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Research Article

Asymptotic Probability Expansions for Random Elements in a Hilbert Space

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Abstract: In this article, we approach a class of problems in probability theory, namely, the asymptotic expansion of probability. We consider an independent, identically distributed, and normalized stochastic process $(X_k)_{k\in\mathbb{N}}$ in a separable Hilbert space H, and associate it with the normalized partial sum

$$S_n = n^{-1/2}\sum_{i=1}^{n} X_i .$$

As a result, we built on the ball with a fixed center asymptotic expansion of non-uniform probabilities; our conditions on the moments are minimal, and the dependency of estimates on the covariance operator is expressed with the terms of the eigenvalue series. Likewise, the covariance operators of the random elements do not coincide. In the open ball set with fixed center a and radius $r, (B_r(a) = \{x \in H : \| x - a \| < r\}$, we estimate the optimal result of the Berry-Esseen type of the remainder, and the terms of the probability $P(\| S_n - a \| < r)$ by the Fourier method.

***Keywords*:** Berry-Esseen, covariance operator, Fourier method, random elements

MSC: 60G50, 34L10, 65R10, 60F17

1. Introduction

Convergence acceleration methods for sequences have been studied for several years. In many fields, sequences are used as tools to solve problems. Some of these sequences converge; others do not. Among those that converge, you will find some with a low convergence rate. Therefore, one area of mathematics was concerned with the theory of the transformation of sequences. However, for a long time, the acceleration of convergence was only illustrated for numerical sequences with methods, such as those of Richardson or Aitken.

Nowadays, not being able to ignore the randomness of dynamic systems, this mathematical research field is now oriented towards stochastic processes and provides answers to many problems. Indeed, with convergence acceleration methods, not only are approximation problems (Riemann sum, integral computation, differential equation) theoretically solved, but also many algorithms can be constructed for the simulation of many phenomena in economics, finance, etc.

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For instance, Devineau and Loise [1] have developed an acceleration algorithm of the method called "simulations dans les simulations" for the computation of solvency economic capital.

Concerning the sequences of random variables, or more generally, stochastic processes, several methods have been proposed and continue to appear in the literature. This is because a universal transformation of sequences to accelerate the convergence of all convergent sequences cannot exist, according to Delahaye and Germain Bonne [2].

However, a fundamental and effective technique for convergence acceleration methods of stochastic processes still remains asymptotic expansions of probability. They provide good estimates in terms of approximation.

These expansions were first examined, without exact foundation, by Tchebycheff [3] for the case of the classical limit theorem. Later, the expansions of the Tchebycheff type were studied by Bruns [4] and Edgeworth [5]. However, the most comprehensive results in this direction were obtained much later by Cramer [6] and Esseen [7].

In finite dimension, for example, results are obtained in Euclidean space by Bikyalis [8]. These results have shown that, in $\mathbb{R}$, if the random variables $\xi_1, \cdots, \xi_k$ are independent and identically distributed and admit an absolute moment of order 3 (meaning that $\mathbb{E}\left[|\xi_i|^3\right] < \infty$), then for any $x \in \mathbb{R}$,

$$F_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du + \frac{e^{-\frac{x^n}{2}}}{\sqrt{2\pi}} \times \frac{(1-x^2)\alpha_3}{6\sigma^3\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right), \tag{1}$$

where σ^2 is the variance of the random variable ξ_1, α_3 the third absolute moment of ξ_1, and F_n is the cumulative distribution function of the sum of the normalized and centered random variables $\xi_1, \cdots, \xi_n$.

If, however, the variables $\xi_1, \cdots, \xi_n$ are in $\mathbb{R}^k$ and admit a non-degenerate covariance matrix Λ, then for all x in $\mathbb{R}^k$, we have:

$$F_n(x) = \varphi(x) + \frac{1}{\sqrt{n}} P_1(-\varphi)(x) + o\left(\frac{1}{\sqrt{n}}\right), \tag{2}$$

where $P_1(-\varphi)(X) = \int_{\{y_1 < x_1\}} \int_{\mathbb{R}^k} \frac{i^3 M \langle t, \xi_1 \rangle^8}{(6-2\pi)^k} e^{-i\langle t, y\rangle - \frac{1}{2} t\Lambda t'} dtdy$; and $t = (t_1, \cdots, t_k), y = (y_1, \cdots, y_k)$ for some vectors of the k-dimensional Euclidean space with norm $\|t\|, \|y\|$, respectively.

In infinite dimension, under the different results in Hilbert space, the construction of the asymptotic probability expansion takes several forms depending on the authors. First, with Bentkus [9], we find a form that is more dependent on the moments in the rest of the expansion. In [10], under condition $\mathbb{E}\left(\| X \|^{2(p+1)}\right) < \infty$, and some other conditions on the covariance operator of the random element X, whose description we omit here, the following result is obtained

$$P\left(\| S_n \| < r\right) = P\left(\| Y \| < r\right) + \sum_{k=1}^{p-2} C_k n^{-k/2} + o\left(n^{-(p-1)/2}\right), \; p > 2, \tag{3}$$

where $C_k = C_k(r)$ are certain functions with a known Fourier transform.

For several years, for more than a decade, a variety of results have emerged in the literature under different conditions on the distribution of random elements, of their moments, and of their covariance operators. The first results of the construction of asymptotic expansions on the metric of the ball are described in Bentkus's papers [9]. Later in [11], his results were extended into Banach spaces. This last article has recently been at the center of studies in a Hilbert space to determine the speed of convergences of stochastic processes (see [12]) and, also, for asymptotic developments for symmetric statistics with degenerate kernels (see [13]).

Motivated by the above work, we will exploit the properties of the covariance operator and moment to improve the previous results by giving a new version of the asymptotic expansion of probabilities on the ball with a fixed center in Hilbert spaces.

Thus, in this article, we pursue two goals. On the one hand, we give an optimal result of Berry-Esseen type in an asymptotic expansion of probability under the hypotheses of weak moments and different covariance operators for two random elements in H. On the other hand, we estimate the terms of this expansion.

The sequel to the paper is the following: in Section 2, we set the mathematical problem of the asymptotic expansion of probability and formulate the various notations and assumptions in the study of the problem. In Section 3, we give our main results. The proofs of our results are discussed in Section 4. We end up with a conclusion.

2. Preliminaries

2.1 *Mathematical model of the problem*

In the theory of the asymptotic expansion of probability for the normalized sum $S_n = n^{-1/2}\sum_{i=1}^{n} X_i$ of independent and identically distributed random elements $(X_i)_{i\in\mathbb{N}}$ in the separable Hilbert space H, the mathematical problem consists in estimating the terms of the following expression:

$$P\left(\| S_n - x \| < r\right), \tag{4}$$

$x \in H, r > 0$.

2.2 *Notations and assumptions*

Before adopting the notations, which will follow in the article, we give these two definitions:

Definition 2.1. Let E be a vector space and $\langle .,. \rangle$ a scalar product on E. The norm associated with $\langle .,. \rangle$ is defined by:

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad \forall x \in E.$$

Definition 2.2. Let $n, m \in \mathbb{N}^*, \left(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\right)$ and $\left(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m)\right)$ be two measurable spaces. The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be a Borel function if it is measurable from $\left(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\right)$ to $\left(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m)\right)$, i.e., with regard to the Borel algebra.

For the construction of the model 4, we use the following notations:

Let H be a separable Hilbert space with norm $\|.\|$. We denote by $\langle . \rangle$, the scalar product in H. The open ball in H centered at a $(a \in H, r > 0)$ is defined as follows:

$$B_r(a) = \left\{x \in H : \| x - a \| < r\right\}. \tag{5}$$

Let $(X_i)_{i\in\mathbb{N}}$ be independent and identically distributed random element in H. The mean and variance of X_1 are denoted by $\mathbb{E}(X_1)$ and σ^2, respectively, and are defined by:

$$\mathbb{E}(X_1) = \mathbb{E}(\| X_1 \|), \;\; \sigma^2 = \mathbb{E}(\| X_1 - \mathbb{E}(X_1) \|^2). \tag{6}$$

Denote by Λ^X the covariance operator of the random element X_1:

$$\langle \Lambda x, y\rangle = \mathbb{E}\left(\langle X_1 - \mathbb{E}(X_1), x\rangle\right)\left(\langle X_1 - \mathbb{E}(X_1), y\rangle\right). \tag{7}$$

We set $\lambda_1^X, \lambda_2^X, \cdots$ (respectively, $e_1, e_2, \cdots$) the eigenvalues (respectively, the eigenvectors) of Λ^X. We denote by c (respectively, $c()$) generic constants (respectively, constants depending on parameters). Considering the results in [14], we notice that $c(\Lambda^X)$ depends only on $\sigma^2\lambda_i^X, i = 1, \cdots, 13$. So, using the result of Nagaev [15], we set:

$$c(\Lambda^X) = c\left(\sigma^6\left(\prod_{i=1}^{7}\lambda_i^{1/2}\right)^{-\frac{6}{7}} + \sigma^4\left(\lambda_1\lambda_2\lambda_7^2\right)^{-1}\right). \tag{8}$$

For any non-zero integer k, let A_0 be the linear operator defined by:

$$A_0 : H \to Hx \to A_0(y) = \sum_{i=1}^{6k-5} \langle y, e_i \rangle \, e_i.$$

Let $Y_1, \cdots, Y_n$ be another sequence of random Gaussian elements of zero mean and with covariance operator Λ^Y. Let's denote by $\lambda_1^Y > \lambda_2^Y > \cdots > \lambda_{13}^Y$ the eigenvalues of Λ^Y.
For all $a, h \in H, r > 0, i = 0, 1, \cdots n$ let's put:

$$\Phi_m(a, r) = P\left(\left\| \left(1 - \frac{m}{n}\right)^{\frac{1}{2}} Y - a \right\| < r \right) \tag{9}$$

If $\Phi_m(a, r)$ is differentiable (assumption that we will assume in the following), we define the operators d_h^k by:

$$d_h^1 \Phi_m(a, r) = d_h \Phi_m(a, r), \; d_h^k \Phi_m(a, r) = d_h\left(d_h^{k-1} \Phi_m(a, r)\right), k \geq 2.$$

The indicator function of the set $\{\| X_j \| < L\}$ is denoted by $\mathbf{1}'_{j,L}$, and we set $\mathbf{1}_{j,t} = \mathbf{1}'_{j,(1+t)n^{1/2}}$ and $\mathbf{1}_j = \mathbf{1}_{j,0}$. We note $\ell^{(s)} = (\ell_1, \cdots, \ell_s)$ the positive integer vectors and we pose

$$Q_s(\ell^{(s)}) = \left(d_{X_1 \mathbf{1}_1}^{\ell_1} - d_{Y_1}^{\ell_1}\right) \cdots \left(d_{X_s \mathbf{1}_s}^{\ell_s} - d_{Y_s}^{\ell_s}\right). \tag{10}$$

For any integer $k \geq 2$, $m \leq k - 2$, we write:

$$A_m(a, r) = n^{-m/2} \sum_{j=1}^{m} \sum n^{-j} C_n^j (\ell^{(j)}!)^{-1} \mathbb{E}(Q_j) \Phi_j(a, r); \tag{11}$$

$$\triangle_n(a, r) = \| P(\| S_n - a \| < r) - P(\| Y - a \| < r) - \sum_{m=1}^{k-2} A_m(a, r) \|; \tag{12}$$

where: $\ell_1 \geq 3, \cdots, \ell_j \geq 3$ and $\ell_1 + \cdots + \ell_j = 2j + m$. $\ell^{(j)}! = \ell_1! \cdots \ell_j!$. $\sum'$ the sum of $\ell^{(j)}$. Let's put $s = \left| \| a \| - r \right|$ and α is a real, such that $\alpha \geq 1/5$. Let u be the integer part of s^α. Our main results will be based on the following hypotheses (Hs):

H1: In order to guarantee the existence of the moments of the random element X_1 in the terms of the asymptotic probability expansion of our mathematical model, we assume the following minimal moment:

$$\mathbb{E}\left[\| X_1 \|^2\right] < \infty$$

and for all $p \geq 0, t \geq 0, k \in \mathbb{N} \setminus \{0; 1\}, L > 0$,

$$\mathbb{E}\left(\| X_1 \|^2\right)\left(1 - \mathbf{1}_{1,L}\right) \leq \frac{\lambda_{6k-5}}{3} < \infty. \tag{13}$$

H2: The eigenvalues of the operators Λ^X and Λ^Y have decreasing order and are strictly positive:

$$\lambda_1^X \geq \lambda_2^X \geq \cdots; \quad \min_{1 \leq i \leq 13} \lambda_i^X > 0; \tag{14}$$

$$\lambda_1^Y \geq \lambda_2^Y \geq \cdots; \quad \min_{1 \leq i \leq 13} \lambda_i^Y > 0. \tag{15}$$

H3: We assume that $\Phi_m(a, r)$ is differentiable with regards to a. This means that:

$$d\Phi_m(a,r) = \lim_{t\to 0} \frac{\Phi_m(a-th,r)-\Phi_m(a,r)}{t}. \tag{16}$$

H4: We also assume that the random element X_1 satisfies the Cramer condition on A_0. That is, A_0 is non-negative and that for all $L > 0$, $\rho < 1$,

$$\lim_{L\to\infty} \sup_{\langle A_0 x, x\rangle \geq L} \{\| \mathbb{E}\left(e^{i\langle x, X_1\rangle}\right) \|\} = \rho. \tag{17}$$

Remark 1. If the operator A_0 is the identity function, then we get a "generalization" of the classical Cramer condition

$$\lim_{L\to\infty} \sup_{\|x\|\geq L^{-1}} \{\| \mathbb{E}\left(e^{i\langle x, X_1\rangle}\right) \|\} = \rho. \tag{18}$$

However, in the case of an infinite dimensional space H, (18) no longer makes sense since for each $r \geq 0$ and for all $X_1 \in H$, we always have $\sup_{\|x\|=r} \{\| \mathbb{E}\left(e^{i\langle x, X_1\rangle}\right)\|\} = 1$. This results from the weak continuity of the characteristic function of the random elements X_i and from the fact that the unit sphere closure contains point O.

H5: Assume that $A_0 \in \sigma(\beta, s)$. That is, the operator A_0 has no less than s eigenvalues (counting their multiplicity) exceeding the number $\beta > 0$.

H6: We assume, thanks to the following inequalities

$$\| \mathbb{E}\left(\exp(it \| Y \|^2)\right) \| \leq \left(1 + 4t^2 (\lambda_k^Y)^4\right)^{-1/4}; \tag{19}$$

$$\| \mathbb{E}\left(\exp(it\langle Y_1, Y_2\rangle)\right) \| \leq \left(1 + t^2 (\lambda_k^Y)^4\right)^{-1/2}; \tag{20}$$

$$\lim_{|t|\to\infty} \| \mathbb{E}\left(\exp(itY_1)\right) \| < 1; \tag{21}$$

(Inequality (21) is called classical Cramer condition has been studied in [11]) that:

$$\rho \equiv \left(1 + 4t^2 \left(\lambda_1^Y\right)^2\right)^{-\frac{1}{4}}. \tag{22}$$

In the following section, we state the main results that we obtained.

3. Main results

As announced in the objectives of this article, we have the following result of Berry-Esseen type for the model.

Theorem 3.1. Under the assumptions, **H1**-**H4** and **H6**, for any $L \leq n^{1/2}$, we have:

$$\begin{aligned}
\vartriangle_n(a,r) \leq\ & A(p,s,t) + c(k)\frac{c(u)}{(1+s)^u}\Big[c(\Lambda)\left(1+\vartheta^3(a,r)\|a\|\langle\Lambda x, y\rangle\right)^{k-2}\left(\frac{L^2}{n}\right)^{\frac{k-1}{2}} \\
& + c(\Lambda)\left(\mathbb{E}\left[B_2(a,r))(1-\mathbf{1}_1)\right] + (1+\vartheta^3(a,r)\langle\Lambda x, y\rangle^{3/2} n^{-1/2})\mathbb{E}\left[B_{k+1}(a,r)\mathbf{1}_1\right]\right) \\
& + \ln\left(\frac{n}{L^2}\right)\left((1+4t^2(\lambda_1^Y)^2)^{-\frac{1}{4}}\right)^{\frac{n}{k\ln(n/L^2)}}\Big],
\end{aligned}$$

where:

$$A(p,s,t)=c(p)(1-s)^{-p}n^{1-\frac{p}{2}}\mathbb{E}(\| X_1\|^p)(\mathbf{1}_{1,t}-\mathbf{1}_1)+n\mathbb{E}(1-\mathbf{1}_{1,t});$$
$$B_j(a,r)=n^{-\frac{j-2}{2}}\left(\| X_1\|^j+\vartheta^j(a,r)\|\langle X_1,a\rangle\|^j\right);$$
$$\vartheta(a,r)=\min\{1,r/\|a\|\};\vartheta(0,r)=0$$

For the estimation of the terms of the model asymptotic development, we have the following result:

Theorem 3.2. Under the assumptions of Theorem 3.1, the terms of the asymptotic expansion satisfy the following estimates:

- For all $\varepsilon>0$, if m is even,

$$\| A_m(a,r)\|\le c(\varepsilon,m)\left(\frac{c(u)}{(1+s)^u}\right)^{\frac{2}{\alpha(2+\varepsilon)^{\alpha/2}}}n^{-m/2}c(\Lambda)\times\Big[\mathbb{E}(\mathbf{1}_1\| X_1\|^{m+2}$$
$$+\|\langle X_1,a\rangle\|^{m+2}\mathbf{1}_1\vartheta^{m+2}(a,r)(1+\vartheta^{2(m-1)}(a,r)\langle\Lambda a,a\rangle^{m-1}))+\left(\vartheta^2(a,r)\langle\Lambda a,a\rangle\right)^{3m/2}\Big].$$

- For all $\varepsilon>0$, if m is odd,

$$\| A_m(a,r)\|\le c(\varepsilon,m)\left(\frac{c(u)}{(1+s)^u}\right)^{\frac{2}{\alpha(2+\varepsilon)^{\alpha/2}}}n^{-m/2}\times\left[c(\Lambda)\left(1+\left(m^2(a,r)\langle\Lambda a,a\rangle\right)^{m-1}\right)\right.$$
$$\times\mathbb{E}\left(\|\langle X_1,a\rangle\|\mathbf{1}_1\right)m(a,r)\left(\|X_1\|^2+\|X_1\|^{m+1}+\|\langle X_1,a\rangle\|^{m+1\vartheta m+1}(a,r)\right)$$
$$\left.+c(\Lambda)\vartheta(a,r)\langle\Lambda a,a\rangle^{1/2}\mathbb{E}\left(\mathbf{1}_1\|X_1\|^{m+2}\right)\right]$$

we give following this theorem the following corollary.

Corollary 3.3. Assume the assumptions of Theorem 3.1. For all $0<L<n^{1/2}$,

$$\Delta_n(a,r)\le A(p,s,t)+c(k)\exp\{-s^{1/5}\}\Bigg[\left(1+\min\left\{\|a\|^{3k-3},r^{3k-3}\right\}\right)$$
$$\times\left(c(\Lambda)\mathbb{E}\left(\|X_1\|^2\right)(1-\mathbf{1}_1)+n^{-(K-1)/2}\mathbb{E}\|X_1\|^{K+1}\mathbf{1}_1\right)$$
$$+c(\Lambda)\left(\frac{L^2}{n}\right)^{\frac{k-1}{2}}+\ln\left(\frac{n}{L^2}\right)\left(\left(1+4t^2\left(\lambda_1^Y\right)^2\right)^{-\frac{1}{4}}\right)^{\frac{n}{k\ln(n/L^2)}}\Bigg].$$

Remark 2. Our results are similar to those in [16], but their particularity is that they give precision on the constants depending on the parameters. Also, contrary to previous works on the subject, we considered random elements having different covariance operators because assuming that two random elements of the same space always have the same covariance operator (see [17] and [11]) does not always seem relevant to us. Also, since any function is not systematically differentiable, in [16], notation $d\Phi_m(a,r)$ as the differential function of $\Phi_m(a,r)$ rather than a hypothesis is not right because it still does not exist.

4. Proofs of theorems

The proofs of our theorems are based on a packaging argument, similar to the spirit of the approach in [9]. We prove the theorems from Esseen inequality and the Fourier-Stieltjes transform. First, we collect and examine, under Section 4.1, some required results from the literature. Then, in Section 4.2, we state and prove some auxiliary results. Finally, we establish the proof of our main results in Section 4.3.

4.1 *Some required results of literature*

The lemma we use below in our Esseen-inequality type estimates, in the proof of the Theorem 3.1, is an adapted version and appears in some variants of literature (see, for example, [11], p.58).

Lemma 4.1. Let $\overline{S}_n = n^{1/2}\sum_{j=1}^{n} X_j \mathbf{1}_j$. For any $k \geq 1$, let's pose:

$$\alpha = \frac{12k-11}{6(6k-5)},\ \ell_n = \frac{L}{n^{1/2}},\ T = \ell_n^{-(1+\alpha)}\left(\ln \ell_n^2\right)^{-1/2}. \tag{23}$$

For any $p \geq 1$ and $L \leq n^{1/2}$, we have:

$$I \equiv \int_T^{\ell_n^{-p}} \frac{1}{|t|} \| \mathbb{E}\left(\exp\left\{it \| \overline{S}_n - a \|^2\right\}\right) \| \, dt \leq c(k,p)\left[\rho^{n/4}(L) \ln \ell_n^{-1} + \left(\prod_{i=1}^{6k-5} (\lambda_i^X)^{-1/2}\right)^{1/2} \ell_n^{k-1}\right].$$

The next lemma, allows us to reformulate, in another way, through Lemma 4.9 of our auxiliary results, the inequality given by Bentkus (see Theorem 3.1 in [11]).

Lemma 4.2. Let $u > 0$. Let's put $\overline{X}_j = X_{j\mathbf{1}}$ and $X'_j = X_{j\mathbf{1}'_{j,u}}$. Let A be a Borel subset of H, f be a real Borel function on $[0; \infty]$. Set $\overline{S}_n = n^{-1/2}(\overline{X}_1 + \cdots \overline{X}_n)$ and $S'_n = n^{-1/2}(X'_1 + \cdots X'_n)$.

We have

$$\triangle' = \| P\left(S_n \in A\right) - P\left(S'_n \in A\right) \| \leq n\mathbb{E}\left(1' - \mathbf{1}'_{1,u}\right)$$

and for all $q \geq 0$ and $a \in H$,

$$\overline{\triangle} = \| \mathbb{E}\left(f(\| S_n - a \|)\right) - \mathbb{E}(f(\| \overline{S}_n - a \|)) \| \leq c(q)(1 + \| a \|^q) B_{f,q} N_q,$$

where $B_{f,q} = \sup_{x \geq 0} \|f(x)\| \left(1 + x^q\right)^{-1}$; $N_q = n\mathbb{E}\left(\|X_1\|^q\right)\mathbb{E}(1 - \mathbf{1}_1) + n^{1-\frac{q}{2}}\mathbb{E}\left(\|X_1\|^q\right)(1 - \mathbf{1}_1)$.

The next lemma appears in [12] and serves as a result in the proof of our auxiliary results (see proof Lemma 4.4).

Lemma 4.3. For all random elements X and Y fulfilling our assumptions, we have:

$$\| \mathbb{E}\left[\exp(it \|X + Y\|_H^2)\right] \| \leq \prod_{k=1}^{\infty} (1 + 4t^2 (\lambda_k^Y)^2)^{-1/4}. \tag{24}$$

4.2 *Auxiliary results*

We state and prove the following auxiliary results:

Lemma 4.4. Inequalities (19) and (20) are true.

Proof of Lemma 4.4. Let us first show that the characteristic function ψ_{Y^2} of the random element Y^2 is:

$$\psi_{Y^2}(y) = \mathbb{E}\left(\exp(itY^2)\right) = \frac{1}{\sqrt{1-2it}}. \tag{25}$$

Let F_{Y^2} and f_{Y^2} respectively, be the cumulative distribution function and the density function of Y^2. We have:

$$\begin{aligned} F_{Y^2} &= P(Y^2 \leq y) = P(-\sqrt{y} \leq Y \leq \sqrt{y}) \\ &= F_Y(\sqrt{y}) - F_Y(-\sqrt{y}). \end{aligned}$$

Thus,

$$
\begin{aligned}
f_{Y^2}(y) &= \frac{1}{2\sqrt{y}} F'_X\left(\sqrt{y}\right) + \frac{1}{2\sqrt{y}} F'_X\left(-\sqrt{y}\right) \\
&= \frac{1}{2\sqrt{y}}\left(f_Y\left(\sqrt{y}\right) + f_Y\left(-\sqrt{y}\right)\right) \\
&= \frac{1}{2\sqrt{y}}\left(\frac{1}{\sqrt{2\pi}}\exp(-1/2)y + \frac{1}{\sqrt{2\pi}}\exp(-1/2)\right) \\
&= \frac{1}{(2\pi x)^{1/2}}\exp\left(-(-1/2)y\right) \\
&= \frac{(1/2)^{1/2}}{\Gamma(1/2)} y^{(1/2)-1}\exp\left(-(-1/2)y\right), \text{ with } \Gamma(k) = \int_0^{+\infty} u^{k-1}\exp(-u)\,du.
\end{aligned}
$$

So,

$$
Y^2 \rightsquigarrow \Gamma(1/2;1/2). \tag{26}
$$

It is known that for any standard random element Z following the gamma distribution of parameters $(k;\theta)\left(Z \rightsquigarrow \Gamma(k;\theta)\right)$, its moment-generating function φ_Z is:

$$
\varphi_Z(t) = \mathbb{E}(\exp(tz)) = \left(\frac{\theta}{\theta - t}\right)^k. \tag{27}
$$

From relations (26) and (27), we can deduce (25).

From Lemma 4.3, we have:

$$
\mathbb{E}\left(\exp(it\,\|Y\|^2)\right) = \prod_{j=1}^{\infty}\left(1 - 2it\lambda_j^Y\right)^{-1/2}
$$

and this clearly implies (19).

Since

$$
\mathbb{E}\left(\exp(it\langle x, Y\rangle)\right) = \exp\left(-\langle \Lambda^Y x, x\rangle / 2\right),
$$

we have

$$
\mathbb{E}\left(\exp(it\langle Y_1, Y_2\rangle)\right) = \mathbb{E}\left(-t^2\langle \Lambda^Y Y, Y\rangle / 2\right) \le \left(\exp(-t^2(\lambda_k^Y)^2 Y^2)\right)^k.
$$

This implies (20).

Lemma 4.5. Let $h_1, \cdots, h_m$ be some elements in H. The function

$$
\begin{aligned}
\varphi &: \mathbb{R} \to \mathbb{C} \\
&\quad t \mapsto \exp\{itw(x)\},
\end{aligned}
$$

were $w: x \mapsto \|x - a\|^2$, is infinitely Frèchet differentiable and we have:

$$
d_{h_1}\cdots d_{h_m}\varphi(x) = \varphi(x)\sum (it)^q d_{A_1}w(x)\cdots d_{A_q}w(x), \tag{28}
$$

where the sum is taken into account over all possible decompositions $A_1 \cup \cdots \cup A_q = \{h_1, \cdots, h_m\}$ into non-empty subsets $A_1, \cdots, A_q$ of cardinality lower or equal than two, and $d_A = d_{g_1}\cdots d_{g_k}$ for $A = \{g_1, \cdots, g_k\}$.

In addition

$$\| d_{h_1}\cdots d_{h_m}\varphi(x)\| \le c(1+\|x\|^m)\|h_1\|\cdots\|h_1\|. \tag{29}$$

Proof of Lemma 4.5. The function $\varphi(x)$ is indefinitely Frèchet differentiable (see Lemma 4.4 in [18]). Since φ is differentiable, then for any fixed *h*, its derivative $d_h\varphi(A)$ is a set function. By applying an induction to *m* (see Lemma 4.3 in [18]), we can define the derivatives marked $d_{h_1}\varphi(A)\cdots d_{h_m}\varphi(A)$. Noting that the third-order derivative of *w* is zero, we can conclude by writing (28). The estimate of (29) is obtained using the inverse Fourier-Stieltjes transform and the proof of Theorem 4.1 in [9].

The following lemma is an adapted version of Lemma 14 in [17].

Lemma 4.6. Let *Y* be a Gaussian random element with value in *H*, with mean zero with a covariance operator $\Lambda^Y; \lambda_1^Y \ge \lambda_2^Y \ge \cdots$ the eigenvalues of Λ^Y. Let *b* in *H*, such that $\|b\| \le \alpha s_1$ with $s_1 = \,|u^{1/2} - \|a\||,\ u \ge 0,\ a \in H,\ \alpha \in [0;1]$. Then, for all $\varepsilon > 0$, $m \in \mathbb{N}$, $\theta \in [0;1]$, *k* even, integers $k', k_1, \cdots k_{k'}$ and $z_1, \cdots z_{k'} \in H$, we have:

$$\left\| \int_{-\infty}^{\infty} \exp\left\{-it\left(u-(1-\theta^2)\|a\|^2\right)\right\} t^m \times \mathbb{E}\left(\exp\left\{it\|Y-a\theta\|^2\right\}\right)\|Y\|^k \prod_{j=1}^{k'}\langle Y, z_j\rangle^{k_j} dt \right\|$$
$$\le c(\bar{k}, m, \varepsilon)c(\Lambda^Y)\prod_{j=1}^{k'}\langle \Lambda z_j, z_j\rangle^{k_j/2}\exp\left\{-\frac{s^2}{2+\varepsilon}\right\},$$

where $\bar{k} = k + \sum_{j=1}^{k'} k_j$.

Lemma 4.7. For all $\varepsilon > 0$, integer $k \ge 1$, $n \ge 1$, *m*, such that $1 \le m \le k$; $h_1, \cdots, h_m \in H$, positive integers $\ell_1, \cdots, \ell_m$ and *j*, such that $j \ge k \ge n$.

- If $l = \ell_1 + \cdots + \ell_m$ is even, we have:

$$\| d_{h_1}^{\ell_1}\cdots d_{h_m}^{\ell_m}\Phi_j(a,r)\| \le c(\varepsilon,k,l)\exp\left\{-\frac{s^2}{2+\varepsilon}\right\}\times c(\Lambda)\prod_{j=1}^{m}\left(\|h_j\|^{\ell_j} + \|\langle a, h_j\rangle\|^{\ell_j}\right). \tag{30}$$

- For odd values of ℓ, we have:

$$\| d_{h_1}^{\ell_1}\cdots d_{h_m}^{\ell_m}\Phi_j(a,r)\| \le c(\varepsilon,k,l)\exp\left\{-\frac{s^2}{2+\varepsilon}\right\}\times\left[c(\Lambda)\sum_{i=1}^{m}\left(\|\langle a,h_i\rangle\|\|\|h_i\|^{\ell_i-1} + \|\langle a,h_i\rangle\|^{\ell_i}\right)\right. \tag{31}$$

Proof of Lemma 4.7. We show that the function $a \mapsto \Theta_a(r) = \Phi_j(a,r)$ is indefinitely Frèchet differentiable and that its derivatives can be estimated by the right-hand side terms of (30) and (31) because of (28),

$$d_{h_1}\cdots d_{h_m}\Theta_a(r) = d_{h_1}^{l_1}\cdots d_{h_m}^{l_m}\Phi_j(a,r).$$

According to Lemma 4.5, $\varphi(x) = \exp\{itw(x)\}$ is indefinitely Frèchet differentiable and (28) is bounded by Lemma 4.4 of [18]. When $|t| \to \infty$, the boundary of (28) and the dimension of the operator Λ^Y, we have a decrease of the function

$$t \mapsto d_{h_1}\cdots d_{h_m}\varphi_t(x). \tag{32}$$

By applying the inversion formula of the Fourier-Stieltjes transform, we see that the inverse transform, inverting the relation (32) which is an infinitely function, having bounded derivatives. Hence, $a \mapsto \Theta_a(r) = \Phi_j(a,r)$ is indefinitely Frèchet differentiable.

We notice that due to the symmetry of the Gaussian distribution, if *G* is a Gaussian measure on *H* with centered, then for all non-negative *m*, and all odd ℓ we have:

$$\int \exp\left\{it\|x\|^2\right\}\|x\|^m\langle x, y\rangle^l G(dx) = 0. \tag{33}$$

We notice that the function $\exp\{it(\|x\|^2+2\langle x,a\rangle)\}$ can be put under the form

$$\exp\{it(\|x\|^2+2\langle x,a\rangle)\}=\exp\{it(\|x\|^2)\}+2\langle x,a\rangle it\int_0^1\exp\{it(\|x\|^2+2\theta\langle x,a\rangle)\}d\theta. \tag{34}$$

- If $s \geq 1$, by repeating the arguments of Lemma 15 in [17], Lemmas 4.5 and 4.6, and relations (33), (34), we obtain the second member.
- If $s < 1$, it suffices to apply (33), (34), Lemma 4.5 and the inverse Fourier Stieltjes transform.

Lemma 4.8. Let a and b be two positive numbers and ε, such that $0 < \varepsilon < 1/4$. We have

$$f=\frac{a}{\exp\left\{|a-b|^2\varepsilon\right\}}\leq\frac{1+\min(a,b)}{\varepsilon^{1/2}}. \tag{35}$$

Proof of Lemma 4.8. It is obvious that $f \leq a$ and also obvious that (35) holds if $b < a \leq 2b$. If $a > 2b$, then $a - b > a/2$ and using the inequality $x\exp\{-x^2\varepsilon/4\}\leq 1/\varepsilon^{1/2}$, for all x, we conclude.

Lemma 4.9. Suppose $\overline{S}_n$ is defined as in Lemma 4.2, and $A_m(a,r)$ satisfying (11). Then,

$$\Delta_n(a,r)\leq\overline{\Delta}(a,r)+A(p,s,t), \tag{36}$$

where

$$\overline{\Delta}(a,r)=\|P(\|\overline{S}_n-a\|<r)-P(\|Y-a\|<r)-\sum_{m=1}^{k-2}A_m(a,r)\|.$$

Proof of Lemma 4.9. Following the main ideas of the proof of Theorem 3.1 in [11] (see also the articles [10] and [17] for more details) and Lemma 4.2, we easily deduce the inequality (36).

Lemma 4.10. Given **H3** and **H5**, if $(\Lambda^X A_0)^2\in\sigma(\beta,s)$, then there exists a constant $c()$, such that:

$$\|P(\|S_n-a\|<r)-P(\|Y-a\|<r)\|\leq\Phi_m(a,r)+c().$$

Proof of Lemma 4.10. The proof of this lemma is a direct consequence of the theorems of Section 3 in [9].

4.3 *Proof of the main results*

Proof of Theorem 3.1. For the proof, we proceed by case for possible values of s.

Case 1: $s < c$.

Let $F_n^{S_n}(x)=P\left(\left\|S_n-a\right\|<x\right), F_n^{\overline{S}_n}(x)=P\left(\left\|\overline{S}_n-a\right\|<x\right), F_0^Y(y)=P\left(\left\|Y_i-a\right\|<y\right), G(y)=\sum_i^{k-2}F_0^Y(y)$, and respectively, $f_{S_n}(t), f_{\overline{S}_n}(t)$ and $h(t)$ the Fourier-Stieltjes transforms of $F_n^{S_n}(x), F_n^{\overline{S}_n}(x)$ and $G(y)$.

According to Theorem 3.1 in [9], the Fourier-Stieltjes transform $h(t)$ of the function $G(y)$ is

$$h(t)=n^{-m/2}\sum_{j=1}^{m}\sum{}'n^jC_n^j(\ell^{(j)}!)^{-1}\mathbb{E}(Q_j)\varphi_j(a),$$

where

$$\varphi_j(a)=\exp\left\{it\left\|\left(1-\frac{j}{n}\right)Y-a\right\|^2\right\}.$$

Using Lemma 4.6 of [18] we can write:

$$\left\|A_m(a,r)\right\| \le c(\Lambda)\left(1+\vartheta^3(a,r)\|a\|\langle\Lambda a,a\rangle\right)^{k-2}\left(\frac{L^2}{n}\right)^{\frac{k-1}{2}} \le c(\Lambda)\left(1+\vartheta^3(a,r)\|a\|\langle\Lambda a,a\rangle\right)^{k-2}.$$

Since according to [9],

$$\| G'(y) \| \le \int_{-\infty}^{+\infty} \|h(t)\| \, dt \le c(a) < \infty,$$

let's put

$$C = \sup_y \| G'(y)(t) \| < \infty.$$

For the remaining of the proof, we state as a lemma the following estimate proved in [16].

Lemma 4.11. (see [16], p.1058). Let $N_0(y) = P(\| Y-a \|^2 < y)$, $N_i(y) = A_i(a, y^{1/2})$, $i \ge 1$, $N(y) = \sum_i^{k-2} N_i(y)$, and respectively, $f_n(t)$, $g_i(t)$, $g(t)$ their Fourier-Stieltjes transform. Let's put $D(t) = \| f_n(t) - g(t) \|$. For any random element $U_i = n^{-1/2} X_{i\mathbf{1}_i}$ where $U_i = n^{-1/2} Y_i$, let $U = (U_1, \cdots, U_n) \in \prod_{i=1}^n \{n^{-1/2} X_i \mathbf{1}_i, n^{-1/2} Y_i\}$. For any other random element $U' = (U'_1, \cdots, U'_n)$, if for any i, U'_i is independent of U_i, let's put $x_j(a) \equiv x_j(a,t,U') = \mathbb{E}\left(\exp\left\{ it \left\| \sum_1^{n-j} U'_k - a \right\|^2 \right\}\right)$ and directional derivatives of higher order are defined by induction:

$$d_h \mathcal{X}_j(a) = \lim_{y \to 0} y^{-1}\left(\mathcal{X}_j(a-yh) - \mathcal{X}_j(a)\right).$$

For integers r and m such that $0 < m \le r \le n$, let the differential operators $Q_{mr}(U,\ell)$ be defined by:

$$Q_{mr}(U,\ell) = \prod_{i=m}^{r} (1-\mathbf{1}s_i)^{\beta(\ell_i)} d_{U_i}^{\ell_i}, \tag{37}$$

where $l = (\ell_1, \cdots, \ell_n)$, $\beta(\ell_i) = 1$, if $\ell_i = 1$ where $\ell_i = 2$ and $\beta(\ell_i) = 0$ otherwise. In (37), we put $U_i = X_i$ if $\ell_i = 1$ or $\ell_i = 2$, and introduce the following relation:

$$\int_H \langle x,b \rangle^l X(dx) = -\int_{\|x\| > \sqrt{n}} \langle x,b \rangle^l \boldsymbol{P}(dx),$$

where X is the difference between the distributions of X_{11} and Y, $\boldsymbol{P}$ the distribution of X_1, $b \in H$, $l = 1,2$.

We obtain the following estimate for $D(t)$:

$$D(t) \le c\mathbb{E}\left(\sup_{U,U'}\left[n^{-(k-1)/2}\sum_{j=1}^{k-1}\sum{}' \left\|Q_{1j}(U,\ell)\mathcal{X}_j(a)\right\| + \left\|(1-\mathbf{1}_1)\left(n^{1/2}d_{X_1} + d_{X_1}^2\right) \times \sum_{j=1}^{k-1}\sum{}' n^{(2j-2-\ell_2-\cdots-\ell_j)/2} Q_{2j}(U,\ell)\mathcal{X}_j(a)\right\|\right]\right), \tag{38}$$

where: $\sum'$ is the sum overall $(\ell_1, \cdots, \ell_j)$, such that $\ell_1 \ge 3, \cdots, \ell_j \ge 3$, $\ell_1 + \cdots + \ell_j = 2$, $j+k-1$ and $\sum''$ is the sum overall $(\ell_2 + \cdots + \ell_j)$, such that $\ell_2 \ge 3, \cdots, \ell_j \ge 3$ and $\ell_2 + \cdots + \ell_j \le k-1+2(j-1)$. The supremum in (37) is taken over all possible combinations that may give U and U'.

We now continue with the proof of our theorem.
Due to the estimated (38), we obtain:

$$\begin{aligned}\left\|A(p,s,t)+c(\Lambda)\left(\mathbb{E}\left(B_2(a,r)\right)(1-\mathbf{1}_1)\right\| &\le \left\|F_n(x)^{S_n}-G(y)\right\| \\ &\le c\mathbb{E}\left(\sup_{U,U'}\left[n^{-(k-1)/2}\sum_{j=2}^{k-1}\sum{}''\left\|Q_{1j}(U,\ell)\mathcal{X}_j(a)\right\|\right.\right. \\ &\left.\left.+\left\|(1-\mathbf{1}_1)\left(n^{1/2}d_{X_1}+d_{X_1}^2\right)\times\sum_{j=2}^{k-1}\sum{}'' n^{(2j-2-\ell_2-\dots-\ell_j)/2}Q_{2j}(U,\ell)\mathcal{X}_j(a)\right\|\right]\right) \\ &+\left\|F_n^{\bar{S}_n}(x)-G(y)\right\|.\end{aligned}$$

Applying Esseen's inequality, for all $0\le T_0=\left(\frac{n}{L^2}\right)^{(k-1)/2}$, $k\ge 2$, we obtain:

$$\left\|F_n^{\bar{S}_n}(x)-G(y)\right\|\le c\left(C/T_1+I_1+I_2+I_3\right), \tag{39}$$

where

$$I_1=\int_{|t|\le T_1}|t|^{-1}\left\|f_{\bar{S}_n}(t)-h(t)\right\|dt,\quad T_1=c\left(\frac{n}{L^2}\right)^{1/2}\left(\text{In}\left(\frac{n}{L^2}\right)\right)^{-1/2},$$
$$I_2=\int_{T_1\le|t|\le T_2}|t|^{-1}\left\|h(t)\right\|dt,\quad T_2=T\left(cf(23)\right),$$
$$I_3=\int_{|t|\le T_2}|t|^{-1}\left\|f_{\bar{S}_n}(t)\right\|dt.$$

It follows from (39) that:

$$\bar{\Delta}(a,r)\le c\left(C/T_1+I_1+I_2+I_3\right). \tag{40}$$

Using Lemma 4.6, Lemma 11 in [17] and inequality (38), we obtain

$$I_1+I_2+I_3\le c(\Lambda)\left(\mathbb{E}\left(B_2(a,r)\right)(1-\mathbf{1}_1)+\left(1+\vartheta^3(a,r)\langle\Lambda a,a\rangle^{3/2}n^{-1/2}\right)\right)\mathbb{E}\left(B_{k+1}(a,r)\mathbf{1}_1\right).$$

To conclude the first case, we smooth the characteristic function of the event $\{X\in H:\|X-a\|<r\}$. Using Lemma 1 in [19], and constructing the following function:

$$f(t)=\begin{cases}0 & \text{if} \quad t<r-n^{-1/2}\\ 1 & \text{if} \quad t>r\end{cases},$$

we make the set $\{X\in H:\|X-a\|<r\}$ smooth because f is three times differentiable and there exists a generic constant c, such that $\|f^{(i)}(t)\|\le cn^{i/2}$ (f being the density of $P(\|X-a\|<r)$).

Case 2: $s\ge c$ (same argument as the proof of Theorem 1.1 in [9]).

Proof of Theorem 3.2. Given (13) and inequality $L<n^{1/2}$, we get from Theorem 1 in [17] that:

$$\mathbb{E}\left(\|X_1\|_{\mathbf{1}_1}^2\right)\ge\frac{2}{3}. \tag{41}$$

With the conditions on the random element Y in Section 2.2, for any positive integer k, we get:

$$\mathbb{E}\left(\|\langle Y,a\rangle\|^k\right)\le c(k)\langle\Lambda a,a\rangle^{k/2},\ \mathbb{E}\left(\|Y\|^k\right)\le c(k). \tag{42}$$

From Lemmas 4.6 and 4.7, Hölder's inequality and estimates (41) and (42), we conclude the theorem.

5. Conclusion

Asymptotic developments in probability, as we have already said, are of notable importance in probability theory. Indeed, they allow for good approximation estimates in the convergence of random elements. The aim of this paper was to give an asymptotic probability expansion on the ball with a fixed center a in a Hilbert space for the sum of independent and identically distributed random elements. In contrast to previous authors, we have considered the covariance operators of the different random elements X and Y with small moments. We used Nagaev [15] on the covariance operator Λ^X. Also, we have relied on the techniques and methods in [16] and [9] to give a new asymptotic probability expansion of model (4).

At the end of this work, we were able to give a solution to the problem of the model (4) with a high degree of accuracy. Indeed, our expansion terms are estimated both in terms of moments and eigenvalues of both covariance operators Λ^X and Λ^Y.

Our perspectives after this article will be as follows: Given the importance of studying the convergence of random elements (presented in the introduction), we intend to study the following convergence problem:

$$\frac{1}{\sqrt{n}} S_n(X) \overset{\mathcal{L}}{\rightarrow} Z_\Lambda, \tag{43}$$

where Z denotes a sequence of Gaussian-centered random elements with covariance operator $\Lambda(.) = E[\langle Z_\Lambda, .\rangle Z_\Lambda]$. In a second perspective, we wish to illustrate the interest of the acceleration of the convergence of random elements in a practical problem: a dynamic system.

Conflict of interest

The authors declare that they have no conflict of interests.

References

[1] Devineau L, Loisel S. Construction d'un algorithme d'accélération de la méthode des "simulations dans les simulations" pour le calcul du capital économique Solvabilité II. *Bulletin Français d'Actuariat*. 2009; 10(17): 188-221.

[2] Delahaye JP, Germain-Bonne B. Résultats négatifs en accélération de la convergence. *Numerische Mathematik*. 1980; 35(4): 443-457.

[3] Tchebycheff P. Sur deux theoremes relatifs aux probabilities. *Acta Mathematica*. 1890-1891; 14: 305-315. Available from: https://doi.org/10.1007/BF02413327.

[4] Bruns H. *Wahrscheinlichkeitsrechnung und kollektivmasslehre*. Cornell University Library; 1906.

[5] Edgeworth FY. The law of errors. *Nature.* 1887; 36: 482-483. Available from: https://doi.org/10.1038/036482d0.

[6] Cramer H. *Random variables and probability distributions*. Cambridge, UK: Cambridge University Press; 1970.

[7] Essen C-G. Fourier analysis of distribution functions. A mathematical study of the Laplace-Gaussian law. *Acta Mathematica*. 1945; 77: 1-125. Available from: https://doi.org/10.1007/BF02392223.

[8] Bikyalis A. Asymptotic expansions for distributions of sums of identically distributed independent lattice random variables. *Theory of Probability and Its Applications*. 1969; 14(3): 481-489. Available from: https://doi.org/10.1137/1114060.

[9] Bentkus VY. Asymptotic expansions for the distribution of the sum of independent random elements in a Hilbert space. *Lietuvos Matematikos Rinkiny*s. 1984; 24(4): 29-48.

[10] Götze F. Asymptotic expansions for bivariate von Mises functionals. *Z. Wahrscheinlichkeitstheorie verw Gebiete*. 1979; 50(3): 333-355. Available from: https://doi.org/10.1007/BF00534154.

[11] Bentkus V, Götze F, Paulauskas V, Račkauskas. The accuracy of Gaussian approximation in Banach spaces. In:

Prokhorov YV, Statulevičius V. (eds.) *Limit theorems of probability theory*. Berlin, Heidelberg: Springer; 2000. p.25-111. Available from: https://doi.org/10.1007/978-3-662-04172-7_2.

[12] Jirak M. Rate of convergence for Hilbert space valued processes. *Bernoulli*. 2018; 24(1): 202-230. Available from: https://doi.org/10.3150/16-BEJ870.

[13] Kanagawa S. Asymptotic expansions for symmetric statistics with degenerate kernels. *Mathematics*. 2022; 10(21): 4158. Available from: https://doi.org/10.3390/math10214158.

[14] Yurinskii VV. On the accuracy of the normal approximation of the probability of hitting a ball. *Theory of Probability & Its Applications*. 1983; 27(2): 280-289. Available from: https://doi.org/10.1137/1127030.

[15] Nagaev SV. On accuracy of normal approximation for distribution of sum of independent Hilbert space valued random variables. In: Prokhorov JV, Itô K. (eds.) *Probability theory and mathematical statistics*. Berlin, Heidelberg: Springer; 1983. p.461-473. Available from: https://doi.org/10.1007/BFb0072941.

[16] Sazonov VV, Ul'yanov VV. Asymptotic expansions of the probability that the sum of independent random variables hits a ball in a Hilbert space. *Russian Mathematical Surveys*. 1995; 50(5): 1045. Available from: https://doi.org/10.1070/RM1995v050n05ABEH002616.

[17] Zalesskii BA, Sazonov VV, Ul'yanov VV. Normal approximation in Hubert space I. *Theory of Probability & Its Applications*. 1988; 33(2): 207-227. Available from: https://doi.org/10.1137/1133037.

[18] Bentkus V. Asymptotic expansions in the central limit theorem in Hilbert space. *Lithuanian Mathematical Journal*. 1984; 24: 210-225. Available from: https://doi.org/10.1007/BF00968039.

[19] Sazonov VV, Ulyanov VV, Zalesskii BA. Asymptotic expansions refining the central limit theorem in Hilbert space. In: *Vol. 1 Probability theory and applications*. Berlin, Boston: De Gruyter; 1987. p.679-688.