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Research Article

Novel Exact and Solitary Wave Solutions for The Time-Fractional Nonlinear Maccari's System

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Abstract: The purpose of this research is to find analytical solutions to the time-fractional nonlinear Maccari system. The double auxiliary equation method, which has never been used before, is used to obtain these solutions. The method is cleverly applied, resulting in the generation of nine new exact solitary wave solutions that have never been found before. We also describe the system's dynamic behavior and the bifurcation of traveling waves. Finally, we show some solutions with different coefficient values that correspond to the nine discovered solutions graphically.

Keywords: three coupled Maccari system, the double auxiliary equation method, bifurcation, conformable fractional derivative, exact solutions

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1. Introduction

Many real-life natural phenomena involve constraints that can only be modeled using partial differential equations. One of the most important and challenging equations is Maccari's system. It was initially introduced in [1] to explain the motion of isolated waves localized in space, but it now has a wide range of applications in physics and optics. The following equations describe Maccari's system:

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$$\begin{cases} iD_t^{\alpha}Q + Q_{xx} + RQ = 0, \\ iD_t^{\alpha}S + S_{xx} + RS = 0, \\ iD_t^{\alpha}N + N_{xx} + RN = 0, \\ D_t^{\alpha}R + R_y + \left(\left|Q + S + N\right|^2\right)_x = 0. \end{cases} \tag{1}$$

D_t^{α} is the conformable derivative of order α, which was first introduced in [2]. It is defined for the function $\Psi : \left(0, +\infty\right) \to R, \alpha \in \left(0, 1\right]$ as follows:

$$D_t^{\alpha}\left(\Psi\left(t\right)\right) = \lim_{\epsilon \to 0} \frac{\Psi\left(t + \epsilon t^{1-\alpha}\right)\Psi - \left(t\right)}{\epsilon}, \text{ for all } t > 0, 0 < \alpha \leq 1. \tag{2}$$

The chain rule, which is the most important formula of the conformable derivative, is defined as follows:

$$D_t^{\alpha} f\left(g\left(t\right)\right) = f_g^{'}\left[g\left(t\right)\right] D_t^{\alpha}(g\left(t\right).$$

Various approaches, such as fixed point theory [3] and fractional calculus, have been used to find exact solutions to nonlinear partial equations. While Maccari's system has received some research attention, there is still a need for efficient approaches to finding new exact solutions. Some of these approaches are as follows: the extended sine-Gordon equation method [4], the generalized Riccati relation [5], the exp-function method [6], the first integral method [7], the multilinear variable separation approach [8], the extended homogeneous balance method [9], the extended trial equation method and generalized Kudryashov method [10], the extended fan sub-equation mapping method [11], the $\exp\left(-\phi\left(\xi\right)\right)$ -expansion method, the sine-cosine approach, and the Riccati-Bernoulli sub-ODE method [12], the modified F-expansion and the generalized projective Riccati equation methods [13], and the extended sinh-Gordon equation expansion method [14]. The traveling wave hypothesis, mapping methods, and the Lie symmetry approach are also used in [15].

Finding exact solutions to partial differential equations remains an open problem that attracts many mathematical researchers. Thus, this paper fills a research gap by describing how to apply the double auxiliary equation method to the time-fractional nonlinear Maccari's system. We contribute to research on Maccari's system in three ways: (1) we describe and analyze the dynamic behavior of the system and the bifurcation of traveling waves; (2) we employ the double auxiliary equation method, developed in [16], which, to the best of our knowledge, has never been used to solve Maccari's system; and (3) we come up with nine new exact solutions with a variety of structures. Furthermore, the findings of this paper also highlight the importance of the double auxiliary equation method as a general and powerful method for solving more complex time-fractional nonlinear equations.

The remainder of the paper is divided into five sections. Section 2 investigates the dynamic behavior of Maccari's system and the bifurcation of traveling waves. Section 3 describes the double auxiliary equation method, and Section 4 presents the application of this method to solve Maccari's system. Section 5 depicts some of the discovered solutions graphically. The paper concludes with Section 6.

2. Analysis of the space-time fractional Maccari's system and bifurcation of traveling waves

To obtain the exact solutions of Maccari's system given by equation 1, the following complex wave transformation is used:

$$
\begin{cases}
Q(x,y,t) = \grave{\mathrm{E}}(\eta)e^{i\psi}, \\
S(x,y,t) = \Xi(\eta)e^{i\psi}, \\
N(x,y,t) = \Lambda(\eta)e^{i\psi}, \\
R(x,y,t) = \Omega(\eta),
\end{cases}
\tag{3}
$$

where

$$
\psi = px + qy + \kappa\frac{t^\alpha}{\alpha} + \lambda_0, \eta = x + y - 2p\frac{t^\alpha}{\alpha},
\tag{4}
$$

p, q, and κ are nonzero constants, λ_0 is an arbitrary constant, and $\alpha \in (0, 1]$. Therefore, Maccari's system can be reduced to the following system:

$$
\begin{cases}
\Theta'' - \left(\kappa + p^2\right)\Theta + \Omega\Theta = 0, \\
\Xi'' - \left(\kappa + p^2\right)\Xi + \Omega\Xi = 0, \\
\Lambda'' - \left(\kappa + p^2\right)\Lambda + \Omega\Lambda = 0, \\
(1-2p)\Omega' - \left((\Theta + \Xi + \Lambda)^2\right)' = 0.
\end{cases}
\tag{5}
$$

When the fourth part of equation 5 is integrated and the integration constant is set to zero, the result is:

$$
\Omega = \frac{1}{(1-2p)}(\Theta + \Xi + \Lambda)^2.
\tag{6}
$$

Putting equation 6 into the first three equations of equation 5, and setting

$$
\Xi = r_1\Theta, \Lambda = r_2\Theta.
\tag{7}
$$

we obtain the following equation:

$$
\Theta'' - \left(\kappa + p^2\right)\Theta - \frac{(1 + r_1 + r_2)^2}{(1-2p)}\Theta^3 = 0, p \neq \frac{1}{2}.
\tag{8}
$$

Thus, equation 8 is equivalent to the following two-dimensional Hamiltonian system:

$$
\begin{cases}
\dfrac{d\Theta}{d\zeta} = \psi, \\
\dfrac{d\psi}{d\zeta} = \left(\kappa + p^2\right)\Theta + \dfrac{(1 + r_1 + r_2)^2}{(1-2p)}\Theta^3, p \neq \dfrac{1}{2}.
\end{cases}
\tag{9}
$$

The Hamiltonian function $H(\Theta, \psi)$ is described in equation 10 where h is an integral constant.

$$
H(\Theta, \psi) = 2\psi^2 - 2\left(\kappa + p^2\right)\Theta^2 - \frac{(1 + r_1 + r_2)^2}{(1-2p)}\Theta^4 = h, p \neq \frac{1}{2}.
\tag{10}
$$

As a result, there is only one equilibrium point $E_0(0,0)$ if $(\kappa+p^2)=0$, as well as three equilibrium points $E_0(0,0), E_1\left(\sqrt{\frac{(\kappa+p^2)(1-2p)}{(1+r_1+r_2)^2}},0\right)$, and $E_2\left(-\sqrt{-\frac{(\kappa+p^2)(1-2p)}{(1+r_1+r_2)^2}},0\right)$ if $(\kappa+p^2)\neq 0$. We can determine the nature of these points by employing the same method used in [16]. At an equilibrium point, the coefficient matrix of the linearized system of equation 9 and $J=\det\left(M\left(\Theta_e,\psi_e\right)\right)$ are as follows:

$$M\left(\Theta_e,\psi_e\right)=\begin{pmatrix} 0 & 1 \\ \left(\kappa+p^2\right)+3\frac{\left(1+r_1+r_2\right)^2}{\left(1-2p\right)}\Theta_e^2 & 0 \end{pmatrix}, J=-\left(\left(\kappa+p^2\right)+3\frac{\left(1+r_1+r_2\right)^2}{\left(1-2p\right)}\Theta_e^2\right). \tag{11}$$

Thus, we have the three cases listed below:

Case 1. If $(\kappa+p^2)>0$ and $(1-2p)<0$, the system described by equation 9 has three equilibrium points $E_0\left(0,0\right), E_1\left(\sqrt{-\frac{\left(\kappa+p^2\right)(1-2p)}{\left(1+r_1+r_2\right)^2}},0\right)$, and $E_2\left(-\sqrt{-\frac{\left(\kappa+p^2\right)(1-2p)}{\left(1+r_1+r_2\right)^2}},0\right)$. The origin point E_0 is a saddle point, and $E_{1,2}$ are two center points. As a result, there are two families of periodic orbits at $E_{1,2}$ and a pair of homoclinic orbits at E_0. Therefore, equation 8 has two solitary wave solutions as well as infinite periodic wave solutions. This case is illustrated in Figure 1, which is generated using Maple 2022.

Case 2. If $(\kappa+p^2)>0$ and $(1-2p)<0$, we have also three equilibrium points $E_0(0, 0)$, $E_1\left(\sqrt{-\frac{\left(\kappa+p^2\right)(1-2p)}{\left(1+r_1+r_2\right)^2}},0\right)$, and $E_2\left(-\sqrt{-\frac{\left(\kappa+p^2\right)(1-2p)}{\left(1+r_1+r_2\right)^2}},0\right)$. The origin point E_0 is a center point, and $E_{1,2}$ are two saddle points. Thus, there is a family of periodic orbits at E_0, two homoclinic orbits at $E_{1,2}$, and two series of bounded open orbits on the left (resp. right) side at E_1 (resp. E_2). As a result, equation 8 has a periodic wave solutions family, two kink (anti-kink) wave solutions, and a series of breaking wave solutions. This case is illustrated in Figure 2, which is generated using Maple 2022.

Case 3. If $(\kappa+p^2)>0$ and $(1-2p)<0$, $E_0(0, 0)$ is the only higher order equilibrium point. As a result, there are two series of bounded open orbits near E_1, implying that equation 8 has two families of breaking wave solutions. This case is illustrated in Figure 3, which is generated using Maple 2022.

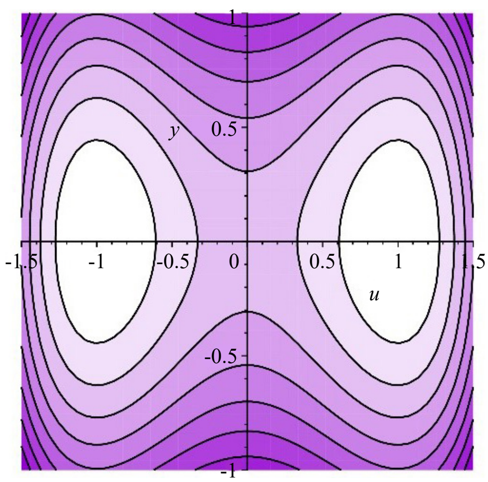


Figure 1. Phase portrait of (9) when $(\kappa+p^2)=1$ and $\frac{(1+r_1+r_2)^2}{(1-2p)}=-1$

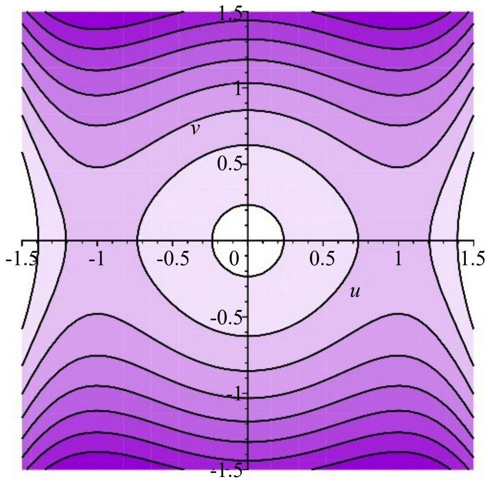


Figure 2. Phase portrait of (9) when $(\kappa + p^2) = 0$ and $\frac{(1+r_1+r_2)^2}{(1-2p)} = 1$

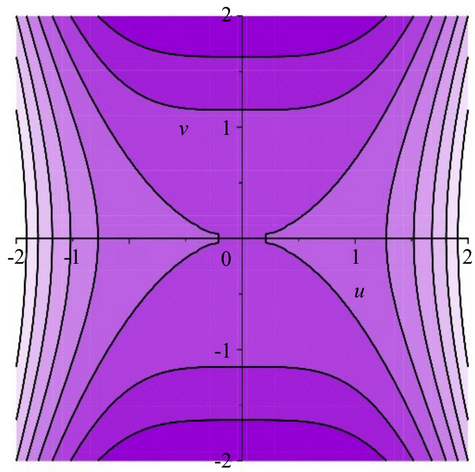


Figure 3. Phase portrait of (9) when $(\kappa + p^2) = 0$ and $\frac{(1+r_1+r_2)^2}{(1-2p)} = 1$

3. Description of the double auxiliary equation method

The double auxiliary equation method that will be used to solve Maccari's system will be described in this section. This method was first proposed in [16] to solve space-time fractional nonlinear equations. Consider the following nonlinear partial differential equation defined by equation 12:

$$H\left(w, D_t^{\alpha} w, w_x, w_y, w_z, \ldots\right) = 0, \tag{12}$$

where w is a function of independent variables (x, y, z, ..., t), D_t^{α} is the conformable derivatives of w with respect to t, and H is a polynomial in w.

Step 1. Convert equation 12 to a nonlinear ordinary differential equation using the wave transformation provided by equation 13:

$$W(x,y,z,\ldots,t)=W(\eta),\eta=kx+ly+hz+\ldots-\frac{vt^{\alpha}}{\alpha}, \tag{13}$$

where k, l, h, ..., and v are nonzero constants that will be determined later. Thus, equation 12 is transformed into a nonlinear fractional partial differential equation with new variable η, where w' represents differentiation with respect to η.

$$H(w,w',w'',w''',\ldots)=0, \tag{14}$$

Step 2. Assume the following solution to equation 14:

$$w(\eta)=\sum_{i=0}^{N}a_i\left(\frac{h(\eta)}{g(\eta)}\right)^{i}+\sum_{i=1}^{N}b_i\left(\frac{h(\eta)}{g(\eta)}\right)^{-i}, \tag{15}$$

where $a_i\,(i\in 0,\ldots,N)$ and $b_j\,(j\in 1,\ldots,N)$ are constants that will be obtained later with $a_N\neq 0$ or $b_N\neq 0$. The expression $\left(\frac{h(\eta)}{g(\eta)}\right)$ is a solution to equation 16 where $g(\eta)=\exp(\eta)(h(\eta))'$.

$$\left(\frac{h(\eta)}{g(\eta)}\right)'=A\left(\frac{h(\eta)}{g(\eta)}\right)^{2}+B\left(\frac{h(\eta)}{g(\eta)}\right)+C. \tag{16}$$

Step 3. Solve equation 16 to obtain the six families of solutions listed below:
Family 1: When $\Delta=B^2-4AC>0$ and $C\neq 0$,

$$\left(\frac{h(\eta)}{g(\eta)}\right)=\frac{-2C\left[1-\tanh\left(\frac{\sqrt{\Delta}}{2}\eta\right)\tanh\left(\frac{\sqrt{\Delta}}{2}k_1\right)\right]}{B-B\tanh\left(\frac{\sqrt{\Delta}}{2}\eta\right)\tanh\left(\frac{\sqrt{\Delta}}{2}k_1\right)-\sqrt{\Delta}\left[\tanh\left(\frac{\sqrt{\Delta}}{2}\eta\right)-\tanh\left(\frac{\sqrt{\Delta}}{2}k_1\right)\right]},k_1\in\mathbb{R}. \tag{17}$$

Family 2: When $\Delta=B^2-4AC<0$ and $C\neq 0$,

$$\left(\frac{h(\eta)}{g(\eta)}\right)=\frac{-2C\left[1+\tan\left(\frac{\sqrt{-\Delta}}{2}\eta\right)\tan\left(\frac{\sqrt{-\Delta}}{2}k_1\right)\right]}{B+B\tan\left(\frac{\sqrt{-\Delta}}{2}\eta\right)\tan\left(\frac{\sqrt{-\Delta}}{2}k_1\right)+\sqrt{-\Delta}\left[\tan\left(\frac{\sqrt{-\Delta}}{2}\eta\right)-\tan\left(\frac{\sqrt{-\Delta}}{2}k_1\right)\right]},k_1\in\mathbb{R}. \tag{18}$$

Family 3: When $\Delta=B^2-4AC=0$, $B>0$, $AC>0$,

$$\left(\frac{h(\eta)}{g(\eta)}\right)=-\frac{C(\eta-k_1)}{\sqrt{AC}(\eta-k_1)-1},k_1\in\mathbb{R}. \tag{19}$$

Family 4: When $B\neq 0$ and $C=0$,

$$\left(\frac{h(\eta)}{g(\eta)}\right)=-\frac{B\exp(B\eta)}{A\exp(B\eta)+Bk_1},k_1\in\mathbb{R}. \tag{20}$$

Family 5: When $B = 0$ and $C = 0$,

$$\left(\frac{h(\eta)}{g(\eta)}\right) = -\frac{1}{A\eta + k_1}, k_1 \in \mathbb{R}. \tag{21}$$

Family 6: When $A = B = 0$,

$$\left(\frac{h(\eta)}{g(\eta)}\right) = C(\eta - k_1), k_1 \in \mathbb{R}. \tag{22}$$

Step 4. Balance both the highest order linear and the highest order nonlinear terms in equation 14 to get the positive integer number N in equation 15.

Step 5. Substitute equation 15 and equation 16 into equation 14 to obtain all required derivatives $w', w'', \ldots$. This gives a polynomial of $\left(\frac{h(\eta)}{g(\eta)}\right)$. We get a system of algebraic equations by setting the coefficients of the same power of $\left(\frac{h(\eta)}{g(\eta)}\right)$ to zero. Then, by resolving this system, we obtain A, B, C, $a_i\ (i = 0, \ldots, N)$, $k, l, h, \ldots$, and v. Finally, the exact solutions of equation 14 are obtained by substituting these results and the solutions to equation 16 into equation 15.

4. Implementation of the method

Using the homogeneous balance method in equation 8, the value $N = 1$ is obtained. The solution to equation 8 is then given using the double auxiliary equation method as follows:

$$\Theta(\eta) = b_1\left(\frac{h(\eta)}{g(\eta)}\right)^{-1} + a_0 + a_1\left(\frac{h(\eta)}{g(\eta)}\right)^{1} \tag{23}$$

We get the following system of algebraic equations by substituting equation 23 into equation 8 and then equating the equal coefficients of different powers of $\left(\frac{h(\eta)}{g(\eta)}\right)$ to zero:

$$\begin{cases} a_1^3 r_1^2 + 2a_1^3 r_1 r_2 + a_1^3 r_2^2 + 4A^2 p a_1 + 2a_1^3 r_1 + 2a_1^3 r_2 - 2A^2 a_1 + a_1^3 = 0, \\ 3a_0 a_1^2 r_1^2 + 6a_0 a_1^2 r_1 r_2 + 3a_0 a_1^2 r_2^2 + 6ABpa_1 + 6a_0 a_1^2 r_1 + 6a_0 a_1^2 r_2 - 3ABa_1 + 3a_0 a_1^2 = 0 \\ 3a_0^2 a_1 r_1^2 + 6a_0^2 a_1 r_1 r_2 + 3a_0^2 a_1 r_2^2 + 3a_1^2 b_1 r_1^2 + 6a_1^2 b_1 r_1 r_2 + \\ 3a_1^2 b_1 r_2^2 + 4ACpa_1 + 2B^2 pa_1 - 2p^3 a_1 + 6a_0^2 a_1 r_1 + 6a_0^2 a_1 r_2 + \\ 6a_1^2 b_1 r_1 + 6a_1^2 b_1 r_2 - 2ACa_1 - B^2 a_1 + p^2 a_1 - 2p\kappa a_1 + 3a_0^2 a_1 + 3a_1^2 b_1 + \kappa a_1 = 0, \\ a_0^3 r_1^2 + 2a_0^3 r_1 r_2 + a_0^3 r_2^2 + 6b_1 a_0 a_1 r_1^2 + 12b_1 a_0 a_1 r_1 r_2 + \\ 6b_1 a_0 a_1 r_2^2 + 2ABpb_1 + 2BCpa_1 - 2a_0 p^3 + 2a_0^3 r_1 + 2a_0^3 r_2 + \\ 12b_1 a_0 a_1 r_1 + 12b_1 a_0 a_1 r_2 - ABb_1 - BCa_1 + a_0 p^2 - 2p\kappa a_0 + a_0^3 + 6b_1 a_0 a_1 + a_0 \kappa = 0, \\ 3b_1 a_0^2 r_1^2 + 6b_1 a_0^2 r_1 r_2 + 3b_1 a_0^2 r_2^2 + 3b_1^2 a_1 r_1^2 + \\ 6b_1^2 a_1 r_1 r_2 + 3b_1^2 a_1 r_2^2 + 4ACpb_1 + 2B^2 pb_1 - 2b_1 p^3 + 6b_1 a_0^2 r_1 + 6b_1 a_0^2 r_2 + 6b_1^2 a_1 r_1 + \\ 6b_1^2 a_1 r_2 - 2ACb_1 - B^2 b_1 + b_1 p^2 - 2p\kappa b_1 + 3b_1 a_0^2 + 3b_1^2 a_1 + b_1 \kappa = 0, \\ 3b_1^2 a_0 r_1^2 + 6b_1^2 a_0 r_1 r_2 + 3b_1^2 a_0 r_2^2 + 6BCpb_1 + 6b_1^2 a_0 r_1 + 6b_1^2 a_0 r_2 - 3BCb_1 + 3b_1^2 a_0 = 0, \\ b_1^3 r_1^2 + 2b_1^3 r_1 r_2 + b_1^3 r_2^2 + 4C^2 pb_1 + 2b_1^3 r_1 + 2b_1^3 r_2 - 2C^2 b_1 + b_1^3 = 0. \end{cases} \tag{24}$$

By solving the above system, we can now get the following nine sets of solutions:

Set 1:

$$\begin{cases} b_1 = -\dfrac{C\sqrt{2(1-2p)}}{r_2 + r_1 + 1}, a_0 = 0, a_1 = \dfrac{\sqrt{2(1-2p)}A}{r_2 + r_1 + 1}, r_1 = r_1, r_2 = r_2 \\ \kappa = 8AC - p^2, p = p, q = q, A = A, B = 0, C = C \end{cases} \tag{25}$$

$$\Theta_1(\eta) = \frac{\sqrt{2(1-2p)}\left(A\left(\dfrac{h(\eta)}{g(\eta)}\right)^2 - C\right)}{(r_2 + r_1 + 1)\left(\dfrac{h(\eta)}{g(\eta)}\right)}$$

Set 2:

$$\begin{cases} b_1 = \dfrac{\sqrt{-4p+2}C}{r_1 + r_2 + 1}, a_0 = \dfrac{Bb_1}{2C}, a_1 = 0, r_1 = r_1, r_2 = r_2, \\ \kappa = 2AC - \dfrac{B^2}{2} - p^2, p = p, q = q, A = A, B = B, C = C \end{cases} \tag{26}$$

$$\Theta_2(\eta) = \frac{\sqrt{-4p+2}\left(B\left(\dfrac{h(\eta)}{g(\eta)}\right) + 2C\right)}{2(l_1 + l_2 + 1)\left(\dfrac{h(\eta)}{g(\eta)}\right)}$$

Set 3:

$$\begin{cases} b_1 = 0, a_0 = \dfrac{B\sqrt{-4p+2}}{2(r_1 + r_2 + 1)}, a_1 = \dfrac{\sqrt{-4p+2}A}{r_1 + r_2 + 1}, r_1 = r_1, r_2 = r_2, \\ \kappa = 2CA - \dfrac{B^2}{2} - p^2, p = p, q = q, A = A, B = B, C = C \end{cases} \tag{27}$$

$$\Theta_3(\eta) = \frac{\sqrt{-4p+2}\left(2A\left(\dfrac{h(\eta)}{g(\eta)}\right) + B\right)}{2r_1 + 2r_2 + 2}$$

Set 4:

$$\begin{cases} b_1 = \dfrac{Ca_1}{A}, a_0 = 0, a_1 = \dfrac{\sqrt{-4p+2}A}{r_2 + r_1 + 1}, r_1 = r_1, r_2 = r_2 \\ \kappa = -4AC - p^2, p = p, q = q, A = A, B = 0, C = C \end{cases} \tag{28}$$

$$\Theta_4(\eta)=\frac{\sqrt{-4p+2}\left(A\left(\frac{h(\eta)}{g(\eta)}\right)^2+C\right)}{(r_2+r_1+1)\left(\frac{h(\eta)}{g(\eta)}\right)}$$

Set 5:

$$\begin{cases} b_1=\frac{Ba_0}{A}, a_0=a_0, a_1=0, r_1=-r_2-1, r_2=r_2, \\ \kappa=B^2-p^2, p=p, q=q, A=A, B=B, C=0 \end{cases} \tag{29}$$

$$\Theta_5(\eta)=\frac{Ba_0}{A\left(\frac{h(\eta)}{g(\eta)}\right)}+a_0$$

Set 6:

$$\begin{cases} b_1=0, a_0=a_0, a_1=a_1, r_1=-r_2-1, r_2=r_2, \\ \kappa=-p^2, p=p, q=q, A=0, B=0, C=C \end{cases} \tag{30}$$

$$\Theta_6(\eta)=a_0+a_1\left(\frac{h(\eta)}{g(\eta)}\right)$$

Set 7:

$$\begin{cases} b_1=0, a_0=a_0, a_1=\frac{Ba_0}{C}, r_1=-r_2-1, r_2=r_2, \\ \kappa=B^2-p^2, p=p, q=q, A=0, B=B, C=C \end{cases} \tag{31}$$

$$\Theta_7(\eta)=\frac{a_0\left(B\left(\frac{h(\eta)}{g(\eta)}\right)+C\right)}{C}$$

Set 8:

$$\begin{cases} b_1=b_1, a_0=a_0, a_1=0, r_1=-r_2-1, r_2=r_2, \\ \kappa=-p^2, p=p, q=q, A=A, B=0, C=0, \end{cases} \tag{32}$$

$$\Theta_8(\eta)=\frac{b_1}{\left(\frac{h(\eta)}{g(\eta)}\right)}+a_0$$

Set 9:

$$\begin{cases} b_1=b_1, a_0=0, a_1=a_1, r_1=-r_2-1, r_2=r_2 \\ \kappa=B^2-p^2, p=p, q=q, A=0, B=B, C=0 \end{cases} \tag{33}$$

$$\Theta_9(\eta)=\frac{a_1\left(\frac{h(\eta)}{g(\eta)}\right)^2+b_1}{\left(\frac{h(\eta)}{g(\eta)}\right)}$$

It is worth noting that $R = 0$ for sets 5, 6, 7, 8, and 9 because $r_1 + r_2 + 1 = 0$. Based on set 1 and using the families solutions of equation 16, the exact solutions of equation 1 are:

Case 1: When $\Delta = -4AC = -\frac{(\kappa+p^2)}{2} > 0$:

$$\begin{cases}
Q_{1.1}(\eta)=\dfrac{\sqrt{2(1-2p)}\left(-\dfrac{C\left(1-\tanh\left(\sqrt{-AC\eta}\right)\tanh\left(\sqrt{-ACw_1}\right)\right)^2}{\left(\tanh\left(\sqrt{-AC\eta}\right)-\left(\sqrt{-ACw_1}\right)^2\right)}-C\right)\sqrt{-AC}\left(\tanh\left(\sqrt{-AC\eta}\right)-\tanh\left(\sqrt{-ACw_1}\right)\right)}{(r_2+r_1+1)C\left(1-\tanh\left(\sqrt{-AC\eta}\right)\tanh\left(\sqrt{-ACw_1}\right)\right)}e^{i\psi}, \\
S_{1.1}(\eta)=r_1\dfrac{\sqrt{2(1-2p)}\left(-\dfrac{C\left(1-\tanh\left(\sqrt{-AC\eta}\right)\tanh\left(\sqrt{-ACw_1}\right)\right)^2}{\left(\tanh\left(\sqrt{-AC\eta}\right)-\left(\sqrt{-ACw_1}\right)^2\right)}-C\right)\sqrt{-AC}\left(\tanh\left(\sqrt{-AC\eta}\right)-\tanh\left(\sqrt{-ACw_1}\right)\right)}{(r_2+r_1+1)C\left(1-\tanh\left(\sqrt{-AC\eta}\right)\tanh\left(\sqrt{-ACw_1}\right)\right)}e^{i\psi}, \\
N_{1.1}(\eta)=r_2\dfrac{\sqrt{2(1-2p)}\left(-\dfrac{C\left(1-\tanh\left(\sqrt{-AC\eta}\right)\tanh\left(\sqrt{-ACw_1}\right)\right)^2}{\left(\tanh\left(\sqrt{-AC\eta}\right)-\left(\sqrt{-ACw_1}\right)^2\right)}-C\right)\sqrt{-AC}\left(\tanh\left(\sqrt{-AC\eta}\right)-\tanh\left(\sqrt{-ACw_1}\right)\right)}{(r_2+r_1+1)C\left(1-\tanh\left(\sqrt{-AC\eta}\right)\tanh\left(\sqrt{-ACw_1}\right)\right)}e^{i\psi}, \\
R_{1.1}(\eta)=2\left(\dfrac{\left(-\dfrac{C\left(1-\tanh\left(\sqrt{-AC\eta}\right)\tanh\left(\sqrt{-ACw_1}\right)\right)^2}{\left(\tanh\left(\sqrt{-AC\eta}\right)-\tanh\left(\sqrt{-ACw_1}\right)^2\right)}-C\right)\sqrt{-AC}\left(\tanh\left(\sqrt{-AC\eta}\right)-\tanh\left(\sqrt{-ACw_1}\right)\right)}{C\left(1-\tanh\left(\sqrt{-AC\eta}\right)\tanh\left(\sqrt{-ACw_1}\right)\right)}\right)^2.
\end{cases} \tag{34}$$

Case 2: $\Delta = -4AC = -\dfrac{(\kappa + p^2)}{2} < 0$

$$\begin{cases} Q_{1.2}(\eta) = -\dfrac{\sqrt{2(1-2p)}\left(\dfrac{C\left(1+\tan\left(\sqrt{AC}\eta\right)\tan\left(\sqrt{ACw_1}\right)\right)^2}{\left(\tan\left(\sqrt{AC}\eta\right)-\tan\left(\sqrt{ACw_1}\right)^2\right)} - C\right)\sqrt{AC}\left(\tan\left(\sqrt{AC}\eta\right)-\tan\left(\sqrt{-ACw_1}\right)\right)}{(r_2+r_1+1)C\left(1-\tan\left(\sqrt{-AC}\eta\right)\tan\left(\sqrt{-ACw_1}\right)\right)}e^{i\psi}, \\ S_{1.2}(\eta) = -r_1\dfrac{\sqrt{2(1-2p)}\left(\dfrac{C\left(1+\tan\left(\sqrt{AC}\eta\right)\tan\left(\sqrt{ACw_1}\right)\right)^2}{\left(\tan\left(\sqrt{-AC}\eta\right)-\tan\left(\sqrt{-ACw_1}\right)^2\right)} - C\right)\sqrt{AC}\left(\tan\left(\sqrt{AC}\eta\right)-\tan\left(\sqrt{-ACw_1}\right)\right)}{(r_2+r_1+1)C\left(1+\tan\left(\sqrt{AC}\eta\right)\tan\left(\sqrt{ACw_1}\right)\right)}e^{i\psi}, \\ N_{1.2}(\eta) = -r_2\dfrac{\sqrt{2(1-2p)}\left(\dfrac{C\left(1+\tan\left(\sqrt{AC}\eta\right)\tan\left(\sqrt{ACw_1}\right)\right)^2}{\left(\tan\left(\sqrt{AC}\eta\right)-\tan\left(\sqrt{ACw_1}\right)^2\right)} - C\right)\sqrt{AC}\left(\tan\left(\sqrt{AC}\eta\right)-\tan\left(\sqrt{ACw_1}\right)\right)}{(r_2+r_1+1)C\left(1+\tan\left(\sqrt{AC}\eta\right)\tan\left(\sqrt{ACw_1}\right)\right)}e^{i\psi}, \\ R_{1.2}(\eta) = 2\left(\dfrac{\left(\dfrac{C\left(1+\tan\left(\sqrt{AC}\eta\right)\tan\left(\sqrt{ACw_1}\right)\right)^2}{\left(\tan\left(\sqrt{AC}\eta\right)-\tan\left(\sqrt{ACw_1}\right)^2\right)} - C\right)\sqrt{AC}\left(\tan\left(\sqrt{AC}\eta\right)-\tan\left(\sqrt{ACw_1}\right)\right)}{C\left(1+\tan\left(\sqrt{-AC}\eta\right)\tan\left(\sqrt{ACw_1}\right)\right)}\right)^2, \end{cases} \tag{35}$$

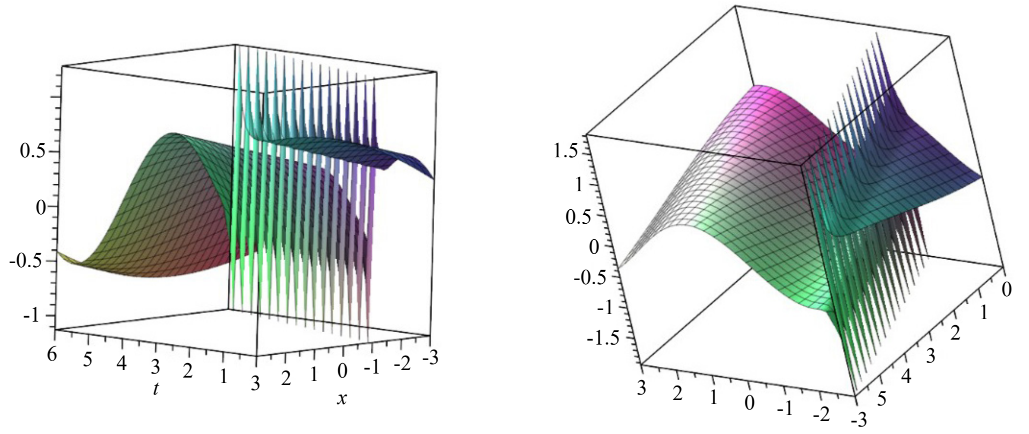
where

$$\psi = px + qy + \kappa\frac{t^\alpha}{\alpha} + \lambda_0, \eta = x + y - 2p\frac{t^\alpha}{\alpha}. \tag{36}$$

Note that p, q, and κ are nonzero constants, w_1 is arbitrary constant. Finally, we can also get the other exact solutions to equation 1 similarly.

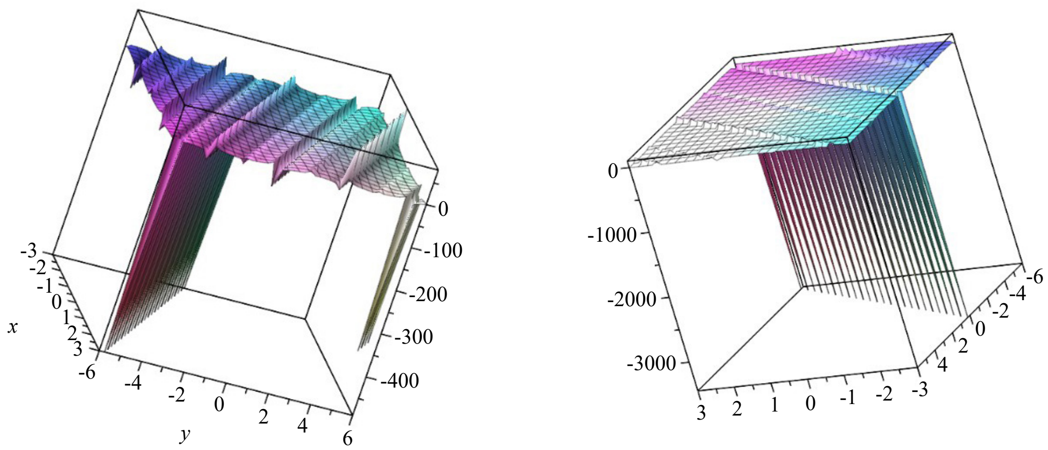
5. Graphical presentation of solutions

This section is devoted to depicting graphically some solutions with varying coefficient values that correspond to the nine discovered solutions in the previous section. In all figures, the solutions are drawn using Maple 2022. Figure 4(A) shows the real part of the solution $Q_{1.1}(\eta)$, while Figure 4(B) depicts the imaginary part. The illustration depicts the existence of two kink waves in both real and imaginary parts.



(A) (B)

Figure 4. (A) Real part and (B) imaginary part of $Q_{1,1}(\eta)$ for $y = 2$, $A = -3$, $B = 0$, $C = 3$, $w_1 = 1$, $r_1 = 8$, $r_2 = 3$, $p = \frac{1}{5}$, $q = 2$, $\alpha = 1$, and $\kappa = \frac{1}{3}$, $y = \frac{1}{2}$ (two kink waves)



(A) (B)

Figure 5. (A) Real part and (B) imaginary part of $Q_{1,2}(\eta)$ for $t = 3$, $A = 2$, $B = 0$, $C = \frac{1}{2}$, $w_1 = 3$, $r_2 = -4$, $p = -\frac{1}{4}$, $q = 4$, $\alpha = -\frac{1}{5}$, and $\kappa = \frac{1}{6}$ (infinite periodic waves)

6. Conclusion

This paper looked into a partial differential equation known as Maccari's system. The double auxiliary equation method is used to solve this system, which has never been used before. One of the contributions of this paper is the clever application of this method, which allowed us to find nine novel solitary wave solutions that had never been obtained before. We also investigated the dynamic behavior of this system and the bifurcation of traveling waves. Some of these solutions are graphically depicted to demonstrate their structures. The generality of our method, combined with its successful application in this paper, heralds a new era for researchers interested in applying it to more difficult partial differential equations.

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Conflict of interest

There is no conflict of interest for this study.

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