1} = x_m - T_m A_m H(x_m). \tag{3}$$

The convergence order of TSHM was established to be four in [12]. This method is the generalization of one of the members of the family of higher-order multi-point methods based on the power mean for solving a single nonlinear equation by Babajee et al. [15]. The benefit of using this method is that it does not require the evaluation of second- or higher-order Fréchet derivatives. Such derivatives are costlier from a computational point of view. This property of a method makes it useful to solve large-scale systems of nonlinear equations. The performance of the comparison with other methods is well explained in [12]. The operator H' satisfies generalized continuity conditions, which include the Lipschitz or Hölder continuity as special cases [2]. These properties of the operators H and H' are used in conditions (C_1)-(C_3) and (H_1)-(H_5), which are stated before the main theorems.

The following problems-limitations constitute our motivation for writing this article:

(P_1) $S_1 = S_2 = \mathbb{R}^j$.

(P_2) The operator H must be at least five times differentiable. Notice that only H' appears on the method.

(P_3) There is no advanced knowledge of how many iterations should be found to achieve pre-decided accuracy. That is, computable upper and lower bounds on the norms $\|x_{m+1} - x_m\|$ or $\xi_m = \|x_m - x^*\|$ are not available.

(P_4) A computable uniqueness domain for x^* is not determined.

(P_5) The results are only local.

(P_6) The selection of the starting guess x_0 is very challenging.

According to the problem (P_2), there exist even simple scalar equations that cannot be solved using TSHM. As an example, let S be any open interval containing the numbers 0 and 1. Define the real function:

$$f(\tau) = \begin{cases} \tau^4 \log(\tau) + 6\tau^7 - 6\tau^6, & \tau \neq 0 \\ 0, & \tau = 0 \end{cases} \tag{4}$$

Notice that $f(1) = 0$. But the function $f^{(5)}(\tau)$ is not continuous at $\tau = 0$. Thus, the result in [12] cannot be applied, although TSHM may converge.

Similar problems exist in the application of other high-convergence order methods [5, 6, 8-11, 15]. The novelty of our article is that the applicability of TSHM is extended to local as well as semi-local convergence by relying only on F and F', which are on (3), and the concept of w-continuity [2].

In particular, extensions to problems (P_1)-(P_6) are:

(F_1) The convergence takes place in a Banach space.

(F_2) Operator H is assumed to be only once differentiable. Notice also that only the first derivative of H appears on TSHM. Moreover, the w-continuity conditions are very weak and cover a wide range of problems.

(F_3) We know in advance the iterations needed to get pre-decided accuracy.

(F_4) Some computable uniqueness region is specified.

(F_5) The new results are local and semi-local.

(F_6) The radius of convergence is determined, making it possible to select initial points starting from which the convergence to x^* is assumed.

(F_7) The convergence order is recovered by the formulas: order of computational convergence (OCC) and order of approximate convergence (OAC), defined respectively by

$$\mathrm{OCC} = \log\left\|\frac{d_{i+2}}{d_{i+1}}\right\| \Big/ \log\left\|\frac{d_{i+1}}{d_i}\right\|, \quad i = 1, 2, \ldots \tag{5}$$

and

$$\mathrm{OAC} = \log\left\|\frac{\bar{d}_{j+2}}{\bar{d}_{j+1}}\right\| \Big/ \log\left\|\frac{\bar{d}_{j+1}}{\bar{d}_j}\right\|, \quad j = 1, 2, \ldots, \tag{6}$$

where $d_i = x_i - x^*$ and $\bar{d}_j = x_j - x_{j-1}$.

Notice that no high-order derivatives are needed but only the iterates and x^*. Moreover, in the case of OAC, the iterate x^* is not involved.

Let k be a fixed natural number. Then, these results also extend the application of the extended TSHM (ETSHM) given as

$$z_0(x_m) = x_m - T_m A_m H(x_m)$$

$$M_m = 2I - s_m,$$

$$z_i(x_m) = z_{i-1}(x_m) - M_m A_m H\left(z_{i-1}(x_m)\right),$$

and

$$x_{m+1} = z_i(x_m). \tag{7}$$

The notation $z_i = z_i(x_n)$ shall also be used. It was shown in [12] that the convergence order is $2k + 4$, $k = 1, 2, \ldots$. Although our approach is employed to handle TSHM and ETSHM, it also applies to other methods requiring linear operators with inverses, since it is so general and does not really depend on these two methods.

The convergence analysis for TSHM and ETSHM is developed in Section 2 and Section 3, respectively. Numerical tests and conclusions appear in Section 4 and Section 5, respectively.

2. Convergence for TSHM

In what follows, the local and semi-local convergence is studied. The former relies on some real functions and the latter on the concept of a majoring sequence [16].

2.1 *Local convergence*

It is appropriate to introduce some real functions on the interval $D = [0, \infty)$. Suppose:

(i) There exists a continuous and nondecreasing (CN) real function φ_0 on the interval D such that the equation $\varphi_0(\tau) - 1$ has the smallest zero $\delta \in D - \{0\}$. Set $D_1 = [0, \delta)$.

(ii) There exists a CN real function φ on the interval D_1, such that the equation $h_1(\tau) - 1 = 0$ has a $R_1 \in D_1 - \{0\}$, where

$$h_1(\tau) = \frac{\int_0^1 \varphi\left((1-\rho)\tau\right) d\rho + \frac{1}{3}\left(1 + \int_0^1 \varphi_0(\rho\tau) d\rho\right)}{1 - \varphi_0(\tau)}.$$

(iii) The equation

$$q(\tau) - 1 = 0$$

admits the smallest solution $\delta_1 \in D_1 - \{0\}$ where

$$q(\tau) = \frac{1}{2}\left(\varphi_0(\tau) + \varphi_0\left(h_1(\tau)\tau\right)\right).$$

Set $\delta_2 = \min\{\delta, \delta_1\}$ and $D_2 = [0, \delta_2)$.

(iv) The equation $h_2(\tau) - 1$ has the smallest solution $R_2 \in D_2 - \{0\}$, where

$$\bar{\varphi}(\tau) = \begin{cases} \varphi\left(1 + \left(h_1(\tau)\right)\tau\right) \\ \text{or} \\ \varphi_0(\tau) + \varphi_0\left(h_1(\tau)\tau\right) \end{cases}$$

$$p(\tau) = \frac{3}{2} + \frac{9}{8}\frac{\bar{\varphi}(\tau)}{1 - \varphi_0(\tau)} + \frac{1}{4}\left(\frac{\bar{\varphi}(\tau)}{1 - \varphi_0(\tau)}\right)^2,$$

and

$$h_2(\tau) = \frac{\int_0^1 \varphi((1-\rho)\tau) d\rho}{1 - \varphi_0(\tau)} + \frac{p(\tau)\left(1 + \int_0^1 \varphi_0(\rho\tau) d\rho\right)}{2(1 - q(\tau))}.$$

Set

$$R = \min\{R_m\}, \quad m = 1, 2, \ldots \tag{8}$$

The parameter R is a convergence radius for TSHM (see Theorem 2.1). Set also $D_3 = [0, R)$. The definition (8) and

D_3 give each $\tau \in D_3$

$$0 \le \varphi_0(\tau) < 1, \tag{9}$$

$$0 \le q(\tau) < 1, \tag{10}$$

$$0 \le p(\tau), \tag{11}$$

and

$$0 \le h_m(\tau) < 1. \tag{12}$$

The symbols $U[x^*, r]$, and $U[x, r]$ represent the open and closed ball in S_1, respectively, with center $x \in S_1$ and radius $r > 0$.

The functions φ_0, φ, and the radius R are connected to the operators on TSHM if $x^* \in S$ is a simple solution for the equation $H(x) = 0$ as follows.

Suppose:

(C_1) $\|H'(x^*)^{-1}(H'(u) - H'(x^*))\| \le \varphi_0(\|u - x^*\|)$ for each $u \in S$. Set $S_0 = S \cap U(x^*, \delta)$.

(C_2) $\|H'(x^*)^{-1}(H'(u_2) - H'(u_1))\| \le \varphi(\|u_2 - u_1\|)$ for each u_1, $u_2 \in S_0$.

(C_3) $U[x^*, R] \subset S$.

We are equipped to show the local convergence of TSHM.

Theorem 2.1 Under the conditions (C_1)-(C_3), further suppose that the starting point $x_0 \in U(x^*, R) - \{x^*\}$. Then, the iterates $\{x_m\}$ generated by TSHM are well defined in $U(x^*, R) - \{x^*\}$, and remain in $U(x^*, R) - \{x^*\}$ for each $m = 0, 1, 2, ...$ and $\lim_{m \to +\infty} x_m = x^*$. Moreover, the following assertions hold for each $m = 0, 1, 2, ...$

$$\left\|y_m - x^*\right\| \le h_1\left(\left\|\xi_m\right\|\right)\left\|\xi_m\right\| \le \left\|\xi_m\right\| < R \tag{13}$$

and

$$\left\|\xi_{m+1}\right\| \le h_2\left(\left\|\xi_m\right\|\right)\left\|\xi_m\right\| \le \left\|\xi_m\right\|, \tag{14}$$

where the radius R is defined by the formula (8) and the functions h_1 and h_2 are as previously given.

Proof. Let $u \in U(x^*, R) - \{x^*\}$ and $\xi_m = \|x_m - x^*\|$ be an arbitrary point. By the conditions (C_1) and (C_3), it follows:

$$\left\|H'\left(x^*\right)^{-1}\left(H'(u) - H'\left(x^*\right)\right)\right\| \le \varphi_0\left(\left\|u - x^*\right\|\right) \le \varphi_0(R) < 1. \tag{15}$$

In view of the estimate (15) and the standard lemma due to Banach on the existence of inverses for linear operators [3], we get that $H'(u)^{-1} \in \mathfrak{L}(S_2, S_1)$ with

$$\left\|H'(u)^{-1} H'\left(x^*\right)\right\| \le \frac{1}{1 - \varphi_0\left(\left\|u - x^*\right\|\right)}. \tag{16}$$

In particular, $H'(x_0)^{-1} \in \mathfrak{L}(S_2, S_1)$, since $x_0 \in U(x^*, R) - \{x^*\}$. It follows from the first subset of the method TSHM that the iterate y_0 exists, and we can write

$$y_0 - x^* = \xi_0 - H'(x_0)^{-1} H(x_0) + \frac{1}{3} H'(x_0)^{-1} H(x_0). \tag{17}$$

Using (8) and (12) (for $m = 1$), (C_1), (C_3), and (16) (for $u = x_0$), we get, on the one hand,

$$\left\| \xi_0 - H'(x_0)^{-1} H(x_0) \right\|$$

$$= \left\| \left[H'(x_0)^{-1} H(x^*) \right] \left[\int_0^1 H'(x^*)^{-1} \left(H'(x^* + \rho(\xi_0)) d\rho - H'(x_0) \right)(\xi_0) \right] \right\|$$

$$\leq \frac{\int_0^1 \varphi\left((1-\rho)\|\xi_0\|\right) d\rho}{1 - \varphi_0\left(\|\xi_0\|\right)}. \tag{18}$$

On the other hand,

$$\frac{1}{3} \left\| \left[H'(x_0)^{-1} H'(x^*) \right] \left[H'(x^*)^{-1} H(x_0) \right] \right\| \leq \frac{1}{3} \left\| H'(x_0)^{-1} H'(x^*) \right\| \left\| \left[H'(x^*)^{-1} H(x_0) \right] \right\|$$

$$\leq \frac{1}{3} \frac{\left\| \int_0^1 H'(x^*)^{-1} \left[H'(x^* + \rho(\xi_0)) d\rho - H'(x_0) + H'(x_0) \right](\xi_0) \right\|}{1 - \varphi_0\left(\|\xi_0\|\right)}$$

$$\leq \frac{\left(1 + \int_0^1 \varphi_0\left(\rho\|\xi_0\|\right) d\rho\right) \|\xi_0\|}{3\left(1 - \varphi_0\left(\|\xi_0\|\right)\right)}. \tag{19}$$

Summing up (18) and (19) in (17), we get

$$\left\| y_0 - x^* \right\| \leq h_1\left(\|\xi_0\|\right) \|\xi_0\| \leq \|\xi_0\| < R. \tag{20}$$

Therefore, the iterate $y_0 \in U(x^*, R) - \{x^*\}$ and the assertion (13) hold for $m = 0$. Next, we establish its existence of B_0^{-1}. It follows from (8), (10), (C_2), and (20) that

$$\left\| \left(2H'(x^*)\right)^{-1} \left(B_0 - 2H'(x^*)\right) \right\| \leq \frac{1}{2} \left[\left\| H'(x^*)^{-1} \left(H'(x_0) - H'(x^*)\right) \right\| + \left\| H'(x^*)^{-1} \left(H'(y_0) - H'(x^*)\right) \right\| \right]$$

$$\leq \frac{1}{2} \left(\varphi_0\left(\|\xi_0\|\right) + \varphi_0\left(\|y_0 - x^*\|\right) \right) \leq q\left(\|\xi_0\|\right) < 1.$$

Hence,

$$\left\| B_0^{-1} H(x^*) \right\| \leq \frac{1}{2\left(1 - q\left(\|\xi_0\|\right)\right)}, \tag{21}$$

the iterate x_1 is well-defined by the second substep of the TSHM, and we can write

$$x_1 - x^* = \xi_0 - H'(x_0)^{-1} H(x_0) + \left[H'(x_0)^{-1} - T_0 A_0\right] H(x_0). \tag{22}$$

The following estimates are needed

$$H'(x_0)^{-1} - T_0 A_0 = H'(x_0)^{-1}\left[I - H'(x_0) T_0 A_0\right] = H'(x_0)^{-1}\left[I - \frac{1}{2} H'(x_0) T_0 B_0^{-1}\right]$$

$$= H'(x_0)^{-1}\left[B_0 - \frac{1}{2} H'(x_0) T_0\right] B_0^{-1}$$

and

$$B_0 - \frac{1}{2} H'(x_0) T_0 = H'(x_0) + H'(y_0) - \frac{1}{2} H'(x_0)\left[I - \frac{1}{4}(s_0 - I) + \frac{1}{2}(s_0 - I)^2\right]$$

$$= H'(y_0) + \frac{1}{2} H'(x_0) + \frac{1}{8} H'(x_0)(s_0 - I) - \frac{1}{4} H'(x_0)(s_0 - I)^2$$

leading to

$$H'(x_0)^{-1}\left[B_0 - \frac{1}{2} H'(x_0) T_0\right] = s_0 + \frac{1}{2} I + \frac{1}{8}(s_0 - I) - \frac{1}{4}(s_0 - I)^2$$

$$= s_0 - I + \frac{3}{2} I + \frac{1}{8}(s_0 - I) - \frac{1}{4}(s_0 - I)^2$$

$$= \frac{3}{2} I + \frac{9}{8}(s_0 - I) - \frac{1}{4}(s_0 - I)^2. \tag{23}$$

Consequently, we obtain

$$\left\| H'(x_0)^{-1}\left[B_0 - \frac{1}{2} H'(x_0) T_0\right]\right\| \le \frac{3}{2} + \frac{9\bar{\varphi}_0}{8\left(1 - \varphi_0\left(\|\xi_0\|\right)\right)} + \frac{1}{4}\left(\frac{\bar{\varphi}_0}{1 - \varphi_0\left(\|\xi_0\|\right)}\right)^2 \le P_0. \tag{24}$$

Returning back to (22) and using (8), (12) (for $m = 2$), (16) (for $u = x_0$), and (24), by

$$\|x_1 - x^*\| \le \left[\frac{\int_0^1 \varphi\left((1-\rho)\|\xi_0\|\right) d\rho}{1 - \varphi_0\left(\|\xi_0\|\right)} + \frac{P_0\left(1 + \int_0^1 \varphi_0\left(\rho\|\xi_0\|\right) d\rho\right)}{2\left(1 - q_0\right)}\right]\|\xi_0\|$$

$$\le h_2\left(\|\xi_0\|\right)\|\xi_0\| \le \|\xi_0\|. \tag{25}$$

Hence, the iterate $x_1 \in U(x^*, R) - \{x^*\}$ and the assertion (14) holds if $m = 0$.

The induction for assertions (13) and (14) is finished by switching x_0, y_0, and x_1 with x_i, y_i, and x_{i+1} in the aforementioned computations. Then, by the estimation

$$\|\xi_{j+1}\| \leq \mu \|\xi_j\| < R, \tag{26}$$

where $\mu = h_2(\|\xi_0\|) \in [0, 1)$, we deduce that the iterates $x_{j+1} \in U(x^*, R) - \{x^*\}$ and $\lim_{j\to\infty} x_j = x^*$. □

Next, the uniqueness region is determined.

Proposition 2.2 Suppose: There exists a solution $\bar{x} \in U(x^*, \xi)$ of the equation $H(x) = 0$ for some $\xi > 0$ the condition (C_1) holds on the ball $U(x^*, \xi)$ and there exists $\xi_1 > \xi$, such that

$$\int_0^1 \varphi_0(\rho\xi_1)\,d\rho < 1. \tag{27}$$

Set $S_2 = S \cap U[x^*, \xi_1]$. Then, the only solution of the equation $H(x) = 0$ in the region S_2 is x^*.

Proof. Let $\bar{y} \in S_2$ be a solution of the equation $H(x) = 0$. Consider the operator

$$Q = \int_0^1 H'(\bar{x} + \rho(\bar{y} - \bar{x}))\,d\rho.$$

Then, from (27), that

$$\left\| H'(x^*)^{-1}\left(Q - H'(x^*)\right)\right\| \leq \int_0^1 \varphi_0(\rho\|\bar{y} - \bar{x}\|)\,d\rho \leq \int_0^1 \varphi_0(\rho\xi_1)\,d\rho < 1,$$

leading to

$$\bar{y} - \bar{x} = Q^{-1}(H(\bar{y}) - H(\bar{x})) = Q^{-1}(0) = 0,$$

thus, we deduce that $\bar{y} = \bar{x} = x^*$. □

Remark 2.3 A possible choice for $\xi = R$ provided that all the conditions of Theorem 2.1 hold.

2.2 *Semi-local convergence*

Let ψ_0 and ψ be continuous and nondecreasing real functions defined on the intervals D and D_1, respectively. Moreover, define the scaler sequences $\{a_m\}$, $\{\beta_m\}$ for $t_0 = 0$, some $s_0 \geq 0$ and each $m = 0, 1, 2, \ldots$ by

$$\bar{\psi}_m = \begin{cases} \psi(\alpha_m + \beta_m) \\ \psi_0(\alpha_m) + \psi_0(\beta_m) \end{cases}$$

$$\mu_m = \frac{1}{2}\left(\varphi_0(\alpha_m) + \varphi_0(\beta_m)\right)$$

$$t_{m+1} = \beta_m + \frac{3}{8}\left(5 + \frac{13}{4}\left(\frac{\bar{\psi}_m}{1 - \psi_0(\alpha_m)}\right) + \frac{1}{2}\left(\frac{\bar{\psi}_m}{1 - \psi_0(\alpha_m)}\right)^2\right)\frac{\left(1 + \psi_0(\alpha_m)\right)}{1 - \mu_m}(\beta_m - \alpha_m)$$

$$r_{m+1} = \left(1 + \int_0^1 \psi_0\left(\alpha_m + \rho\left(t_{m+1} - \alpha_m\right) d\rho\right)\left(t_{m+1} - \alpha_m\right) + \frac{3}{2}\left(1 + \psi_0\left(\alpha_m\right)\right)\left(s_m - \alpha_m\right)\right), \tag{28}$$

and

$$s_{m+1} = t_{m+1} + \frac{2}{3}\frac{r_{m+1}}{1 - \psi_0\left(t_{m+1}\right)}.$$

These sequences are shown to be majorizing for TSHM (see Theorem 2.6). But first, a general auxiliary convergence result for these sequences is developed.

Lemma 2.4 Assume there exists $t^{**} \in [0, \delta)$ so for each $m = 0, 1, 2, ...,$

$$\psi_0\left(\alpha_m\right) < 1, \ \mu_m < 1 \ \text{ and } \ \alpha_m < t^{**}. \tag{29}$$

Then,

$$\alpha_m \leq \beta_m \leq t_{m+1} < t^{**}. \tag{30}$$

and

$$\lim_{m \to \infty} \alpha_m = t^* \leq t^{**}. \tag{31}$$

Proof. It follows from the formula (28) and the conditions (29) that the assertion (30) holds. Then, the assertion (31) follows (30). □

Remark 2.5

(i) The limit t^* is the unique least upper bound of the sequences $\{\alpha_m\}$.

(ii) A possible choice for $t^{**} = \psi^{-1}(1)$, provided that the function ψ_0 is strictly increasing.

As in the local case, we connect the functions ψ_0, ψ, and the limit t^* to operators on TSHM, provided that there exists $x_0 \in S$, such that $H'(x_0)^{-1} \in \mathfrak{L}(S_2, S_1)$. Suppose:

(H_1) $\frac{2}{3} \| H'(x_0)^{-1} H(x_0) \| \leq s_0$.

(H_2) $\|H'(x_0)^{-1}(H'(u) - H'(x_0))\| \leq \varphi_0(\|u - x_0\|)$ for each $u \in S$. Set $S_3 = S \cap U(x_0, \delta)$.

(H_3) $\|H'(x_0)^{-1}(H'(u_2) - H'(u_1))\| \leq \varphi(\|u_2 - u_1\|)$ for each $u_1, u_2 \in S_3$.

(H_4) The conditions of the Lemma 2.4 are fulfilled and

(H_5) $U[x_0, t^*] \subset S$.

Next, the semi-local convergence of the TSHM is presented.

Theorem 2.6 Under the conditions (H_1)-(H_5) the iteration $\{x_j\}$ exists in $U(x_0, t^*)$, stays in $U(x_0, t^*)$, for each $j = 0, 1, 2,$ ... and converges to a solution $x^* \in U[x_0, t^*]$ of the equation $H(x) = 0$, such that

$$\left\|x^* - x_j\right\| \leq t^* - \alpha_j, \ \text{ for each } \ j = 0, 1, 2, \ldots, \tag{32}$$

Proof. Induction shall first establish the assertions

$$\left\|y_j - x_j\right\| \leq \beta_j - \alpha_j \tag{33}$$

and

$$\left\| x_{j+1} - y_j \right\| \le t_{j+1} - \beta_j. \tag{34}$$

Formula (28) and the condition (H_1) give

$$\left\| y_0 - x_0 \right\| = \frac{2}{3} \left\| H'(x_0)^{-1} H(x_0) \right\| \le s_0 = s_0 - t_0 < t^*$$

implying the assertion (33) for $j = 0$ as well as that the iterate $y_0 \in U(x_0, t)$. As in the local case, but by exchanging x^*, φ_0 by x_0, ψ_0, respectively, we get

$$\left\| B_j H'(x_0) \right\| \le \frac{1}{2(1 - \mu_j)}, \tag{35}$$

and x_{j+1} exists by the second substep of TSHM. Moreover,

$$x_{j+1} - y_j = \left(\frac{2}{3} H'(x_j)^{-1} - T_j A_j \right) H(x_j). \tag{36}$$

We also need the estimates

$$\frac{2}{3} H'(x_j)^{-1} - T_j A_j = H'(x_j)^{-1} \left[\frac{3}{2} I - \frac{1}{2} H'(x_j) T_j B_j^{-1} \right]$$

$$= \frac{1}{2} H'(x_j)^{-1} \left[3B_j - H'(x_j) T_j \right] B_j^{-1}$$

and

$$3B_j - H'(x_j) T_j = 3H'(x_j) + 3H'(y_j) - H'(x_j) \left[I - \frac{1}{4}(s_j - I) + \frac{1}{2}(s_j - I)^2 \right]$$

$$= 2H'(x_j) + 3H'(y_j) + \frac{1}{4} H'(x_j)(s_j - I) - \frac{1}{2} H'(x_j)(s_j - I)^2,$$

so

$$H'(x_j)^{-1} \left(3B_j - H'(x_j) T_j \right) = 2I + 3s_j + \frac{1}{4}(s_j - I) - \frac{1}{2}(s_j - I)^2$$

$$= 5I + 3(s_j - I) + \frac{1}{4}(s_j - I) - \frac{1}{2}(s_j - I)^2$$

$$= 5I + + \frac{13}{4}(s_j - I) - \frac{1}{2}(s_j - I)^2,$$

leading (36) to

$$x_{j+1}-y_j=\frac{1}{2}\left(5I+\frac{13}{4}\left(s_j-I\right)-\frac{1}{2}\left(s_j-I\right)^2\right)B_j^{-1}H\left(x_j\right). \tag{37}$$

Thus, we get

$$\left\|x_{j+1}-y_j\right\|\le\frac{1}{2}\left(5+\frac{13}{4}\left(\frac{\bar{\psi}_j}{1-\psi_0\left(\alpha_j\right)}\right)+\frac{1}{2}\left(\frac{\bar{\psi}_j}{1-\psi_0\left(\alpha_j\right)}\right)^2\right)\frac{1}{2\left(1-\xi_j\right)}\left\|-\frac{3}{2}H'\left(x_j\right)\left(y_j-x_j\right)\right\|$$

$$=\frac{3}{8}\left(5+\frac{13}{4}\left(\frac{\bar{\psi}_j}{1-\psi_0\left(\alpha_j\right)}\right)+\frac{1}{2}\left(\frac{\bar{\psi}_j}{1-\psi_0\left(\alpha_j\right)}\right)^2\right)\frac{\left(1+\psi_0\left(\alpha_j\right)\right)}{1-\mu_j}\left(\beta_j-\alpha_j\right)$$

$$=t_{j+1}-\beta_j$$

and

$$\left\|x_{j+1}-x_0\right\|\le\left\|x_{j+1}-y_j\right\|+\left\|y_j-x_0\right\|\le t_{j+1}-\beta_j+\beta_j-t_0=t_{j+1}<t^*.$$

Therefore, the assertion (31) holds and the iterate $x_{j+1}\in U(x_0, t^*)$.
By the first substep of TSHM, it follows

$$H\left(x_{j+1}\right)=H\left(x_{j+1}\right)-H\left(x_j\right)-\frac{3}{2}H'\left(x_j\right)\left(y_j-x_j\right)$$

$$=\int_0^1 H'\left(x_j+\rho\left(x_{j+1}-x_j\right)\right)d\rho\left(x_{j+1}-x_j\right)-\frac{3}{2}H'\left(x_j\right)\left(y_j-x_j\right)$$

$$=\int_0^1\left(H'\left(x_j+\rho\left(x_{j+1}-x_j\right)\right)d\rho-H'\left(x_0\right)+H'\left(x_0\right)\right)\left(x_{j+1}-x_j\right)-\frac{3}{2}\left(H'\left(x_j-H'\left(x_0\right)+H'\left(x_0\right)\right)\left(y_j-x_j\right)\right.$$

leading to

$$\left\|H'\left(x_0\right)^{-1}H\left(x_{j+1}\right)\right\|\le\left(1+\int_0^1\psi_0\left(\alpha_j+\rho\left(t_{j+1}-\alpha_j\right)\right)d\rho\right)\left(t_{j+1}-\alpha_j\right)+\frac{3}{2}\left(1+\psi_0\left(\alpha_j\right)\right)\left(\beta_j-\alpha_j\right)=\gamma_{j+1}. \tag{38}$$

Consequently, we have

$$\left\|y_{j+1}-x_{j+1}\right\|\le\frac{2}{3}\left\|H'\left(x_{j+1}\right)^{-1}H'\left(x_0\right)\right\|\left\|H'\left(x_0\right)^{-1}H'\left(x_{j+1}\right)\right\|$$

$$\leq \frac{2}{3}\frac{\gamma_{j+1}}{1-\varphi_0\left(t_{j+1}\right)} = s_{j+1} - t_{j+1}$$

and

$$\left\|y_{j+1} - x_0\right\| \leq \left\|y_{j+1} - x_{j+1}\right\| + \left\|x_{j+1} - x_0\right\|$$

$$\leq s_{j+1} - t_{j+1} + t_{j+1} - t_0 = s_{j+1} < t^*.$$

Thus, the induction for the assertions (33) and (34) is completed, and x^*, $y_j \in U[x_0, t^*]$ for each $j = 0, 1, 2, \ldots$.

Notice that the sequence $\{\alpha_j\}$ is fundamentally convergent under the condition (H_4). Therefore, the sequence $\{x_j\}$ is also fundamental in the Banach space S_1. Thus, it is convergent to some $x^* \in U[x_0, t^*]$. By letting $j \to \infty$ in the estimate (38), the continuity of the operator gives $H(x^*) = 0$. □

Next, the uniqueness of the solution region is specified.

Proposition 2.7 Suppose: There exists a solution $\overline{x} \in U(x_0, r_0)$ of the equation $H(x) = 0$ for some $r_0 > 0$; the condition (H_2) holds on the ball $U(x_0, r)$ and there exists $r_1 > r_0$, such that

$$\int_0^1 \psi_0\left((1-\rho) + \rho r_1\right) d\rho < 1. \tag{39}$$

Set $S_4 = S \cap U[x_0, r_1]$. Then, the point $\overline{x}$ is the only solution of the equation $H(x) = 0$ in the region S_4.

Proof. Let $\overline{y} \in S_4$ be a solution of the equation $H(x) = 0$. Consider the operator

$$Q = \int_0^1 H'(\overline{x} + \rho(\overline{y} - \overline{x}))d\rho.$$

Then, by (39) that

$$\left\|H'\left(x_0\right)^{-1}\left(Q - H'\left(x_0\right)\right)\right\| \leq \int_0^1 \psi_0\left((1-\rho)\left\|\overline{x} - x_0\right\| + \rho\left\|\overline{y} - x_0\right\|\right) d\rho \leq \int_0^1 \psi_0\left((1-\rho)r_0 + \rho r_1\right) d\rho < 1,$$

Thus, we conclude that $\overline{y} = \overline{x}$. □

Remark 2.8

(i) The point t^* may be switched with δ or t^{**} in the condition (H_5).

(ii) Under Theorem 2.6, we can take $\overline{x} = x^*$ and $r_0 = t^*$ in Proposition 2.7.

3. Convergence for ETSHM

As in the local convergence of TSHM, some real functions are defined. Define the functions on interval D_1 by

$$a(\tau) = 1 + \int_0^1 \varphi_0(\rho\tau)d\rho$$

$$b(\tau) = \frac{a(\tau)}{4(1-q(\tau))}\left(1 + \frac{\overline{\varphi}(\tau)}{1-\varphi_0(\tau)}\right)$$

$$c(\tau) = 1 + b(\tau)$$

and

$$d(\tau) = c(\tau)h_2(\tau).$$

Assume: the equation $d(\tau) - 1 = 0$ has the smallest solution in the interval (0, R]. Denote such a solution by R^*. The condition (C_3) is replaced by

$$(C_3)' \, U\left[x^*, R^*\right] \subset S.$$

Then, we have the local convergence analysis for ETSHM.

Theorem 3.1 Under the conditions (C_1), (C_2), and (C_3)′, the sequence $\{x_m\}$ is convergent to x^*.

Proof. As in Theorem 2.1, we have in turn the estimates

$$H\left(z_{i-1}\right) = H\left(z_{i-1}\right) - H\left(x^*\right) = \int_0^1 \left[H'\left(x^* + \rho\left(z_{i-1} - x^*\right)\right) d\rho - H'\left(x^*\right) + H\left(x^*\right)\right],$$

so

$$\left\|H'\left(x^*\right)^{-1} H\left(z_{i-1}\right)\right\| \le \left(1 + \int_0^1 \varphi_0\left(\rho\left(z_{i-1} - x^*\right)\right) d\rho\right)\left\|z_{i-1} - x^*\right\|$$

$$= a_{i-1}\left\|z_{i-1} - x^*\right\|$$

and

$$\left\|M_m A_m H\left(z_{i-1}\right)\right\| = \left\|\left[1 + \left(1 - s_m\right)\right]\right\|\frac{1}{2}\left[B_m^{-1} H'\left(x^*\right)\right]\left[H'\left(x^*\right)^{-1} H\left(z_{i-1}\right)\right]\right\|$$

$$\le \frac{1}{4}\left(1 + \frac{\bar{\varphi}_m}{1 - \varphi_0\left(\left\|x_m - x^*\right\|\right)}\right)\frac{a_{i-1}}{1 - q\left(\left\|x_m - x^*\right\|\right)}\left\|z_{i-1} - x^*\right\|$$

$$\le b\left(R^*\right)\left\|z_{i-1} - x^*\right\|.$$

By Theorem 2.1, we have

$$\left\|z_0 - x^*\right\| \le h_2\left(\left\|x_m - x^*\right\|\right)\left\|x_m - x^*\right\|.$$

Hence, by the rest of the substeps, we get in turn that

$$\left\|z_1 - x^*\right\| \le C\left(R^*\right)\left\|z_0 - x^*\right\|$$

$$\left\|z_2 - x^*\right\| \le C\left(R^*\right)\left\|z_1 - x^*\right\| \le C^2\left(R^*\right)\left\|z_0 - x^*\right\|.$$

$$\|\xi_{m+1}\| = \|z_k - x^*\| \le C^k\left(R^*\right)\|z_0 - x^*\|$$

$$\le C^k\left(R^*\right)h_2\left(\|\xi_m\|\right)\|\xi_m\| \le d\left(R^*\right)\|\xi_m\|.$$

Therefore, we get $\lim_{m\to\infty} x_m = x^*$ and $x_m \in U(x^*, R)$ for each $m = 0, 1, 2, \ldots$.
Clearly, the uniqueness domain for the solution x^* is already established in Proposition 2.2.

4. Experiments

We take into account some numerical examples to estimate the real parameters in order to validate the theoretical deductions. The parameters R_1 and R_2 are obtained as solutions of the nonlinear equations $h_1(t) - 1 = 0$, $h_2(t) - 1 = 0$, defined earlier, respectively, for specialized functions φ_0 and φ. Then, the radius R is given by the formula (8). Numerical computations are performed in Mathematica software, with multi-precision arithmetic that uses a floating-point representation of 50 decimal digits of mantissa.

Example 1 Let us consider the example given in the introduction (see (4)). Note that $t^* = 1$ is the zero of this function. Then, pick $\varphi_0(\tau) = L\tau$ and $\varphi(\tau) = L\tau$ where $L = 146.66290$. So, we obtain the radii

$$R_1 = 0.00272734,\ R_2 = 0.000659923,\ R = 0.000659923.$$

Example 2 Consider (see [17]) the function, $H := (f_1, f_2, f_3)^T : D \to \mathbb{R}^3$ defined by

$$H(l) = \left(10l_1 + \sin\left(l_1 + l_2\right) - 1, 8l_2 - \cos^2\left(l_3 - l_2\right) - 1, 12l_3 + \sin\left(l_3\right) - 1\right)^T,$$

where $l = (l_1, l_2, l_3)^T$.

Fréchet-derivative of $H(l)$ is given by

$$H'(l) = \begin{bmatrix} 10 + \cos\left(l_1 + l_2\right) & \cos\left(l_1 + l_2\right) & 0 \\ 0 & 8 + \sin 2\left(l_2 - l_3\right) & -\sin 2\left(l_2 - l_3\right) \\ 0 & 0 & 12 + \cos\left(l_3\right) \end{bmatrix}.$$

Set $\varphi_0(\tau) = \varphi(\tau) = 0.269812\tau$. So, we obtain

$$R_1 = 1.48251,\ R_2 = 0.358717,\ R = 0.358717.$$

Example 3 We study Kepler's equation [18]

$$H(l) = l - \beta\sin(l) - K = 0,$$

where $\beta \in [0, 1)$, and $K \in [0, \pi]$. In [18], β and K are studied for many values. We fix $K = 0.1$ and $\beta = 0.27$. So, $l^* \approx 0.13682853547099\ldots$ and

$$H'(l) = 1 - \beta\cos(l)$$

Thus,

$$\left|H'(\alpha)^{-1}\left(H'(l)-H'(q)\right)\right|=\frac{\left|\beta\left(\cos(l)-\cos(q)\right)\right|}{|1-\beta\cos(\alpha)|}$$

$$=\frac{2\beta\left|\sin\left(\frac{l+q}{2}\right)\sin\left(\frac{l-q}{2}\right)\right|}{|1-\beta\cos(\alpha)|}$$

$$\leq\frac{\beta}{|1-\beta\cos(\alpha)|}|l-q|,$$

$$\left|H'(\alpha)^{-1}H'(l)\right|=\frac{|1-\beta\cos(l)|}{|1-\beta\cos(\alpha)|}\leq\frac{1+\beta}{|1-\beta\cos(\alpha)|}.$$

So, we can set $\varphi_0(\tau)=\varphi(\tau)=0.3685888\tau$. Then, we obtain

$$R_1=1.08522,\ R_2=0.262586,\ R=0.262586.$$

Example 4 See [17]. Let, $C[0,1]=Y=X$ and $D=\bar{U}(0,1)$. Let the operator H on D be

$$H(\varphi)(l)=\varphi(l)-10\int_0^1 l\rho\varphi(\rho)^3\,d\rho.$$

Thus, it follows

$$H'(\varphi(\xi))(l)=\xi(l)-30\int_0^1 l\rho\varphi(\rho)^2\xi(\rho)d\rho,$$

for each $\xi\in D$.

But $l^*=0$, so we can set $\varphi_0(\tau)=15\tau$, $\varphi(\tau)=30\tau$. Then, we obtain

$$R_1=0.0205128,\ R_2=0.00420457,\ R=0.00420457.$$

Example 5 Introducing the integral equation (see [3]),

$$l(d)=\int_0^1 T(d,\omega)\left(l(\omega)^{3/2}+\frac{l(\omega)^2}{2}\right)d\omega,$$

$$T(d,\omega)=\begin{cases}(1-d)\omega, & \omega\leq d,\\ d(1-\omega), & d\leq\omega.\end{cases}\tag{40}$$

But $\alpha(d)=0$. Define $H:D\subseteq[0,1]\to C[0,1]$ is as

$$H(l)(d)=l(d)-\int_0^1 T(d,\omega)\left(l(\omega)^{3/2}+\frac{l(\omega)^2}{2}\right)d\omega.$$

Notice that

$$\left\| \int_0^1 T(d,\omega)d\omega \right\| \leq \frac{1}{8}.$$

Thus, we have

$$H'(l)q(d) = q(d) - \int_0^1 T(d,\omega)\left(\frac{3}{2}l(\omega)^{1/2} + l(\omega)\right)d\omega,$$

since $H'(\alpha(d)) = 1$, it follows that

$$\left\| H'(\alpha)^{-1}\left(H'(l) - H'(q)\right)\right\| \leq \frac{5}{16}\|l - q\|. \tag{41}$$

In (41), switch q by l_0

$$\left\| H'(\alpha)^{-1}\left(H'(l) - H'\left(l_0\right)\right)\right\| \leq \frac{5}{16}\|l - l_0\|.$$

Thus, we take

$$\varphi_0(l) = \varphi(l) = L_0 l, \text{ where } L_0 = \frac{5}{16}.$$

Hence, we obtain

$$R_1 = 1.28, R_2 = 0.309716, R = 0.309716.$$

5. Conclusion

This paper thoroughly discusses the convergence (local and semi-local) of a fourth-order technique in Banach space. We have simply taken into account the first derivative in our process, unlike other techniques that rely on higher derivatives and the Taylor series. In this way, the method's uses are broadened because it may be used for a wider class of functions. The production of an error estimate and convergence ball, within which the iterates lie, is another benefit of convergence analysis. The theoretical outcomes of this approach are verified through numerical testing on a few problems. The idea is applicable to other methods cited in the introduction and can be extended to handle more advanced equations [19].

Conflict of interest

The authors declare no conflict of interest.

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