Review of a Trilogy about Sylow Theory in Locally Finite Groups

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MSC: 20D20, 20F50, 20E32, 20D05, 20D06, 20G15, 20G40, 20G07

1. Introduction

The books are based on the author’s Diplomarbeit [1] and the continuation of his paper “Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime $p$,” which appeared in Advances in Group Theory and Applications (AGTA; see MR4441631 and Zbl 1496.20065). Since they contain a lot of material that cannot be published in a mathematical journal, they have been published as Books on Demand (BoD; see https://www.bod.de/buchshop/ and https://www.bod.fr/librairie/). In the following, we describe the mathematical subject matter. Regarding the non-mathematical contents, including dedications to famous or beloved persons, we refer to the contained four recensions: Part 1, page (v); Part 2, page (i); Part 3, page (i) and page 42.
2. Part 1

Part 1, “Characterising Locally Finite Groups Satisfying the Strong Sylow Theorem for the Prime $p$” of the trilogy [2–4] is based on the BoD-Book “Characterising locally finite groups satisfying the strong Sylow Theorem for the prime $p$ – Revised edition” (see under ISBN 978-3-7562-3416-5), which in turn has been based on the author’s quite pioneering research paper in Volume 13 of AGTA (see https://www.advgroupthtory.com/journal/Volumes/13/Flemisch.pdf and the summary in Zbl 1496.20065). The first edition of Part 1 removes the highlights in light green of the revised edition and adds the albeit very considerably improved pages from (i) to (vi) and pages 27 to 34 to the AGTA paper. In addition, Part 1 adds the 10 new pages from 35 to 44 to the revised edition and therefore had to renumber the pages from (xv) to (xviii) into pages 45 to 48. It includes the second paper of Professor Otto H. Kegel about Sylow theory in locally finite groups (see MR981832 and Zbl 0659.20024) as Appendix 1 and his first paper on the subject (see MR414714 and Zbl 0328.20029) as Appendix 2. Finally, it calls to mind Kegel’s contribution to the conference Ischia Group Theory 2016.

Most importantly, Part 1 introduces the basic concepts of the (first) trilogy about Sylow theory in locally finite groups: locally finite groups satisfying the strong Sylow theorem for the prime $p$ (or the strong Sylow $p$-Theorem); singular (Sylow) $p$-subgroup; (very) good Sylow $p$-subgroup; $p$-uniqueness subgroup; minimal (w.r.t. order) $p$-unique subgroup; (numerical) Sylow $p$-invariant $a_p$.

The second edition of Part 1 introduces uniform page numbering, adds page numbers to the appendices, improves pages (iv) and (v), page 22, pages 25 to 34 and pages 39, 45, 49, 50, 75, 76, 105 and 106, adds pages 109 to 112, and adds a two-page Table of Contents of the trilogy. Part 1 has $6 + 48 + 27 + 31 + 8 = 120$ pages and includes 15 references according to the AGTA paper followed by • a two-page postscript about Kegel’s theorem/conjecture on simple locally finite groups satisfying the strong Sylow $p$-Theorem (see Part 2) and about the (first) trilogy, by • the section “About the author”, by • a note on Kegel’s theorem/conjecture, by • a section about “Sylow $p$-intersections and $p$-uniqueness subgroups”, by • the illuminating section “Comparison of Kegel’s two papers on Sylow theory in locally finite groups”, by • the section “Five personal notes about mathematics”, and finally followed by • the insight-gaining section “Augus-tin-Louis Cauchy’s and Évariste Galois’ contributions to Sylow theory in finite groups” describing and summarising a follow-up paper. It also adds the sections ”Mathematical Reviews (MR; MathSciNet) and zbMATH Open (Zentralblatt MATH”).

3. Part 2

Part 2, “About the Strong Sylow Theorem for the Prime $p$ in Simple Locally Finite Groups” of the trilogy [5] is based on Part 1. It first gives a profound overview of the structure of simple groups, and in particular of the simple locally finite groups, and reduces their Sylow theory for the prime $p$ to an old (1987) conjecture of Professor Otto H. Kegel, contained in his second paper: “Let the $p$-subgroup $P$ be a $p$-uniqueness subgroup in the finite simple group $S$ which belongs to one of the seven rank-unbounded families. Then the rank of $S$ is bounded in terms of $P$”, about the rank-unbounded ones of the known 19 families of finite simple groups.

Part 2 then introduces a new scheme to describe these 19 families, the family $T:= \{\text{abelian}^p, A^*, A = \text{PSL}_{\infty}, B = P\Omega^\bullet_{odd}^{+}, C = \text{PSp}_{\infty}, D = P\Omega^\bullet_{even}^{+}, \hat{A} = \text{PSU}_{\infty}^{+}, \hat{D} = P\Omega^\bullet_{even}^{+}, E_{\infty}, E_8, F_4, G_2, \hat{B}_2, \hat{D}_4, \hat{E}_6, \hat{F}_4, \hat{G}_2, \text{sporadic}^* \}$ of types, defines the rank of each type, and emphasises the role of Kegel covers. This introductory part presents a unified picture of known results with proofs only by reference, whence the title of Part 2 starts with “About”.

Subsequently, Part 2 applies new ideas to prove the conjecture for the alternating groups. An abstract of this approach is presented on page (ii).

Thereupon, Part 2 remembers the central role played by Kegel covers and $\star$-sequences. Finally, it presents a plan how to prove and even how to optimise the conjecture step-by-step or peu à peu by suggesting a way 1) and a way 2) which leads to further tough conjectures thereby happily unifying Sylow theory in locally finite simple groups with Sylow theory in locally finite and $p$-soluble groups being the work by adored Professors Otto H. Kegel and Brian Hartley (see page 27 of Part 1). Part 2 has $4 + 20 = 24$ pages including 24 references, an overview by 12 references and two figures of the classification of the finite simple groups. Any unexplained terminology refers to Part 1.
4. Part 3

Part 3, “The Strong Sylow Theorem for the Prime $p$ in Projective Special Linear Locally Finite Groups” of the trilogy [6] continues the program begun in Part 2 to optimise along way 1), the theorem about the first type $\Xi = \{A_n\}$ of infinite families of finite simple groups step-by-step to further types by proving it for the second type $\Xi = \{\text{PSL}_n\}$. It first proves the Conjecture 2 from Part 2 about the General Linear Groups over (commutative) locally finite fields, stating that their rank is bounded in terms of their $p$-uniqueness, and then breaks down this insight into the Special Linear Groups and the Projective Special Linear (PSL) Groups over locally finite fields. An abstract of this three-stage-approach with a description of the new ideas regarding the proof for the General Linear Groups, a figure showing the transitions to the Special Linear Groups and the PSL Groups is presented on page (ii).

Part 3 closes with suggestions for future research • regarding the remaining rank-unbounded types (the “Classical Groups”) and the way 2) • regarding the (locally) finite and $p$-soluble groups by announcing a proof of Conjecture 3 from Part 2 thereby somehow rounding off the classical Hall-Higman theory and • regarding Augustin-Louis Cauchy’s and Évariste Galois’ pioneering contributions to Sylow theory in finite groups showing three rectangles/tableaux. These three suggestions culminate in the announcement of the second trilogy about Sylow Theory in Locally Finite Groups.

Part 3 has $4 + 42 = 46$ pages and includes 32 references followed by • an insightful Postscript about • the Alternating Groups, • the PSL Groups, • the first trilogy and the 19 families of finite simple groups, and finally followed by • the four-page section “A Contribution to the Mathematical Theory of Big Game Hunting”. Its last page 42, shows the first trilogy as a triptych (see https://en.wikipedia.org/wiki/Triptych and page 49 of Part 1).

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Conflict of interest

The author declares that there are no competing personal or organisational or financial conflicts of interest with this original work or other conflicts of interest regarding the publication of this meticulous review paper.

References