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Research Article

Fuzzy Metric Spaces: Optimizing Coincidence and Proximity Points

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Abstract: Our manuscript puts forward two novel fuzzy proximal contractive conditions. First, we present two variants of fuzzy α-proximal quasi-H-contractions and establish optimal coincidence point outcomes for these contractions in fuzzy metric space. This manuscript's second part proposes the fuzzy ψ-contraction for a multivalued mapping equipped with fuzzy weak P-property and achieves the best proximity point outcome in strong fuzzy metric space. The findings of this study broaden and generalize some existing research results.

***Keywords*:** fuzzy metric space, optimal coincidence points, best proximity points, fuzzy proximal contractions

MSC: 47H10, 54E50

1. Introduction

Zadeh [1] made the pioneering contribution of introducing fuzzy sets in 1965. In 1975, the integration of metric space and fuzzy sets led Kramosil et al. [2] to put forward fuzzy metric space. A modified version of this concept is presented by George et al. [3] to attain Hausdorff topology in this space. The investigation of fixed point theory in the mathematical framework of fuzzy metric space began with Grabiec's seminal research [4]. Consequently, researchers studied various contractions and obtained their fixed point results in these spaces (see [5-9]). Fixed point theory is a powerful tool that has various applications in different branches of mathematics, engineering, and economics. The central objective of this theory is to identify the presence and singularity of a solution $\Upsilon z = z$, where $\Upsilon : Z \to Z$ is a mapping and Z is a designated abstract space. Assuming Ω and Θ are nonempty closed subsets of Z, if $\Upsilon : \Omega \to \Theta$ is non-self mapping, it is possible to have no solution for $\Upsilon z = z$ for any $z \in \Omega$. In such a scenario, one tries to locate an element that results in the smallest possible distance between z and Υz. The term best proximity point of Υ is used to refer to this point.

Fan [10] introduced the best approximation theorem in 1969, which asserts that a nonempty compact convex subset Ω of Banach space Z and a continuous mapping $P : \Omega \to Z$ implies the best proximity point of P is present in Ω. This classical result has become a fundamental result in best approximation theory. Since then, research on proximity point problems has been developed rapidly in metric spaces; for instance, see [11-13]. It can be seen that the best

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approximation theorem generalizes the fixed point theorem. Indeed, when Ω is equal to Θ, the best proximity point is identical to a fixed point. Recently, researchers have had a great interest in extending the best approximation results to fuzzy metric spaces. In 2013, Vetro et al. [14] demonstrated that non-self mappings with diverse proximal contractions on fuzzy metric spaces have a unique best proximity point. In non-Archimedean fuzzy metric spaces, Saleem et al. [15] established some outcomes for optimal coincidence points; see also [16-18] for similar works. By using the concept of admissible mapping [19] on b-fuzzy metric spaces, Saleem et al. [20] established the presence of optimal coincidence points for fuzzy (α-η)-generalized and fuzzy (β-ψ)-generalized proximal contractions.

The focus of this research is to uncover some original best approximation outcomes under a fuzzy metric space framework. Firstly, we demonstrate the emergence and exclusivity of optimal coincidence points for two variants of fuzzy α-proximal quasi-H-contraction in the fuzzy metric space context. Our results expand and generalize the results in [14, 17, 20] and [21]. Secondly, we bring forth fuzzy ψ-proximal contraction and establish the manifestation of the best proximity point for multivalued mapping satisfying this contractive condition in a complete strong fuzzy metric space.

2. Preliminaries

This section revisits known definitions that will have significance in this manuscript.

Definition 1 [22] Binary operation $*: \mathcal{I} \times \Lambda \to \Lambda$, in which $\Lambda = [0, 1]$ is referred to as continuous t-norm when it satisfies the conditions below:

1. $e*1 = e$ for each $e \in \Lambda$;
2. $e*l = l*e$, $(e*l)*o = e*(l*o)$;
3. $e \leq o$ and $l \leq w$ implies $e*l \leq o*w$ across $e, l, o, w \in \Lambda$;
4. $*$ is continuous.

Examples for the definition above are $e*_{\min} l = \min\{e, l\}$ (minimum t-norm), $e*_p l = e \cdot l$(product t-norm), and $e*_L l = \max\{e + l - 1, 0\}$(Lukasiewicz t-norm).

The notion of fuzzy metric space from [3] is adopted throughout this research.

Definition 2 [3] Suppose Z is an arbitrary nonempty set, $*$ is a continuous t-norm and Γ is a fuzzy set defined on $Z \times Z \times (0, \infty)$ meets subsequent conditions:

FMS1. $\Gamma(\omega, \theta, \varkappa) > 0$;

FMS2. $\Gamma(\omega, \theta, \varkappa) = 1$ for all $\varkappa > 0$ if and only if $\omega = \theta$;

FMS3. $\Gamma(\omega, \theta, \varkappa) = \Gamma(\theta, \omega, \varkappa)$;

FMS4. $\Gamma(\omega, \theta, \varkappa + \varsigma) \geq \Gamma(\omega, \theta, \varkappa) * \Gamma(\theta, \varpi, \varsigma)$;

FMS5. $\Gamma(\omega, \theta, \cdot) : (0, \infty) \to (0, 1]$ is continuous,

for all $\omega, \theta, \varpi \in Z$ and $\varkappa, \varsigma > 0$. Then, $(Z, \Gamma, *)$ is referred to as fuzzy metric space.

$\Gamma(\omega, \theta, \varkappa)$ represents the extent to which ω and θ are close with respect to $\varkappa$. In Definition 2, if we replace FMS4 with the following condition:

FMS6. $\Gamma(\omega, \varpi, \max\{\varkappa, \varsigma\}) \geq \Gamma(\omega, \theta, \varkappa) * \Gamma(\theta, \varpi, \varsigma)$,

for all $\omega, \theta, \varpi \in Z$ and $\varkappa, \varsigma > 0$, then we said that $(Z, \Gamma, *)$ can be viewed as non-Archimedean fuzzy metric space. Additionally, set $\varsigma = \varkappa$ in FMS6, the inequality $\Gamma(\omega, \varpi, \varkappa) \geq \Gamma(\omega, \theta, \varkappa) * \Gamma(\theta, \varpi, \varkappa)$ holds for every $\omega, \theta, \varpi \in Z$. In this context, $(Z, \Gamma, *)$ is referred to as strong fuzzy metric space. The next lemma is proven by Grabiec [4].

Lemma 1 [4] Assume that $(Z, \Gamma, *)$ is a fuzzy metric space. Then, for any ω, θ in Z, $\Gamma(\omega, \theta, \cdot)$ is nondecreasing.

Definition 3 [3] Let $(Z, \Gamma, *)$ be a fuzzy metric space and $\{\omega_n\}$ be a sequence in Z. It follows that

1. $\{\omega_n\}$ is convergent provided there exists $\omega \in Z$ satisfies $\lim_{n\to\infty}\Gamma(\omega_n, \omega, \varkappa) = 1$ for all $\varkappa > 0$;
2. sequence $\{\omega_n\}$ is Cauchy provided for each $\iota \in (0, 1)$ and $\varkappa > 0$, we have $n_0 \in \mathbb{N}$ satisfies $\Gamma(\omega_n, \omega_m, \varkappa) > 1 - \iota$ for any $n, m \geq n_0$;
3. $(Z, \Gamma, *)$ is considered complete provided each Cauchy sequence in Z converges.

Wardowski [23] defined a class H as follows.

Definition 4 [23] Denote H as the collection of continuous functions $\eta : (0, 1] \to [0, \infty)$ that fulfil the criteria below:

1. η is decreasing on (0, 1].
2. $\eta(\varkappa) = 0 \Leftrightarrow \varkappa = 1$;
3. given a continuous t-norm $*$, we have $\eta(\varkappa) + \eta(\varsigma) \geq \eta(\varkappa * \varsigma)$ for any $\varkappa, \varsigma \in [0, 1)$.

Proposition 1 [23] Suppose $(Z, \Gamma, *)$ is fuzzy metric space and $\eta \in H$. A sequence $\{\omega_n\}$ in Z is Cauchy implies and is implied by for any $\iota > 0$ and $\varkappa > 0$, one may find $n_0 \in N$ satisfies $\eta(\Gamma(\omega_n, \omega_m, \varkappa)) < \iota$ for each $m, n \geq n_0$.

Definition 5 [14] Suppose Ω and Θ are nonempty subsets of a fuzzy metric space $(Z, \Gamma, *)$. For all $\varkappa > 0$, the sets $\Omega_0(\varkappa)$ and $\Theta_0(\varkappa)$ is define as:

$$\Omega_0(\varkappa) = \{\omega \in \Omega : \Gamma(\omega, \theta, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \text{ for some } \theta \in \Theta\},$$

$$\Theta_0(\varkappa) = \{\theta \in \Theta : \Gamma(\omega, \theta, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \text{ for some } \omega \in \Omega\}$$

where

$$\Gamma(\Omega, \Theta, \varkappa) = \sup\{\Gamma(a, b, \varkappa) \text{ for } a \in \Omega \text{ and } b \in \Theta\}.$$

Also, we have $\Gamma(\omega, \Omega, \varkappa) = \sup_{a \in \Omega} \Gamma(\omega, a, \varkappa)$ for all $\varkappa > 0$.

Definition 6 [21] Given a function $\alpha : Z \times Z \times (0, \infty) \to [0, \infty)$. A mapping $g : Z \to Z$ is said to be

1. α-admissible when

$$\alpha(\omega, \theta, \varkappa) \geq 1 \Rightarrow \alpha(g\omega, g\theta, \varkappa) \geq 1;$$

2. α_R-admissible when

$$\alpha(g\omega, g\theta, \varkappa) \geq 1 \Rightarrow \alpha(\omega, \theta, \varkappa) \geq 1,$$

for any $\omega, \theta \in Z$ and $\varkappa > 0$.

Definition 7 [24] A mapping $T : \Omega \to \Theta$ is considered as a α-proximal admissible provided that for all ω, θ, μ, $v \in \Omega$ and $\varkappa > 0$,

$$\left.\begin{array}{l} \alpha(\omega, \theta, \varkappa) \geq 1 \\ \Gamma(\mu, T\omega, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \\ \Gamma(v, T\theta, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \end{array}\right\} \Rightarrow \alpha(\mu, v, \varkappa) \geq 1.$$

Definition 8 [15] Let Ω be a nonempty subset of a fuzzy metric space $(Z, \Gamma, *)$. A mapping $g : \Omega \to \Omega$ is known as fuzzy isometry if for any $\omega, \theta \in \Omega$ and $\varkappa > 0$, we have

$$\Gamma(g\omega, g\theta, \varkappa) = \Gamma(\omega, \theta, \varkappa).$$

Definition 9 [15] Let Ω be a nonempty subset of a fuzzy metric space $(Z, \Gamma, *)$. A mapping $g : \Omega \to \Omega$ is known as fuzzy expansive if for any $\omega, \theta \in \Omega$ and $\varkappa > 0$, we have

$$\Gamma(g\omega, g\theta, \varkappa) \leq \Gamma(\omega, \theta, \varkappa).$$

Definition 10 [15] Let Ω and Θ be nonempty subsets of a fuzzy metric space $(Z, \Gamma, *)$. The set Θ is referred to as fuzzy approximately compact with respect to Ω if for all sequence $\{\theta_n\}$ in Θ and some $\omega \in \Omega$ in which $\Gamma(\omega, \theta_n, \varkappa) \to \Gamma(\omega, \Theta, \varkappa)$, one has $\omega \in \Omega_0(\varkappa)$.

A point $\omega^* \in \Omega$ is termed best proximity point if $\Gamma(\omega^*, T\omega^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$ hold for all $\varkappa > 0$.

Definition 11 [16] Let Ω and Θ be two nonempty subsets of a fuzzy metric space $(Z, \Gamma, *)$. A point $\omega^* \in \Omega$ is termed optimal coincidence point of mappings g and T, where $T : \Omega \to \Theta$ and $g : \Omega \to \Omega$ if $\Gamma(g\omega^*, T\omega^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$ hold for all $\varkappa > 0$.

Amini-Harandi et al. [25] generalized fuzzy H-contraction under the setting of fuzzy metric space as follows.

Definition 12 [25] Let $(Z, \Gamma, *)$ be a fuzzy metric space. A mapping $T : Z \to Z$ is known as fuzzy H-quasi-contraction with respect to $\eta \in H$ provided that some $k \in (0, 1)$ meeting the following criteria:

$$\eta\left(\Gamma(T\omega, T\theta, \varkappa)\right) \le k \max\left\{\eta\left(\Gamma(\omega, \theta, \varkappa)\right), \eta\left(\Gamma(\omega, T\omega, \varkappa)\right), \eta\left(\Gamma(\theta, T\theta, \varkappa)\right), \eta\left(\Gamma(\omega, T\theta, \varkappa)\right), \eta\left(\Gamma(\theta, T\omega, \varkappa)\right)\right\} \quad (1)$$

for all $\omega, \theta \in Z$ and any $\varkappa > 0$.

3. Coincidence point optimization in fuzzy metric spaces: Fuzzy α-proximal quasi H-contraction

In the following, we define new proximal contractions within the context of fuzzy metric space.

Definition 13 Let $T: \Omega \to \Theta$ and $\alpha: Z \times Z \times (0, \infty) \to [0, \infty)$ be mappings. T is known as a fuzzy α-proximal quasi-H-contraction of type I if for any $\omega, \theta, \mu, \nu \in \Omega$ and $\varkappa > 0$, it is possible to find a mapping $\eta \in H$ and a constant $k \in (0, 1)$ such that

$$\left.\begin{array}{l}\Gamma(\mu, T\omega, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \\ \Gamma(\nu, T\theta, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{array}\right\} \Rightarrow \alpha(\omega, \theta, \varkappa)\eta\left(\Gamma(\mu, \nu, \varkappa)\right) \le k\Gamma^*(\mu, \nu, \omega, \theta, \varkappa), \quad (2)$$

in which $\Gamma^*(\mu, \nu, \omega, \theta, \varkappa) = \max\left\{\eta\left(\Gamma(\omega, \theta, \varkappa)\right), \eta\left(\Gamma(\omega, \mu, \varkappa)\right), \eta\left(\Gamma(\theta, \nu, \varkappa)\right), \eta\left(\Gamma(\theta, \mu, \varkappa)\right)\right\}$.

Definition 14 Let $T: \Omega \to \Theta$, $g: \Omega \to \Omega$ and $\alpha: Z \times Z \times (0, \infty) \to [0, \infty)$ be mappings. The pair (g, T) is known as a fuzzy α-proximal quasi-H-contraction of type II if for any $\omega, \theta, \mu, \nu \in \Omega$ and $\varkappa > 0$, it is possible a mapping $\eta \in H$ and a constant $k \in (0, 1)$ such that

$$\left.\begin{array}{l}\Gamma(g\mu, T\omega, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \\ \Gamma(g\nu, T\theta, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{array}\right\} \Rightarrow \alpha(\omega, \theta, \varkappa)\eta\left(\Gamma(g\mu, g\nu, \varkappa)\right) \le k\Gamma^*(\mu, \nu, \omega, \theta, \varkappa), \quad (3)$$

in which $\Gamma^*(\mu, \nu, \omega, \theta, \varkappa) = \max\left\{\eta\left(\Gamma(\omega, \theta, \varkappa)\right), \eta\left(\Gamma(\omega, \mu, \varkappa)\right), \eta\left(\Gamma(\theta, \nu, \varkappa)\right), \eta\left(\Gamma(\theta, \mu, \varkappa)\right)\right\}$.

Remark 1 If we let $g = I_\omega$, that is, g is an identity mapping on Ω, this results in the fuzzy α-proximal quasi-H-contraction of type II becoming a fuzzy α-proximal quasi-H-contraction of type I.

We are now prepared to present and verify our outcomes for fuzzy α-proximal quasi-H-contraction in the domain of fuzzy metric space.

Theorem 1 Given that Ω, Θ are nonempty closed subsets of a complete fuzzy metric space $(Z, \Gamma, *)$ in which $\Omega_0(\varkappa) \neq \varnothing$ and Θ is fuzzy approximately compact with respect to Ω. Assume $g: \Omega \to \Omega$ and $T: \Omega \to \Theta$ satisfying conditions below:

1. g is continuous and fuzzy expansive;
2. both g and T are α_R-admissible and α-proximal admissible mappings respectively;
3. (g, T) can be regarded as fuzzy α-proximal quasi-H-contraction of type II;
4. $\Omega_0(\varkappa) \subseteq g\left(\Omega_0(\varkappa)\right)$ and $T\left(\Omega_0(\varkappa)\right) \subseteq \Theta_0(\varkappa)$;
5. there exist elements ω_0 and ω_1 in $\Omega_0(\varkappa)$ satisfying

$$\Gamma(g\omega_1, T\omega_0, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \text{ and } \alpha(\omega_0, \omega_1, \varkappa) \ge 1$$

for all $\varkappa > 0$ and $\left\{\eta\left(\Gamma(\omega_0, \omega_1, \varkappa_i)\right)\right\}$ is bounded for any sequence $(\varkappa_i)$ in $(0, \infty)$ with $\varkappa_i \to 0$ as $i \to \infty$;

6. if a sequence $\{\omega_n\}$ in Ω satisfying $\alpha(\omega_n, \omega_{n+1}, \varkappa) \ge 1$ whenever $n \in \mathbb{N}, \varkappa > 0$ and $\omega_n \to \omega^*$ as $n \to \infty$, then $\alpha(\omega_n, \omega^*, \varkappa) \ge 1$.

It follows that g and T possess optimal coincidence point ω^* in $\Omega_0(\varkappa)$. Furthermore, if for every $\omega^*, \theta^* \in \Omega_0(\varkappa)$ such that

$$\left.\begin{aligned}\Gamma(g\omega^*, T\omega^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\\ \Gamma(g\theta^*, T\theta^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{aligned}\right\} \Rightarrow \alpha(\omega^*, \theta^*, \varkappa) \geq 1 \tag{4}$$

for each $\varkappa > 0$, then the optimal coincidence point is uniquely determined.

Proof. Let $\omega_0, \omega_1 \in \Omega_0(\varkappa)$ such that $\Gamma(g\omega_1, T\omega_0, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$ and $\alpha(\omega_0, \omega_1, \varkappa) \geq 1$ for all $\varkappa > 0$. Since $T\omega_1 \in T(\Omega_0(\varkappa)) \subseteq \Theta_0(\varkappa)$ and $\Omega_0 \subseteq g(\Omega_0(\varkappa))$, there exists $\omega_2 \in \Omega_0(\varkappa)$ for which $\Gamma(g\omega_2, T\omega_1, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$ for all $\varkappa > 0$. Seeing that T is α-proximal admissible, for all $\varkappa > 0$, it follows that

$$\left.\begin{aligned}&\alpha(\omega_0, \omega_1, \varkappa) \geq 1\\ &\Gamma(g\omega_1, T\omega_0, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\\ &\Gamma(g\omega_2, T\omega_1, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{aligned}\right\} \Rightarrow \alpha(g\omega_1, g\omega_2, \varkappa) \geq 1.$$

Since g is α_R-admissible, $\alpha(g\omega_1, g\omega_2, \varkappa) \geq 1$ implies $\alpha(g\omega_1, g\omega_2, \varkappa) \geq 1$ for all $\varkappa > 0$. Again, since $T\omega_2 \in T(\Omega_0(\varkappa)) \subseteq \Theta_0(\varkappa)$ and $\Omega_0 \subseteq g(\Omega_0(\varkappa))$, there exists $\omega_3 \in \Omega_0(\varkappa)$ for which $\Gamma(g\omega_3, T\omega_2, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$. Seeing that T is α-proximal admissible, for all $\varkappa > 0$, it follows that

$$\left.\begin{aligned}&\alpha(\omega_1, \omega_2, \varkappa) \geq 1\\ &\Gamma(g\omega_2, T\omega_1, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\\ &\Gamma(g\omega_3, T\omega_2, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{aligned}\right\} \Rightarrow \alpha(g\omega_2, g\omega_3, \varkappa) \geq 1.$$

Since g is α_R-admissible, $\alpha(g\omega_2, g\omega_3, \varkappa) \geq 1$ implies $\alpha(\omega_2, \omega_3, \varkappa) \geq 1$ for all $\varkappa > 0$. Repeating this process, we obtain a sequence $\{\omega_n\}$ in $\Omega_0(\varkappa)$ satisfying

$$\Gamma(g\omega_{n+1}, T_{\omega n}, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \text{ and } \alpha(\omega_n, \omega_{n+1}, \varkappa) \geq 1$$

for each $n \in \mathbb{N} \cup \{0\}$ and $\varkappa > 0$. Due to the fact that the pair (g, T) satisfy the conditions of fuzzy α-proximal quasi-H-contraction of type II, using (3) with $\mu = \omega_n, \nu = \omega_{n+1}, \omega = \omega_{n-1}$ and $\theta = \omega_n$ we have

$$\alpha(\omega_{n-1}, \omega_n, \varkappa)\eta(\Gamma(g\omega_n, g\omega_{n+1}, \varkappa)) \leq k\Gamma^*(\omega_n, \omega_{n+1}, \omega_{n-1}, \omega_n, \varkappa) \tag{5}$$

where

$$\begin{aligned}\Gamma^*(\omega_n, \omega_{n+1}, \omega_{n-1}, \omega_n, \varkappa) &= \max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_n, \varkappa)\right)\right\}\\ &= \max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right), \eta(1)\right\}\\ &= \max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right), 0\right\}\\ &\leq \max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right)\right\}\end{aligned}$$

for all $\varkappa > 0$. By fuzzy expansive property of g and decreasing property of η, we have $\Gamma(g\omega_n, g\omega_{n+1}, \varkappa) \leq \Gamma(\omega_n, \omega_{n+1}, \varkappa)$ and $\eta(\Gamma(g\omega_n, g\omega_{n+1}, \varkappa)) \geq \eta(\Gamma(\omega_n, \omega_{n+1}, \varkappa))$ respectively. Therefore, using (5) and $\alpha(\omega_{n-1}, \omega_n, \varkappa) \geq 1$ we obtain

$$\begin{aligned}\eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right) \leq \eta\left(\Gamma(g\omega_n, g\omega_{n+1}, \varkappa)\right) &\leq \alpha(\omega_{n-1}, \omega_n, \varkappa)\eta\left(\Gamma(g\omega_n, g\omega_{n+1}, \varkappa)\right)\\ &\leq k\Gamma^*(\omega_{n-1}, \omega_{n+1}, \omega_{n-1}, \omega_n, \varkappa)\\ &\leq k \max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right)\right\}\end{aligned}$$

which implies

$$\eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right) \le k \max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right)\right\} \tag{6}$$

for all $\varkappa > 0$. If $\max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right)\right\} = \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right)$ for all $\varkappa > 0$, then by inequality (6) and $k \in (0, 1)$ we have

$$\eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right) \le k\eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right) < \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right)$$

which conflict with our findings. As a result, we obtain $\max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right)\right\} = \eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right)$ for all $\varkappa > 0$ and inequality (6) becomes

$$\eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right) \le k\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right)$$

for every $n \in \mathbb{N} \cup \{0\}$ as well as $\varkappa > 0$. Employing the inequality above successively for any $n \in \mathbb{N} \cup \{0\}$ and $\varkappa > 0$, we obtain

$$\begin{aligned}\eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right) &\le k\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right)\\ &\le k^2\eta\left(\Gamma(x_{n-2}, \omega_{n-1}, \varkappa)\right)\\ &\le \cdots\\ &\le k^n\eta\left(\Gamma(\omega_0, \omega_1, \varkappa)\right)\end{aligned}$$

which implies that

$$\eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right) \le k^n\eta\left(\Gamma(\omega_0, \omega_1, \varkappa)\right).$$

After letting n approach infinity in (7), we yield

$$\lim_{n\to\infty} \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right) = 0.$$

Our task now is to demonstrate sequence $\{\omega_n\}$ is Cauchy. Suppose that $m, n \in \mathbb{N}$ where $m > n$. Construct a sequence of positive numbers $\{a_i\}$ that is strictly decreasing in which $\sum_{i=1}^{\infty} a_i = 1$. Subsequently, for all $\varkappa > 0$, we yield

$$\begin{aligned}\Gamma(\omega_n, \omega_m, \varkappa) &= M\left(\omega_n, \omega_m, \varkappa - \sum_{i=n}^{m-1} a_i\varkappa + \sum_{i=n}^{m-1} a_i\varkappa\right)\\ &\ge M\left(\omega_n, \omega_m, \varkappa - \sum_{i=n}^{m-1} a_i\varkappa\right) * M\left(\omega_n, \omega_m, \sum_{i=n}^{m-1} a_i\varkappa\right)\\ &= 1 * M\left(\omega_n, \omega_m, \sum_{i=n}^{m-1} a_i\varkappa\right)\\ &= 1 * M\left(\omega_n, \omega_m, \sum_{i=n}^{m-1} a_i\varkappa\right)\\ &\ge \Gamma(\omega_n, \omega_{n+1}, a_n\varkappa) * \Gamma(\omega_{n+1}\ \omega_{n+2}, a_{n+1}\varkappa) * \cdots * \Gamma(\omega_{m-1}, \omega_m, a_{m-1}\varkappa).\end{aligned}$$

By the inequality above, condition (3) from Definition 4 and Equation (7), we yield

$$\begin{aligned}\eta\big(\Gamma(\omega_n, \omega_m, \varkappa)\big) &\le \eta\big(\Gamma(\omega_n, \omega_{n+1}, a_n\varkappa) * \Gamma(\omega_{n+1}\ \omega_{n+2}, a_{n+1}\varkappa) * \cdots * \Gamma(\omega_{m-1}, \omega_m, a_{m-1}\varkappa)\big) \\ &\le \eta\big(\Gamma(\omega_n, \omega_{n+1}, a_n\varkappa)\big) + \eta\big(\Gamma(\omega_{n+1}\ \omega_{n+2}, a_{n+1}\varkappa)\big) + \cdots + \eta\big(\Gamma(\omega_{m-1}, \omega_m, a_{m-1}\varkappa)\big) \\ &\le k^n\eta\big(\Gamma(\omega_0, \omega_1, a_n\varkappa)\big) + k^{n+1}\eta\big(\Gamma(\omega_0, \omega_1, a_{n+1}\varkappa)\big) + \cdots + k^{m-1}\eta\big(\Gamma(\omega_0, \omega_1, a_{m-1}\varkappa)\big) \\ &= \sum^{m-1} k^i\eta\left(\Gamma(\omega_0, \omega_1, a_i\varkappa)\right).\end{aligned} \tag{8}$$

As the sequence of positive numbers $\{a_i\}$ is strictly decreasing, by Lemma 1 and the fact η is decreasing, we can verify that the sequence $\{\eta\left(\Gamma(\omega_0, \omega_1, a_i\varkappa)\right)\}$ is nondecreasing. In addition to that, by condition (5) the sequence is bounded. As a result, the series $\sum_{i=1}^{\infty} k^i\eta\left(\Gamma(\omega_0, \omega_1, a_i\varkappa)\right)$ is convergent. In light of this, for any $\iota > 0$ one can find $n_0 \in \mathbb{N}$ satisfying

$$\sum_{i=n}^{m-1} k^i\eta\left(\Gamma(\omega_0, \omega_1, a_i\varkappa)\right) < \iota \quad \text{whenever } m, n \ge n_0, m > n.$$

Using the inequality above and (8), we have

$$\eta\left(\Gamma(\omega_n, \omega_m, \varkappa)\right) < \iota \quad \text{whenever } m, n \ge n_0, m > n.$$

Therefore, by Proposition 1, sequence $\{\omega_n\}$ is Cauchy.

As Ω is closed and $(Z, \Gamma, *)$ is complete, there are $\omega^* \in \Omega$ such that

$$\lim_{n\to\infty} \Gamma(\omega_n, \omega^*, \varkappa) = 1$$

for each $\varkappa > 0$. Note that

$$\Gamma(g\omega_{n+1}, \Theta, \varkappa) \ge \Gamma(g\omega_{n+1}, T\omega_n, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \ge \Gamma(g\omega_{n+1}, \Theta, \varkappa).$$

Since g is continuous along with $\omega_n \to \omega^*$ as $n \to \infty$, this implies that $g\omega_n \to g\omega^*$ as $n \to \infty$. Taking the limit as $n \to \infty$ for inequality above, it leads to

$$\Gamma(g\omega^*, T\omega_n, \varkappa) \to \Gamma(g\omega^*, \Theta, \varkappa).$$

Since Θ is fuzzy approximately compact with respect to Ω, $g\omega^* \in \Omega_0(\varkappa)$. As $\Omega_0(\varkappa) \subseteq g\left(\Omega_0(\varkappa)\right)$, there exists $u \in \Omega_0(\varkappa)$ such that $g\omega^* = gu$. By fuzzy expansive property of g and decreasing property of η on (0, 1], we deduce that for all $\varkappa > 0$,

$$0 \le \eta\Big(\Gamma(\omega^*, u, \varkappa)\Big) \le \eta\Big(\Gamma(g\omega^*, gu, \varkappa)\Big) = \eta(1) = 0$$

which implies $\eta\Big(\Gamma(\omega^*, u, \varkappa)\Big) = 0$. Furthermore, $\Gamma(\omega^*, u, \varkappa) = 1$ whenever $\varkappa > 0$, that is, $\omega^* = u \in \Omega_0(\varkappa)$.

We now show that ω^* represents optimal coincidence point of the g and T. Since $T\omega^* \in T\left(\Omega_0(\varkappa)\right) \subseteq \Theta_0(\varkappa)$, there exists a point $z \in \Omega_0(\varkappa)$ satisfying $\Gamma(gz, T\omega^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$. Given that the pair (g, T) satisfy (3), we have

$$\left.\begin{aligned}\Gamma(g\omega_{n+1}, T\omega_n, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \\ \Gamma(gz, T\omega^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{aligned}\right\} \Rightarrow \alpha(\omega_n, \omega^*, \varkappa)\eta\left(\Gamma(g\omega_{n+1}, gz, \varkappa)\right) \le kM^*(\omega_{n+1}, z, \omega_n, \omega^*, \varkappa) \tag{9}$$

where

$$M^*(\omega_{n+1}, z, \omega_n, \omega^*, \varkappa) = \max\left\{\eta\left(\Gamma(\omega_n, \omega^*, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right), \eta\left(\Gamma(\omega^*, z, \varkappa)\right), \eta\left(\Gamma(\omega^*, \omega_{n+1}, \varkappa)\right)\right\}.$$

As $\alpha(\omega_n, \omega^*, \varkappa) \geq 1$ by condition (6), fuzzy expansive property of g as well as decreasing property of η on (0, 1], it follows that

$$\eta\left(\Gamma(\omega_{n+1}, z, \varkappa)\right) \leq \eta\left(\Gamma(g\omega_{n+1}, gz, \varkappa)\right) \leq \alpha(\omega_n, \omega^*, \varkappa)\eta\left(\Gamma(g\omega_{n+1}, gz, \varkappa)\right) \leq kM^*(\omega_{n+1}, z, \omega_n, \omega^*, \varkappa).$$

Let $n \to \infty$ in the inequality above, these yields

$$\begin{aligned}\eta\left(\Gamma(\omega^*, z, \varkappa)\right) &\leq k \max\left\{\eta\left(\Gamma(z, \omega^*, \varkappa)\right), 0, \eta\left(\Gamma(\omega^*, z, \varkappa)\right), \eta(1)\right\} \\ &= k \max\left\{\eta\left(\Gamma(\omega^*, z, \varkappa)\right), 0, \eta\left(\Gamma(\omega^*, z, \varkappa)\right), 0\right\} \\ &\leq k\eta\left(\Gamma(\omega^*, z, \varkappa)\right).\end{aligned}$$

If $\eta\left(\Gamma(\omega^*, z, \varkappa)\right) > 0$, then we have $k = 0$ which contradict with the fact that $k \in (0, 1)$. Thus, $\eta(\Gamma(\omega^*, z, \varkappa)) = 0$ and we obtain $\Gamma(\omega^*, z, \varkappa) = 1$ which implies $\omega^* = z$. Further,

$$\Gamma(g\omega^*, T\omega^*, \varkappa) = \Gamma(gz, T\omega^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$$

which suggests ω^* is an optimal coincidence point of g and T.

For the uniqueness part, consider $\theta^* \in \Omega_0(\varkappa)$ as optimal coincidence point of g and T such that $\omega^* \neq \theta^*$. Then, we have $\Gamma(g\omega^*, T\omega^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$ and $\Gamma(g\theta^*, T\theta^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$ for all $\varkappa > 0$. By (4), we have $\alpha(\omega^*, \theta^*, \varkappa) \geq 1$ for all $\varkappa > 0$. By (3), fuzzy expansive property of g and decreasing property of η, we have

$$\eta\left(\Gamma(\omega^*, \theta^*, \varkappa)\right) \leq \eta\left(\Gamma(g\omega^*, g\theta^*, \varkappa)\right) \leq \alpha(\omega^*, \theta^*, \varkappa)\eta\left(\Gamma(g\omega^*, g\theta^*, \varkappa)\right) \leq kM^*(\omega^*, \theta^*, \omega^*, \theta^*, \varkappa) \tag{10}$$

where

$$\begin{aligned}M^*(\omega^*, \theta^*, \omega^*, \theta^*, \varkappa) &= \max\left\{\eta\left(\Gamma(\omega^*, \theta^*, \varkappa)\right), \eta\left(\Gamma(\omega^*, \omega^*, \varkappa)\right), \eta\left(\Gamma(\theta^*, \theta^*, \varkappa)\right), \eta\left(\Gamma(\theta^*, \omega^*, \varkappa)\right)\right\} \\ &= \max\left\{\eta\left(\Gamma(\omega^*, \theta^*, \varkappa)\right), \eta(1), \eta(1), \eta\left(\Gamma(\theta^*, \omega^*, \varkappa)\right)\right\} \\ &= \max\left\{\eta\left(\Gamma(\omega^*, \theta^*, \varkappa)\right), 0, 0, \eta\left(\Gamma(\omega^*, \theta^*, \varkappa)\right)\right\} \\ &= \max\left\{\eta\left(\Gamma(\omega^*, \theta^*, \varkappa)\right), 0, 0, \eta\left(\Gamma(\omega^*, \theta^*, \varkappa)\right)\right\} \\ &= \eta\left(\Gamma(\omega^*, \theta^*, \varkappa)\right).\end{aligned}$$

From inequality (10) and $k \in (0, 1)$, we obtain

$$\eta\left(\Gamma(\omega^*, \theta^*, \varkappa)\right) \leq k\eta\left(\Gamma(\omega^*, \theta^*, \varkappa)\right) < \eta\left(\Gamma(\omega^*, \theta^*, \varkappa)\right)$$

which is a contradiction. For this reason, the optimal coincidence point is unique.

We present an example below to illustrate the above theorem.

Example 1 Let $Z = [0, 1] \times \mathbb{R}$, $\Omega = \{(0, \omega) : \omega \geq 0, \omega \in \mathbb{R}\}$ and $\Theta = \{(1, \theta) : \theta \geq 0, \theta \in \mathbb{R}\}$. We define a metric $d : Z \times Z \to [0, \infty)$ by

$$d\big((\omega_1, \theta_1), (\omega_2, \theta_2)\big) = \sqrt{(\omega_1 - \omega_2)^2 + (\theta_1 - \theta_2)^2}.$$

Then, $(Z, \Gamma, *_p)$ is a complete fuzzy metric space with $\Gamma : Z \times Z \times (0, \infty) \to [0, 1]$ is defined as

$$\Gamma(\omega, \theta, \varkappa) = \frac{\varkappa}{\varkappa + d(\omega, \theta)}.$$

We can check that

$$\Gamma(\Omega, \Theta, \varkappa) = \frac{\varkappa}{\varkappa + 1}$$

for all $\varkappa > 0$. In addition to that, we have $\Omega_0(\varkappa) = \Omega$ and $\Theta_0(\varkappa) = \Theta$. $T : \Omega \to \Theta$ and $g : \Omega \to \Omega$ is defined by

$$T(0, \omega) = \left(1, \frac{\omega}{5}\right) \text{ and } g(0, \omega) = (0, 5\omega).$$

One can observed that g can be identified as fuzzy expansive, $T(\Omega_0(\varkappa)) \subseteq \Theta_0(\varkappa)$ and $\Omega_0(\varkappa) \subseteq g(\Omega_0(\varkappa))$ for all $\varkappa > 0$. For points $\mu = (0, \omega_1), \nu = (0, \omega_2) \in \Omega$, there exist $\omega = (0, \theta_1), \theta = (0, \theta_2) \in \Omega$ such that

$$\Gamma(g\mu, T\omega, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \text{ and } \Gamma(g\nu, T\theta, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$$

for all $\varkappa > 0$ if $\omega_1 = \frac{\theta_1}{25}$ and $\omega_2 = \frac{\theta_2}{25}$. Consider $\eta(\varsigma) = \frac{1}{\varsigma} - 1$ for all $\varsigma \in (0, 1]$. Also, define $\alpha : Z \times Z \times (0, \infty) \to [0, \infty)$ as

$$\alpha(\omega, \theta, \varkappa) = \begin{cases} 1, & \omega, \theta \in [0, 1] \times [0, \infty), \\ 0, & \text{otherwise.} \end{cases}$$

for all $\varkappa > 0$. We can verify that (g, T) corresponds to fuzzy α-proximal quasi-H-contraction of type II. Consequently, every criteria of Theorem 1 is fulfilled. Furthermore, $(0, 0)$ in $\Omega_0(\varkappa)$ is the only coincidence point that is optimal for both g and T.

Corollary 1 Given that Ω, Θ are nonempty closed subsets of a complete fuzzy metric space $(Z, \Gamma, *)$ in which $\Omega_0(\varkappa) \neq \varnothing$ and Θ is fuzzy approximately compact with respect to Ω. Assume $T : \Omega \to \Theta$ satisfying conditions below:

1. T is α-proximal admissible as well as fuzzy α-proximal quasi-H-contraction of type I;
2. $T(\Omega_0(\varkappa)) \subseteq \Theta_0(\varkappa)$;
3. here exist elements ω_0 and ω_1 in $\Omega_0(\varkappa)$ satisfying

$$\Gamma(g\omega_1, T\omega_0, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \text{ and } \alpha(\omega_0, \omega_1, \varkappa) \geq 1$$

for all $\varkappa > 0$ and $\{\eta(\Gamma(\omega_0, \omega_1, \varkappa_i))\}$ is bounded for any sequence $\{\varkappa_i\}$ in $(0, \infty)$ with $\varkappa_i \to 0$ as $i \to \infty$;

4. if a sequence $\{\omega_n\}$ in Ω satisfying $\alpha(\omega_n, \omega_{n+1}, \varkappa) \geq 1$ for every $n \in \mathbb{N}$, $\varkappa > 0$ and $\omega_n \to \omega^*$ as $n \to \infty$, then $\alpha(\omega_n, \omega^*, \varkappa) \geq 1$.

It follows that T possess best proximity point ω^* in $\Omega_0(\varkappa)$. Furthermore, if for all $\omega^*, \theta^* \in \Omega_0(\varkappa)$ such that

$$\left.\begin{array}{l}\Gamma(\omega^*,\ T\omega^*, \varkappa)=\Gamma(\Omega, \Theta, \varkappa)\\ \Gamma(\theta^*,\ T\theta^*, \varkappa)=\Gamma(\Omega, \Theta, \varkappa)\end{array}\right\}\Rightarrow \alpha(\omega^*,\ \theta^*, \varkappa)\geq 1$$

for each $\varkappa > 0$, then the best proximity point is uniquely determined.

Proof. The deduction follows from Theorem 1 by letting $g = I_\omega$.

Corollary 2 Given that Ω, Θ are nonempty closed subsets of a complete fuzzy metric space $(Z, \Gamma, *)$ such that $\Omega_0(\varkappa) \neq \emptyset$ and Θ is fuzzy approximately compact with respect to Ω. Assume that $g: \Omega \to \Omega$ and $T: \Omega \to \Theta$ satisfying conditions below:

1. g is continuous and fuzzy isometry;
2. both g and T are α_R-admissible mapping and α-proximal admissible mappings respectively;
3. (g, T) can be regarded as fuzzy α-proximal quasi-H-contraction of type II;
4. $\Omega_0(\varkappa) \subseteq g(\Omega_0(\varkappa))$ and $T(\Omega_0(\varkappa)) \subseteq \Theta_0(\varkappa)$;
5. there exist elements ω_0 and ω_1 in $\Omega_0(\varkappa)$ satisfying

$$\Gamma(g\omega_1, T\omega_0, \varkappa)=\Gamma(\Omega, \Theta, \varkappa) \ \text{ and } \ \alpha(\omega_0, \omega_1, \varkappa)\geq 1$$

for all $\varkappa > 0$ and $\{\eta(\Gamma(\omega_0, \omega_1, \varkappa_i))\}$ is bounded for any sequence $\{\varkappa_i\}$ in $(0, \infty)$ with $\varkappa_i \to 0$ as $i \to \infty$;

6. if a sequence $\{\omega_n\}$ in Ω satisfying $\alpha(\omega_n, \omega_{n+1}, \varkappa) \geq 1$ for every $n \in \mathbb{N}, \varkappa > 0$ and $\omega_n \to \omega^*$ as $n \to \infty$ then $\alpha(\omega_n, \omega^*, \varkappa) \geq 1$.

It follows that g and T possess optimal coincidence point ω^* in $\Omega_0(\varkappa)$. Furthermore, if for all $\omega^*, \theta^* \in \Omega_0(\varkappa)$ such that

$$\left.\begin{array}{l}\Gamma(g\omega^*, T\omega^*, \varkappa)=\Gamma(\Omega, \Theta, \varkappa)\\ \Gamma(g\theta^*, T\theta^*, \varkappa)=\Gamma(\Omega, \Theta, \varkappa)\end{array}\right\}\Rightarrow \alpha(\omega^*,\ \theta^*, \varkappa)\geq 1 \tag{11}$$

for all $\varkappa > 0$, then the optimal coincidence point is uniquely determined.

Proof. Given that g is fuzzy isometry, for every $\omega, \theta \in \Omega$ and $\varkappa > 0$ we have $\Gamma(g\omega, g\theta, \varkappa) = \Gamma(\omega, \theta, \varkappa)$ which implies that g exhibit fuzzy expansive property. The conclusion can be derived from Theorem 1.

Corollary 3 Given that Ω, Θ are nonempty closed subsets of a complete fuzzy metric space $(Z, \Gamma, *)$ in which $\Omega_0(\varkappa) \neq \emptyset$ and Θ is fuzzy approximately compact with respect to Ω. Assume that $g: \Omega \to \Omega$ and $T: \Omega \to \Theta$ satisfying conditions below:

1. g is continuous and fuzzy expansive;
2.

$$\left.\begin{array}{l}\Gamma(\mu, T\omega, \varkappa)=\Gamma(\Omega, \Theta, \varkappa)\\ \Gamma(\nu, T\theta, \varkappa)=\Gamma(\Omega, \Theta, \varkappa)\end{array}\right\}\Rightarrow \eta\left(\Gamma(g\mu, g\nu, \varkappa)\right)\leq k\Gamma^*(\mu, \nu, \omega, \theta, \varkappa), \tag{12}$$

where $\Gamma^*(\mu, \nu, \omega, \theta, \varkappa) = \max\left\{\eta\left(\Gamma(\omega, \theta, \varkappa)\right), \eta\left(\Gamma(\omega, \mu, \varkappa)\right),\ \eta\left(\Gamma(\theta, \nu, \varkappa)\right),\ \eta\left(\Gamma(\theta, \mu, \varkappa)\right)\right\}, \eta \in H$ and $k \in (0, 1)$;

3. $\Omega_0(\varkappa) \subseteq g(\Omega_0(\varkappa))$ and $T(\Omega_0(\varkappa)) \subseteq \Theta_0(\varkappa)$;
4. there exist ω_0 and ω_1 in $\Omega_0(\varkappa)$ satisfying

$$\Gamma(g\omega_1, T\omega_0, \varkappa)=\Gamma(\Omega, \Theta, \varkappa) \text{ and } \alpha(\omega_0, \omega_1, \varkappa)\geq 1$$

for all $\varkappa > 0$ and $\left\{\eta\left(\Gamma(\omega_0, \omega_1, \varkappa_i)\right)\right\}$ is bounded for any sequence $\{\varkappa_i\}$ in $(0, \infty)$ with $\varkappa_i \to 0$ as $i \to \infty$;

It follows that the mappings g and T possess unique optimal coincidence point ω^* in $\Omega_0(\varkappa)$.

Proof. The deduction follows from Theorem 1 by letting $\alpha(\omega, \theta, \varkappa) = 1$ whenever $\omega, \theta \in \Omega_0(\varkappa)$ and $\varkappa > 0$.

Corollary 4 Given that Ω, Θ are nonempty closed subsets of a complete fuzzy metric space $(Z, \Gamma, *)$ in which $\Omega_0(\varkappa) \neq \emptyset$ and Θ is fuzzy approximately compact with respect to Ω. Assume that $g: \Omega \to \Omega$ and $T: \Omega \to \Theta$ satisfying conditions below:

1. g is continuous and fuzzy expansive;

2. both g and T are α_R-admissible mapping and α-proximal admissible mappings respectively;

3.

$$\left.\begin{aligned}\Gamma(g\mu, T\omega, \varkappa) &= \Gamma(\Omega, \Theta, \varkappa)\\ \Gamma(g\nu, T\theta, \varkappa) &= \Gamma(\Omega, \Theta, \varkappa)\end{aligned}\right\} \Rightarrow \alpha(\omega, \theta, t)\eta\left(\Gamma(g\mu, g\nu, \varkappa)\right) \le k\eta\left(M(\omega, \theta, \varkappa)\right);$$

4. $\Omega_0(\varkappa) \subseteq g\left(\Omega_0(\varkappa)\right)$ and $T\left(\Omega_0(\varkappa)\right) \subseteq \Theta_0(\varkappa)$;

5. there exist elements ω_0 and ω_1 in $\Omega_0(\varkappa)$ satisfying

$$\Gamma(g\omega_1, T\omega_0, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \text{ and } \alpha(\omega_0, \omega_1, \varkappa) \ge 1$$

for all $\varkappa > 0$ and $\left\{\eta\left(\Gamma(\omega_0, \omega_1, \varkappa_i)\right)\right\}$ is bounded for any sequence $\{\varkappa_i\}$ in $(0, \infty)$ with $\varkappa_i \to 0$ as $i \to \infty$;

6. if a sequence $\{\omega_n\}$ in Ω satisfying $\alpha(\omega_n, \omega_{n+1}, \varkappa) \ge 1$ whenever $n \in \mathbb{N}$, $\varkappa > 0$ and $\omega_n \to \omega^*$ as $n \to \infty$, then $\alpha(\omega_n, \omega^*, \varkappa) \ge 1$.

It follows that g and T possess optimal coincidence point ω^* in $\Omega_0(\varkappa)$. Furthermore, if for all $\omega^*, \theta^* \in \Omega_0(\varkappa)$ such that

$$\left.\begin{aligned}\Gamma(g\omega^*, T\omega^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\\ \Gamma(g\theta^*, T\theta^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{aligned}\right\} \Rightarrow \alpha(\omega^*, \theta^*, \varkappa) \ge 1 \tag{13}$$

for all $\varkappa > 0$, then the optimal coincidence point is uniquely determined.

Proof. The deduction follows from Theorem 1 by letting $\Gamma^*(\mu, \nu, \omega, \theta, \varkappa) = \eta\left(\Gamma(\omega, \theta, \varkappa)\right)$.

Remark 2 The corollary above is Theorem 1 in Saleem et al. [21]. This shows that our result generalizes the existed results in literature. Note that they didn't mention anything about condition (6) and Equation (13). However, these conditions must be included in order to complete their proofs.

Remark 3 Replacing the space with fuzzy b-metric space, Theorem 1 could provide an extension of findings obtained by Abbas et al. [17] and Saleem et al. [20].

Corollary 5 Given that Ω, Θ are nonempty closed subsets of a complete fuzzy metric space $(Z, \Gamma, *)$ in which $\Omega_0(\varkappa) \neq \varnothing$ and Θ is fuzzy approximately compact with respect to Ω. Assume that $g: \Omega \to \Omega$ and $T: \Omega \to \Theta$ satisfying conditions below:

1. g is continuous and fuzzy expansive;

2.

$$\left.\begin{aligned}\Gamma(g\mu, T\omega, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\\ \Gamma(g\nu, T\theta, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{aligned}\right\} \Rightarrow \eta\left(\Gamma(g\mu, g\nu, \varkappa)\right) \le k\eta\left(\Gamma(\omega, \theta, \varkappa)\right);$$

3. $\Omega_0(\varkappa) \subseteq g\left(\Omega_0(\varkappa)\right)$ and $T\left(\Omega_0(\varkappa)\right) \subseteq \Theta_0(\varkappa)$;

4. there exist elements ω_0 and ω_1 in $\Omega_0(\varkappa)$ satisfying

$$\Gamma(g\omega_1, T\omega_0, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \text{ and } \alpha(\omega_0, \omega_1, \varkappa) \ge 1$$

for all $\varkappa > 0$ and $\left\{\eta\left(\Gamma(\omega_0, \omega_1, \varkappa_i)\right)\right\}$ is bounded for any sequence $\{\varkappa_i\}$ in $(0, \infty)$ with $\varkappa_i \to 0$ as $i \to \infty$;

It follows that g and T possess optimal coincidence point ω^* in $\Omega_0(\varkappa)$.

Proof. The deduction follows from Theorem 1 by letting $\alpha(\omega, \theta, \varkappa) = 1$, $\Gamma^*(\mu, \nu, \omega, \theta, \varkappa) = \eta\ (\Gamma(\omega,\ \theta,\ \varkappa))$ whenever $\omega, \theta \in \Omega_0(\varkappa)$ and $\varkappa > 0$.

Corollary 6 Given that Ω, Θ are nonempty closed subsets of a complete fuzzy metric space $(Z, \Gamma, *)$ in which $\Omega_0(\varkappa) \neq \varnothing$ and Θ is fuzzy approximately compact with respect to Ω. Assume that $T: \Omega \to \Theta$ be mapping satisfying $T\left(\Omega_0(\varkappa)\right) \subseteq \Theta_0(\varkappa)$ and

$$\left.\begin{aligned}\Gamma(\mu, T\omega, \varkappa) &= \Gamma(\Omega, \Theta, \varkappa)\\ \Gamma(\nu, T\theta, \varkappa) &= \Gamma(\Omega, \Theta, \varkappa)\end{aligned}\right\} \Rightarrow \eta\left(\Gamma(\mu, \nu, \varkappa)\right) \le k\eta\left(\Gamma(\omega, \theta, \varkappa)\right)$$

where $\eta \in H$ and $k \in (0,1)$. It follows that T possess unique best proximity point ω^* in $\Omega_0(\varkappa)$.

Proof. The deduction follows from Theorem 1 by letting $\alpha(\omega, \theta, \varkappa) = 1$, and $\Gamma^*(\mu, \nu, \omega, \theta, \varkappa) = \eta(\Gamma(\omega, \theta, \varkappa))$ and $g = I_\omega$ whenever $\omega, \theta \in \Omega_0(\varkappa)$ and $\varkappa > 0$.

Remark 4 Note that non-Archimedean fuzzy metric spaces implied fuzzy metric space. Therefore, the corollary above is applicable to the results in Vetro et al. [14] and Saleem et al. [15].

We present an alternate condition for contractive mapping in the result below.

Theorem 2 Given that Ω, Θ are nonempty closed subsets of a complete fuzzy metric space $(Z, \Gamma, *)$ in which $\Omega_0(\varkappa) \neq \varnothing$ and Θ is fuzzy approximately compact with respect to Ω. Let $g: \Omega \to \Omega$ and $T: \Omega \to \Theta$ satisfies the following conditions:

1. g is continuous and fuzzy expansive;
2. both g and T are α_R-admissible and α-proximal admissible mappings respectively;
3.

$$\left.\begin{array}{l}\Gamma(g\mu, T\omega, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\\ \Gamma(g\nu, T\theta, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{array}\right\} \Rightarrow \alpha(\omega, \theta, \varkappa)\eta(\Gamma(g\mu, g\nu, \varkappa)) \leq k\Gamma^*(\mu, \nu, \omega, \theta, \varkappa), \tag{14}$$

where $\eta \in H$, $k \in (0, 1)$ and

$$\Gamma^*(\mu, \nu, \omega, \theta, \varkappa) = \max\left\{\eta(\Gamma(\omega, \theta, \varkappa)), \eta\left(\frac{\Gamma(\omega, \mu, \varkappa)[1 + \Gamma(\theta, \nu, \varkappa)]}{1 + \Gamma(\mu, \nu, \varkappa)}\right), \eta\left(\frac{\Gamma(\theta, \nu, \varkappa)[1 + \Gamma(\omega, \mu, \varkappa)]}{1 + \Gamma(\omega, \theta, \varkappa)}\right),\right.$$

$$\left.\eta\left(\frac{\Gamma(\theta, \mu, \varkappa)[1 + \Gamma(\omega, \mu, \varkappa)]}{1 + \Gamma(\omega, \theta, \varkappa)}\right)\right\};$$

4. $\Omega_0(\varkappa) \subseteq g(\Omega_0(\varkappa))$ and $T(\Omega_0(\varkappa)) \subseteq \Theta_0(\varkappa)$;
5. there exist elements ω_0 and ω_1 in $\Omega_0(\varkappa)$ such that

$$\Gamma(g\omega_1, T\omega_0, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \text{ and } \alpha(\omega_0, \omega_1, \varkappa) \geq 1$$

for all $\varkappa > 0$ and $\{\eta(\Gamma(\omega_0, \omega_1, \varkappa_i))\}$ is bounded for any sequence $\{\varkappa_i\}$ in $(0, \infty)$ with $\varkappa_i \to 0$ as $i \to \infty$;

6. if a sequence $\{\omega_n\}$ in Ω satisfying $\alpha(\omega_n, \omega_{n+1}, \varkappa) \geq 1$ whenever $n \in \mathbb{N}$, $\varkappa > 0$ and $\omega_n \to \omega^*$ as $n \to \infty$, then $\alpha(\omega_n, \omega^*, \varkappa) \geq 1$.

It follows that g and T possess optimal coincidence point ω^* in $\Omega_0(\varkappa)$. Furthermore, if for all $\omega^*, \theta^* \in \Omega_0(\varkappa)$ such that

$$\left.\begin{array}{l}\Gamma(g\omega^*, T\omega^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\\ \Gamma(g\theta^*, T\theta^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{array}\right\} \Rightarrow \alpha(\omega^*, \theta^*, \varkappa) \geq 1$$

for each $\varkappa > 0$, then the optimal coincidence point is uniquely determined.

Proof. The present theorem is proven using methodology as Theorem 1 with the replacement of (3) with (14). Additionally, on (5), we have $\Gamma^*(\omega_n, \omega_{n+1}, \omega_{n-1}, \omega_n, \varkappa)$ changed as follows:

$$\Gamma^*(\omega_n, \omega_{n+1}, \omega_{n-1}, \omega_n, \varkappa) = \max\left\{\eta(\Gamma(\omega_{n-1}, \omega_n, \varkappa)), \eta\left(\frac{\Gamma(\omega_{n-1}, \omega_n, \varkappa)[1 + \Gamma(\omega_n, \omega_{n+1}, \varkappa)]}{1 + \Gamma(\omega_n, \omega_{n+1}, \varkappa)}\right),\right.$$

$$\left.\eta\left(\frac{\Gamma(\omega_n, \omega_{n+1}, \varkappa)[1 + \Gamma(\omega_{n-1}, \omega_n, \varkappa)]}{1 + \Gamma(\omega_{n-1}, \omega_n, \varkappa)}\right), \eta\left(\frac{\Gamma(\omega_n, \omega_n, \varkappa)[1 + \Gamma(\omega_{n-1}, \omega_n, \varkappa)]}{1 + \Gamma(\omega_{n-1}, \omega_n, \varkappa)}\right)\right\}$$

$$= \max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right), \eta(1)\right\}$$

$$= \max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right), 0\right\}$$

$$\leq \max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right)\right\}$$

for each $\varkappa > 0$.

For the next result, we relax the contractive condition further and establish the presence of unique optimal coincidence point for such contraction with continuous t-norm restricted to $*_{\min}$.

Theorem 3 Given that Ω, Θ are nonempty closed subsets of a complete fuzzy metric space $(X, M, *_{\min})$ in a way that $\Omega_0(\varkappa) \neq \emptyset$ and Θ is fuzzy approximately compact with respect to Ω. Let $g: \Omega \to \Omega$ and $T: \Omega \to \Theta$ meet conditions below:

1. g is continuous and fuzzy expansive;
2. both g and T are α_R-admissible and α-proximal admissible mappings respectively;
3.

$$\left.\begin{array}{l}\Gamma(g\mu, T\omega, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\\ \Gamma(g\nu, T\theta, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{array}\right\} \Rightarrow \alpha(\omega, \theta, \varkappa)\eta\left(\Gamma(g\mu, g\nu, \varkappa)\right) \leq k\Gamma^*(\mu, \nu, \omega, \theta, \varkappa), \tag{15}$$

where $\Gamma^*\left(\mu, \nu, \omega, \theta, \varkappa\right) = \max\left\{\eta\left(\Gamma(\omega, \theta, \varkappa)\right), \eta\left(\Gamma(\omega, \mu, \varkappa)\right), \eta\left(\Gamma(\theta, \nu, \varkappa)\right), \eta\left(\Gamma(\theta, \mu, \varkappa)\right), \eta\left(\Gamma(\omega, \nu, 2\varkappa)\right)\right\}$, $\eta \in H$ and $k \in (0, 1)$;

4. $\Omega_0(\varkappa) \subseteq g\left(\Omega_0(\varkappa)\right)$ and $T\left(\Omega_0(\varkappa)\right) \subseteq \Theta_0(\varkappa)$;
5. there exist elements ω_0 and ω_1 in $\Omega_0(\varkappa)$ such that

$$\Gamma(g\omega_1, T\omega_0, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \text{ and } \alpha(\omega_0, \omega_1, \varkappa) \geq 1$$

for all $\varkappa > 0$ and $\left\{\eta\left(\Gamma(\omega_0, \omega_1, \varkappa_i)\right)\right\}$ is bounded for any sequence $\{\varkappa_i\}$ in $(0, \infty)$ with $\varkappa_i \to 0$ as $i \to \infty$;

6. if a sequence $\{\omega_n\}$ in Ω satisfying $\alpha(\omega_n, \omega_{n+1}, \varkappa) \geq 1$ whenever $n \in \mathbb{N}$, $\varkappa > 0$ and $\omega_n \to \omega^*$ as $n \to \infty$ it follows that $\alpha(\omega_n, \omega^*, \varkappa) \geq 1$.

It follows that g and T possess optimal coincidence point ω^* in $\Omega_0(\varkappa)$. Furthermore, if for all $\omega^*, \theta^* \in \Omega_0(\varkappa)$ such that

$$\left.\begin{array}{l}\Gamma(g\omega^*, T\omega^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\\ \Gamma(g\theta^*, T\theta^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{array}\right\} \Rightarrow \alpha(\omega^*, \theta^*, \varkappa) \geq 1$$

for all $\varkappa > 0$, then the optimal coincidence point is uniquely determined.

Proof. The present theorem is proven using methodology as Theorem 1 with the replacement of (3) with (15). Additionally, on (5), we have $\Gamma^*(\omega_n, \omega_{n+1}, \omega_{n-1}, \omega_n, \varkappa)$ changed as follows:

$$\Gamma^*(\omega_n, \omega_{n+1}, \omega_{n-1}, \omega_n, \varkappa)$$

$$= \max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_{n-1}, \omega_{n+1}, 2\varkappa)\right)\right\}$$

$$\leq \max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right), \eta(1), \eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa) * \Gamma(\omega_n, \omega_{n+1}, \varkappa)\right)\right\}$$

$$\leq \max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right), \eta\left(\min\left\{\Gamma(\omega_{n-1}, \omega_n, \varkappa), \Gamma(\omega_n, \omega_{n+1}, \varkappa)\right\}\right)\right\}$$

$$\leq \max\left\{\eta\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right), \eta\left(\Gamma(\omega_n, \omega_{n+1}, \varkappa)\right)\right\}$$

Remark 5 The above theorem extends Amini-Harandi et al. work [25] from fixed point theory to best proximity point theory. Note that if fuzzy metric space imposes non-Archimedean property, then arbitrary t-norm can be applied for the result above.

4. Best proximity points of multivalued proximal contraction in strong fuzzy metric spaces: Fuzzy ψ-proximal contraction

The current section deals with obtaining the best proximity point outcome for new multivalued proximal contraction within the parameters of strong fuzzy metric space. Our starting point is the definition of Hausdorff fuzzy metric, along with a lemma by Rodríguez-López et al. [26].

Definition 15 [26] Let $K(Z)$ consist all non-null compact subsets of Z. For any $\omega \in Z$ and $\Omega, \Theta \in K(Z)$, we define the Hausdorff fuzzy metric derived from the fuzzy metric Γ in the following way:

$$H(\Omega, \Theta, \varkappa) = \min\left\{\inf_{a\in\Omega} \Gamma(a, \Theta, \varkappa), \inf_{b\in\Theta} \Gamma(\Omega, b, \varkappa)\right\}$$

for each $\varkappa > 0$, where $\Gamma(\omega, \Omega, \varkappa) = \sup_{a\in\Omega} \Gamma(\omega, a, \varkappa)$.

Lemma 2 [26] Considering a fuzzy metric space, $(Z, \Gamma, *)$. Assume that $a \in Z, \Theta \in K(Z)$ as well as $\varkappa > 0$, one can find $b_0 \in \Theta$ satisfying $\Gamma(a, \Theta, \varkappa) = \Gamma(a, b_0, \varkappa)$.

Next, we have a class Ψ given by Miheţ [27].

Definition 16 [27] Denote Ψ as the collection of continuous function $\psi : [0, 1] \to [0, 1]$ that meet requirements below:

1. ψ is nondecreasing function;
2. $\psi(\varsigma) > \varsigma$ for all $\varsigma \in (0, 1)$;
3. $\psi(0) = 0$ and $\psi(1) = 1$;
4. $\lim_{n\to\infty} \psi^n(\varsigma) = 1$ for all $\varsigma \in (0, 1)$, where ψ^n refers to the mapping obtained by composing ψ with itself n times.

Definition 17 [28] Nonempty subsets Ω, Θ of $(Z, \Gamma, *)$ in which $\Omega_0 \neq \varnothing$ are considered. The fuzzy weak P-property is attributed to the pair (Ω, Θ) if and only if

$$\left.\begin{aligned}\Gamma(\omega_1, \theta_1, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\\ \Gamma(\omega_2, \theta_2, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{aligned}\right\} \Rightarrow \Gamma(\omega_1, \omega_2, \varkappa) \geq \Gamma(\theta_1, \theta_2, \varkappa)$$

for all $\omega_1, \omega_2 \in \Omega_0, \theta_1, \theta_2 \in \theta_0$ and $\varkappa > 0$.

We define a new fuzzy proximal contraction for multivalued mapping as follow.

Definition 18 A multivalued mapping $T : \Omega \to K(\Theta)$ is identified as fuzzy ψ-proximal contraction provided that at least one $\psi \in \Psi$ exists such that for all $\omega, \theta \in \Omega$ and $\varkappa > 0$,

$$H(T\omega, T\theta, \varkappa) \geq \psi(\Gamma(\omega, \theta, \varkappa)).$$

Our task now is to prove the appearance of optimal proximity point on multivalued mapping with such contractive condition within the context of strong fuzzy metric space.

Theorem 4 Given that Ω, Θ are nonempty closed subsets of a complete strong fuzzy metric space $(Z, \Gamma, *)$ in which $\Omega_0(\varkappa) \neq \varnothing$. Consider $T : \Omega \to K(\Theta)$ that meet the following conditions:

1. T is a continuous fuzzy ψ-proximal contraction,
2. $T(\omega) \subseteq \Theta_0(\varkappa)$ for each $\omega \in \Omega_0(\varkappa)$ and $\varkappa > 0$,
3. The fuzzy weak P-property is present in the pair (Ω, Θ).

Consequently, best proximity point of T can be found in Ω.

Proof. As $\Omega_0(\varkappa) \neq \varnothing$, select an arbitrary element $\omega_0 \in \Omega_0(\varkappa)$ and choose $\theta_1 \in T\omega_0$. Since $\theta_1 \in T\omega_0 \subseteq \Theta_0(\varkappa)$ for each

$\varkappa > 0$, one can find $\omega_1 \in \Omega_0(\varkappa)$ such that $\Gamma(\omega_1, \theta_1, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$. If $\theta_1 \in T\omega_1$, then

$$\Gamma(\Omega, \Theta, \varkappa) \geq \Gamma(\omega_1, T\omega_1, \varkappa) \geq \Gamma(\omega_1, \theta_1, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$$

which implies $\Gamma(\omega_1, T\omega_1, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$ and T attains its best proximity point at ω_1. Now consider the case where $\theta_1 \notin T\omega_1$. Since T is fuzzy ψ-proximal contraction, it follows that

$$\Gamma(\theta_1, T\omega_1, \varkappa) \geq H(T\omega_0, T\omega_1, \varkappa) \geq \psi\left(\Gamma(\omega_0, \omega_1, \varkappa)\right) \tag{16}$$

for all $\varkappa > 0$. Given that $T\omega_1$ is compact, by Lemma 2, one can find $\theta_2 \in T\omega_1$ in a way that

$$\Gamma(\theta_1, T\omega_1, \varkappa) = \Gamma(\theta_1, \theta_2, \varkappa) \tag{17}$$

for all $\varkappa > 0$. From (16) and (17), we obtain

$$\Gamma(\theta_1, \theta_2, \varkappa) \geq \psi\left(\Gamma(\omega_0, \omega_1, \varkappa)\right) \tag{18}$$

for all $\varkappa > 0$.

Again, as $\theta_2 \in T\omega_1 \subseteq \Theta_0(\varkappa)$, one can find $\omega_2 \in \Omega_0(\varkappa)$ such that $\Gamma(\omega_2, \theta_2, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$ for each $\varkappa > 0$. If $\omega_1 = \omega_2$, then

$$\Gamma(\Omega, \Theta, \varkappa) \geq \Gamma(\omega_1, T\omega_1, \varkappa) \geq \Gamma(\omega_1, \theta_2, \varkappa) = \Gamma(\omega_2, \theta_2, \varkappa) = \Gamma(\Omega, \Theta, \varkappa).$$

Thus, we have $\Gamma(\omega_1, T\omega_1, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$ for all $\varkappa > 0$ and T attains its best proximity point at ω_1. Suppose $\omega_1 \neq \omega_2$, the fuzzy weak P-property of (Ω, Θ) leads to

$$\left.\begin{array}{l}\Gamma(\omega_1, \theta_1, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\\ \Gamma(\omega_2, \theta_2, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{array}\right\} \Rightarrow \Gamma(\omega_1, \omega_2, \varkappa) \geq \Gamma(\theta_1, \theta_2, \varkappa) \tag{19}$$

for all $\varkappa > 0$. From (18) and (19), it is evident that

$$\Gamma(\omega_1, \omega_2, \varkappa) \geq \psi\left(\Gamma(\omega_0, \omega_1, \varkappa)\right)$$

for all $\varkappa > 0$. Assume that $\theta_2 \notin T\omega_2$, otherwise T attains its best proximity point at ω_2. By the contractive contraction of T,

$$\Gamma(\theta_2, T\omega_2, \varkappa) \geq H(T\omega_1, T\omega_2, \varkappa) \geq \psi\left(\Gamma(\omega_1, \omega_2, \varkappa)\right) \tag{20}$$

for all $\varkappa > 0$. Since $T\omega_2$ is compact, by Lemma (2), it is possible to find a point $\theta_3 \in T\omega_2$ satisfying

$$\Gamma(\theta_2, T\omega_2, \varkappa) = \Gamma(\theta_2, \theta_3, \varkappa) \tag{21}$$

for each $\varkappa > 0$. Using (20) and (21), it follows that

$$\Gamma(\theta_2, \theta_3, \varkappa) \geq \psi\left(\Gamma(\omega_1, \omega_2, \varkappa)\right) \quad \text{for all } \varkappa > 0. \tag{22}$$

Since $\theta_3 \in T\omega_2 \subseteq \Theta_0(\varkappa)$, one can find $\omega_3 \in \Omega_0(\varkappa)$ such that $\Gamma(\omega_3, \theta_3, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$ for each $\varkappa > 0$. Suppose $\omega_2 \neq \omega_3$; otherwise, T attains its best proximity point at ω_2. The fuzzy weak P-property of (Ω, Θ) give us

$$\left.\begin{array}{l}\Gamma(\omega_2, \theta_2, \varkappa) = \Gamma(\Omega, \Theta, \varkappa) \\ \Gamma(\omega_3, \theta_3, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)\end{array}\right\} \Rightarrow \Gamma(\omega_2, \omega_3, \varkappa) \geq \Gamma(\theta_2, \theta_3, \varkappa) \tag{23}$$

for all $\varkappa > 0$. From (22) and (23), we have

$$\Gamma(\omega_2, \omega_3, \varkappa) \geq \psi\left(\Gamma(x_1, \omega_2, \varkappa)\right).$$

Continuing in this manner, we obtain two sequences, $\{\omega_n\} \subseteq \Omega_0(\varkappa)$ and $\{\theta_n\} \subseteq \Theta_0(\varkappa)$ for each $n \in \mathbb{N}$ and $\varkappa > 0$ satisfying:

1. $\omega_n \neq \omega_{n+1}$;
2. $\theta_n \in T\omega_{n-1}$ and $\theta_n \notin T\omega_n$;
3. $\Gamma(\omega_n, \theta_n, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$.

Also, it can be observed that

$$\Gamma(\omega_n, \omega_{n+1}, \varkappa) \geq \psi\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right) \tag{24}$$

for each $\varkappa > 0$ and $n \in \mathbb{N}$. Successively apply (24) will leads to

$$\begin{aligned}\Gamma(\omega_n, \omega_{n+1}, \varkappa) &\geq \psi\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right) \\ &\geq \psi^2\left(\Gamma(x_{n-2}, \omega_{n-1}, \varkappa)\right) \\ &\geq \ldots \\ &\geq \psi^n\left(\Gamma(\omega_0, \omega_1, \varkappa)\right)\end{aligned}$$

for each $\varkappa > 0$ and $n \in \mathbb{N}$. Considering the above inequality as $n \to \infty$, it becomes clear that

$$\lim_{n\to\infty} \Gamma(\omega_n, \omega_{n+1}, \varkappa) = 1 \quad \forall \varkappa > 0.$$

Now, we will show that $\{\omega_n\}$ and $\{\theta_n\}$ are Cauchy sequences. Let $\iota \in (0, 1)$ as well as $m, n \in \mathbb{N}$ and $m > n$. Then

$$\begin{aligned}\Gamma(\omega_n, \omega_m, \varkappa) &\geq \Gamma(\omega_n, \omega_{n+1}, \varkappa) * \Gamma(\omega_{n+1}, \omega_{n+2}, \varkappa) * \cdots * \Gamma(\omega_{m-1}, \omega_m, \varkappa) \\ &\geq \psi^n\left(\Gamma(\omega_0, \omega_1, \varkappa)\right) * \psi^{n+1}\left(\Gamma(\omega_0, \omega_1, \varkappa)\right) * \cdots * \psi^{m-1}\left(\Gamma(\omega_0, \omega_1, \varkappa)\right).\end{aligned}$$

Taking limit as $n, m \to \infty$ in inequality above, it is evident that

$$\lim_{n, m\to\infty} \Gamma\left(\omega_n, \omega_m, \varkappa\right) = 1.$$

Therefore, given every $\iota \in (0,1)$ and $\varkappa > 0$ one can find $n_0 \in \mathbb{N}$ satisfying $\Gamma(\omega_n, \omega_m, \varkappa) > 1 - \iota$ for each $m > n > n_0$. It can be inferred that $\{\omega_n\}$ is Cauchy. A similar approach is applied to show sequence $\{\theta_n\}$ is Cauchy.

Since Ω, Θ are closed and $(Z, \Gamma, *)$ is complete, there exist $\omega^* \in \Omega$ and $\theta^* \in \Theta$ satisfying $\omega_n \to \omega^*$ and $\theta_n \to \theta^*$. As $\Gamma(\omega_n, \theta_n, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$ for every $n \in \mathbb{N}$, we deduce that $\Gamma(\omega^*, \theta^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$. Owning to the continuity of T, it can be shown that $T\omega_n \to T\omega^*$ as $n \to \infty$. We will show that $\theta^* \in T\omega^*$. Note that

$$M(\theta_n, T\omega_n, \varkappa) \geq H(T\omega_{n-1}, T\omega_n, \varkappa) \geq \psi\left(\Gamma(\omega_{n-1}, \omega_n, \varkappa)\right).$$

For inequality above, taking limit as $n \to \infty$ and it leads to

$$\Gamma(\theta^*, T\omega^*, \varkappa) = \psi(1) = 1$$

which implies $\theta^* \in T\omega^*$. Furthermore,

$$\Gamma(\Omega, \Theta, \varkappa) \geq \Gamma(\omega^*, T\omega^*, \varkappa) \geq \Gamma(\omega^*, \theta^*, \varkappa) = \Gamma(A, B, \varkappa)$$

which implies that $\Gamma(\omega^*, T\omega^*, \varkappa) = \Gamma(\Omega, \Theta, \varkappa)$. Hence ω qualifies as best proximity point of T.

The practical application of Theorem 4 can be exemplified by example below.

Example 2 Consider $Z = [0, \infty]$ and a fuzzy metric Γ defined as follow:

$$\Gamma(\omega, \theta, \varkappa) = \frac{\min\{\omega, \theta\}}{\max\{\omega, \theta\}}$$

for all $\omega, \theta \in Z$ and $\varkappa > 0$. Clearly, $(Z, \Gamma, *_p)$ forms a strong fuzzy metric space. Consider $\Omega = [0, 10]$ and $\Theta = [0, 5]$. It is obvious that $\Gamma(\Omega, \Theta, \varkappa) = 1$, $\Omega_0(\varkappa) = [0, 5]$ and $\theta_0 = [0, 5]$. Let $\psi \in \Psi$ define by $\psi(\varrho) = \sqrt{\varrho}$, where $\varrho \in [0, 1]$. Moreover, let continuous multivalued mapping $T : \Omega \to K(\Theta)$ define by

$$T(\omega) = [\sqrt{\omega}, \sqrt{\omega + 2}], \quad \omega \in \Omega.$$

It is straightforward to see that (Ω, Θ) exhibits fuzzy weak P-property. Also, we have $T(\omega_0(\varkappa) \subseteq \Theta_0(\varkappa))$ for all $\varkappa > 0$. Now, to verify T is the desired contraction, we shall consider the following cases for every $\omega, \theta \in \Omega$ and $\varkappa > 0$:

Case 1 $\sqrt{\omega + 2} < \sqrt{\theta}$. We yield

$$\Gamma(a, T\theta, \varkappa) = \sup_{b \in T\theta} \Gamma(a, b, \varkappa) = \frac{a}{\sqrt{\theta}}$$

for all $a \in T\omega$. From this, we can infer that $\inf_{a \in T\omega} \Gamma(a, T\theta, \varkappa) = \frac{\sqrt{\omega}}{\sqrt{\theta}}$. Likewise, we have $\inf_{b \in T\theta} \Gamma(T\omega, b, \varkappa) = \frac{\sqrt{\omega + 2}}{\sqrt{\theta + 2}}$.

As a result, we have

$$H(T\omega, T\theta, \varkappa) = \min\left\{\inf_{a \in T\omega} \Gamma(a, T\theta, \varkappa), \inf_{b \in T\theta} \Gamma(T\omega, b, \varkappa)\right\}$$

$$= \min\left\{\frac{\sqrt{\omega}}{\sqrt{\theta}}, \frac{\sqrt{\omega + 2}}{\sqrt{\theta + 2}}\right\}$$

$$= \frac{\sqrt{\omega}}{\sqrt{\theta}}$$

$$\geq \psi(\Gamma(\omega, \theta, \varkappa)).$$

Case 2 $\sqrt{\omega} < \sqrt{\theta} \leq \sqrt{\omega + 2}$. We yield $\inf_{a \in T\omega} \Gamma(a, T\theta, \varkappa) = \frac{\sqrt{\omega}}{\sqrt{\theta}}$. In addition, we observe that $\inf_{b \in T\theta} \Gamma(T\omega, b, \varkappa) = \frac{\sqrt{\omega + 2}}{\sqrt{\theta + 2}}$. As a result, we have

$$H(T\omega, T\theta, \varkappa) = \min\left\{\inf_{a \in T\omega} \Gamma(a, T\theta, \varkappa), \inf_{b \in T\theta} \Gamma(T\omega, b, \varkappa)\right\}$$

$$= \min\left\{\frac{\sqrt{\omega}}{\sqrt{\theta}}, \frac{\sqrt{\omega+2}}{\sqrt{\theta+2}}\right\}$$

$$= \frac{\sqrt{\omega}}{\sqrt{\theta}}$$

$$\geq \psi\left(\Gamma(\omega, \theta, \varkappa)\right).$$

Case 3 $\sqrt{\theta} < \sqrt{\omega} < \sqrt{\theta+2}$. We yield $\inf_{a \in T\omega}\Gamma(a, T\theta, \varkappa) = \frac{\sqrt{\theta+2}}{\sqrt{\omega+2}}$. In addition, we observe that $\inf_{b \in T\theta}\Gamma(T\omega, b, \varkappa) = \frac{\sqrt{\theta}}{\sqrt{\omega}}$. As a result, we have

$$H(T\omega, T\theta, \varkappa) = \min\left\{\inf_{a \in T\omega} \Gamma(a, T\theta, \varkappa), \inf_{b \in T\theta} \Gamma(T\omega, b, \varkappa)\right\}$$

$$= \min\left\{\frac{\sqrt{\theta+2}}{\sqrt{\omega+2}}, \frac{\sqrt{\theta}}{\sqrt{\omega}}\right\}$$

$$= \frac{\sqrt{\theta}}{\sqrt{\omega}}$$

$$\geq \psi\left(\Gamma(\omega, \theta, \varkappa)\right).$$

Case 4 $\sqrt{\theta+2} < \sqrt{\omega}$. It can be seen that

$$\Gamma(T\omega, b, \varkappa) = \sup_{a \in T\omega} \Gamma(a, b, \varkappa) = \frac{b}{\sqrt{\omega}}$$

for all $b \in T\theta$. This implies that $\inf_{b \in T\theta}\Gamma(T\omega, b, \varkappa) = \frac{\sqrt{\theta}}{\sqrt{\omega}}$. Likewise, we have $\inf_{a \in T\omega} \Gamma(a, T\theta, \varkappa) = \frac{\sqrt{\theta+2}}{\sqrt{\omega+2}}$. As a result, we have

$$H(T\omega, T\theta, \varkappa) = \min\left\{\inf_{a \in T\omega} \Gamma(a, T\theta, \varkappa), \inf_{b \in T\theta} \Gamma(T\omega, b, \varkappa)\right\}$$

$$= \min\left\{\frac{\sqrt{\theta+2}}{\sqrt{\omega+2}}, \frac{\sqrt{\theta}}{\sqrt{\omega}}\right\}$$

$$= \frac{\sqrt{\theta}}{\sqrt{\omega}}$$

$$\geq \psi\left(\Gamma(\omega, \theta, \varkappa)\right).$$

Consequently, all conditions of Theorem 4 holds which implies that T admits optimal point of closest proximity in Ω. It is to be remarked that $[1, 2] \cup \{0\}$ are the best proximity points of T.

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Conflict of interest

The authors declare no competing interests in this study.

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