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Review

Review of a Trilogy about Sylow Theory in Locally Finite Groups

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Abstract: This comprehensive review concerns the trilogy about Sylow Theory in Locally Finite Groups which has been published in four books by Books on Demand (BoD), Norderstedt, Germany, namely: Part 1 - ISBN 978-3-7543-6087-3 (November 2022); Part 1 - Second edition - ISBN 978-3-7568-0801-4 (March 2023); Part 2 - ISBN 978-3-7568-3892-9 (December 2022); Part 3 - ISBN 978-3-7568-9853-4 (January 2023).

***Keywords*:** locally finite group satisfying the Strong Sylow Theorem for the prime p (or the Strong Sylow p-Theorem), singular (Sylow) p-subgroup, (very) good Sylow p-subgroup, p-uniqueness subgroup, minimal (w.r.t. order) p-unique subgroup, (numerical) Sylow p-invariant a_p, simple group; nested local system, family $\mathcal{T}$ of types of known finite simple groups; simple locally finite group of type $\Xi \in \mathcal{T}$, rank of a locally finite simple group, P-invariant Sylow p-subgroup, Kegel cover, ★-sequence, Kegel sequence, conjugacy class, P-isomorphic P-orbit, p-length of a p-soluble finite group, Hall-Higman Theory, locally finite field $\mathcal{F}$, algebraic closure of the prime field in characteristic p, General Linear Group, Special Linear Group, Projective Special Linear (PSL) Group, G-module over a (locally finite) field $\mathcal{F}$, irreducibility, complete reducibility, non-modular G-module, modular G-module, G-isomorphic G-modules, Jordan normal form, Classical Group, group of Lie type, twisted Chevalley group

MSC: 20D20, 20F50, 20E32, 20D05, 20D06, 20G15, 20G40, 20G07

1. Introduction

The books are based on the author's Diplomarbeit [1] and the continuation of his paper "Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p," which appeared in Advances in Group Theory and Applications (AGTA; see MR4441631 and Zbl 1496.20065). Since they contain a lot of material that cannot be published in a mathematical journal, they have been published as Books on Demand (BoD; see https://www.bod.de/buchshop/ and https://www.bod.fr/librairie/). In the following, we describe the **mathematical** subject matter. Regarding the **non-mathematical** contents, including dedications to famous or beloved persons, we refer to the contained four recensions: **Part 1**, page (v); **Part 2**, page (i); **Part 3**, page (i) and page 42.

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2. Part 1

Part 1, "Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime p" of the trilogy [2-4] is based on the BoD-Book "Characterising locally finite groups satisfying the strong Sylow Theorem for the prime p – Revised edition" (see under ISBN 978-3-7562-3416-5), which in turn has been based on the author's quite pioneering research paper in Volume 13 of AGTA (see https://www.advgrouptheory.com/journal/Volumes/13/Flemisch.pdf and the summary in Zbl 1496.20065). The **first edition** of **Part 1** removes the highlights in light green of the revised edition and adds the albeit very considerably improved pages from (i) to (vi) and pages 27 to 34 to the AGTA paper. In addition, **Part 1** adds the 10 new pages from 35 to 44 to the revised edition and therefore had to renumber the pages from (xv) to (xviii) into pages 45 to 48. It includes the second paper of **Professor Otto H. Kegel** about Sylow theory in locally finite groups (see MR981832 and Zbl 0659.20024) as **Appendix 1** and his first paper on the subject (see MR414714 and Zbl 0328.20029) as **Appendix 2**. Finally, it calls to mind Kegel's contribution to the conference Ischia Group Theory 2016.

Most importantly, **Part 1** introduces **the basic concepts of the (first) trilogy about Sylow theory in locally finite groups**: locally finite groups satisfying the strong Sylow theorem for the prime p (or the strong Sylow p-Theorem); singular (Sylow) p-subgroup; (very) good Sylow p-subgroup; p-uniqueness subgroup; minimal (w.r.t. order) p-unique subgroup; (numerical) Sylow p-invariant a_p.

The **second edition** of **Part 1** introduces uniform page numbering, adds page numbers to the appendices, improves pages (iv) and (v), page 22, pages 25 to 34 and pages 39, 45, 49, 50, 75, 76, 105 and 106, adds pages 109 to 112, and adds a two-page **Table of Contents of the trilogy**. **Part 1** has 6 + 48 + 27 + 31 + 8 = **120 pages** and includes **15 references** according to the AGTA paper followed by • a two-page postscrip about **Kegel**'s theorem/conjecture on simple locally finite groups with the Strong Sylow p-Theorem (see Part 2) and about the (first) trilogy, by • the section "About the author", by • a note on Kegel's theorem/conjecture, by • a section about "Sylow p-intersections and p-uniqueness subgroups", by • the illuminating section "Comparison of Kegel's two papers on Sylow theory in locally finite groups", by • the section "Five personal notes about mathematics", and finally followed by • the insight-gaining section "**Augustin-Louis Cauchy**'s and **Évariste Galois**' contributions to Sylow theory in finite groups" describing and summarising a follow-up paper. It also adds the sections "Mathematical Reviews (MR; MathSciNet) and zbMATH Open (Zentralblatt MATH)".

3. Part 2

Part 2, "About the Strong Sylow Theorem for the Prime p in Simple Locally Finite Groups" of the trilogy [5] is based on **Part 1**. It first gives a profound overview of the structure of simple groups, and in particular of the simple locally finite groups, and reduces their Sylow theory for the prime p to an old (1987) conjecture of **Professor Otto H. Kegel**, contained in his second paper: "Let the p-subgroup P be a p-uniqueness subgroup in the finite simple group S which belongs to one of the seven rank-unbounded families. Then the rank of S is bounded in terms of P", about the rank-unbounded ones of the known 19 families of finite simple groups.

Part 2 then introduces a new scheme to describe these 19 families, the **family** $\mathcal{T}$:= {abelian$_p$, $\underline{A}^n$, A = PSL_n, B = $P\Omega_{\text{odd n}}$, C = PSp_n, D = $P\Omega^+_{\text{even}}$, ^{2}A = PSU_n, ^{2}D = $P\Omega^-_{\text{even n}}$, E_6, E_7, E_8, F_4, G_2, $^2B_2$, $^3D_4$, $^2E_6$, $^2F_4$, $^2G_2$, sporadic$_\star$} **of types**, defines the rank of each type, and emphasises the role of **Kegel** covers. This introductory part presents a unified picture of known results with proofs only by reference, whence the title of **Part 2** starts with "About".

Subsequently, **Part 2** applies ***new ideas*** to prove the conjecture for the alternating groups. An **abstract** of this approach is presented on **page (ii)**.

Thereupon, **Part 2** remembers the central role played by **Kegel** covers and ⋆-sequences. Finally, it presents a plan how to prove and even how to optimise the conjecture step-by-step or peu à peu by suggesting **a way 1)** and **a way 2)** which leads to further tough conjectures thereby happily unifying Sylow theory in locally finite **simple** groups with Sylow theory in locally finite and ***p*-soluble** groups being the work by adored **Professors Otto H. Kegel** and **Brian Hartley** (see page 27 of **Part 1**). **Part 2** has 4 + 20 = **24 pages** including **24 references**, an overview by 12 references and two figures of **the classification of the finite simple groups**. Any unexplained terminology refers to **Part 1**.

4. Part 3

Part 3, "The Strong Sylow Theorem for the Prime p in Projective Special Linear Locally Finite Groups" of the trilogy [6] continues the program begun in **Part 2** to optimise along **way 1)**, the theorem about **the first type** Ξ = "A_n" of infinite families of finite simple groups step-by-step to further types by proving it for **the second type** Ξ = "$A = PSL_n$". It **first** proves the **Conjecture 2 from Part 2** about the General Linear Groups over (commutative) locally finite fields, stating that their rank is bounded in terms of their p-uniqueness, and **then** breaks down this insight into the Special Linear Groups and the Projective Special Linear (PSL) Groups over locally finite fields. An **abstract** of this three-stage-approach with a description of the ***new ideas*** regarding the proof for the General Linear Groups, a figure showing the transitions to the Special Linear Groups and the PSL Groups is presented on **page (ii)**.

Part 3 closes with suggestions for future research • regarding the remaining rank-unbounded types (the "Classical Groups") and **the way 2)** • regarding the (locally) finite and p-soluble groups by announcing a proof of **Conjecture 3 from Part 2** thereby somehow rounding off the classical Hall-Higman theory and • regarding **Augustin-Louis Cauchy**'s and **Évariste Galois**' pioneering contributions to Sylow theory in finite groups showing three rectangles/tableaux. These three suggestions culminate in the announcement of **the second trilogy** about Sylow Theory in Locally Finite Groups. **Part 3** has 4 + 42 = **46 pages** and includes **32 references** followed by • an insightful Postscript about ‣ the Alternating Groups, ‣ the PSL Groups, ‣ the first trilogy and the 19 families of finite simple groups, and finally followed by • the four-page section "A Contribution to the Mathematical Theory of Big Game Hunting". Its last **page 42**, shows **the first trilogy** as a triptych (see https://en.wikipedia.org/wiki/Triptych and **page 49** of **Part 1**).

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Conflict of interest

The author declares that there are no competing personal or organisational or financial conflicts of interest with this original work or other conflicts of interest regarding the publication of this meticulous review paper.

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