$$f(t_1, r_1(t_1,t_2), r_2(t_1,t_2)) = F(t_1, t_1\varphi_1(t_1^{-1},t_2), t_1\varphi_2(t_1^{-1},t_2),1) =$$
$$F(1,\varphi_1(t_1^{-1},t_2),\varphi_2(t_1^{-1},t_2),t_1^{-1}) = F(1,\varphi_1(t_1,t_2),\varphi_2(t_1,t_2),t_1) = 0$$

around $t_1 = 0$, and F is the implicit polynomial of V^*. Observe that in this section, we are given the parametrization P^* of V^* and then, $F(P^*(s_1, s_2)) =$

$$F\left(1, \frac{p_{21}(s_1,s_2)}{p_{22}(s_1,s_2)}\cdot\frac{p_{12}(s_1,s_2)}{p_{11}(s_1,s_2)}, \frac{p_{31}(s_1,s_2)}{p_{32}(s_1,s_2)}\cdot\frac{p_{12}(s_1,s_2)}{p_{11}(s_1,s_2)}, \frac{p_{12}(s_1,s_2)}{p_{11}(s_1,s_2)}\right) = 0.$$

Hence, to compute the branches of V, and in particular φ_i, $i = 1, 2$, one rewrite $P^*(s_1, s_2)$ in the form of $(1 : \varphi_1(t_1, t_2) : \varphi_2(t_1, t_2) : t_1)$ around $t_1 = 0$. This is a search for a value of the parameters (t_1, t_2), say $\ell(t_1,t_2) \in \mathbb{C} \ll t_1,t_2 \gg^2$, such that $P^*(\ell(t_1, t_2)) = (1 : \varphi_1(t_1, t_2) : \varphi_2(t_1, t_2) : t_1)$ around $t_1 = 0$.

There exist solutions $\ell_1(t_1,t_2), \ell_2(t_1,t_2), \ldots, \ell_k(t_1,t_2) \in \mathbb{C} \ll t_1,t_2 \gg^2$ such that, $p_{12}(\ell_i(t_1, t_2)) - t_1 p_{11}(\ell_i(s, t)) = 0$, $i = 1, ..., k$, in a neighborhood of $t_1 = 0$. Note that

$$\ell_i(t_1,t_2) = (\ell_{1i}(t_1,t_2), \ell_{2i}(t_1,t_2)) = (\ell_{1,i,0}(t_2) + \ell_{1,i,1}(t_2)t_1^{n_{1,i,1}} + \ell_{1,i,2}(t_2)t_1^{n_{1,i,2}} + \cdots,$$
$$\ell_{2,i,0}(t_2) + \ell_{2,i,1}(t_2)t_1^{n_{2,i,1}} + \ell_{2,i,2}(t_2)t_1^{n_{2,i,2}} + \cdots), 0 < n_{r,i,1} < n_{r,i,2} < \cdots, r = 1,2$$

where $h(\ell_{1,i,0}(t_2), \ell_{2,i,0}(t_2)) = 0$ and $\ell_{r,i,j}(t_2) \in \mathbb{C} \ll t_2 \gg^2$ for $i = 1, ..., k$ and $j = 0, 1, ..., r = 1, 2$.

Hence, for $i = 1, ..., k$, there exists $M_i \in \mathbb{R}^+$ such that the points $(1 : \varphi_1(t_1, t_2) : \varphi_2(t_1, t_2) : t_1)$ or similarly, the points $(t_1^{-1} : t_1^{-1}\varphi_1(t_1,t_2) : t_1^{-1}\varphi_2(t_1,t_2) : 1)$, where

$$\varphi_{1i}(t_1,t_2) = \frac{p_{21}(\ell_i(t_1,t_2))}{p_{22}(\ell_i(t_1,t_2))}, \quad \varphi_{2i}(t_1,t_2) = \frac{p_{31}(\ell_i(t_1,t_2))}{p_{32}(\ell_i(t_1,t_2))}, \tag{2}$$

are in V^* for $|t_1| < M_i (P^*(\ell_i(t_1,t_2)) \in V^*$ since P^* is a parametrization of V^*). Observe that $\varphi_{ji}(t_1, t_2)$, $j = 1, 2$, is Puiseux series, since $p_{k1}(\ell_i(t_1,t_2))$ and $p_{k2}(\ell_i(t_1,t_2)), k = 2,3$, can be expressed as Puiseux series and $\mathbb{C} \ll t_1,t_2 \gg$ is a field.

Finally, we set $t_1 \to t_1^{-1}$ and we have that the points $(t_1, r_{1i}(t_1, t_2), r_{2i}(t_1, t_2))$, where $r_{ji}(t_1,t_2) = t_1\varphi_{ji}(t_1^{-1},t_2)$, are in V for $|t_1| > M_i^{-1}$. Hence, the infinity branches of V are the sets $B_i = \{(t_1, r_{1i}(t_1,t_2), r_{2i}(t_1,t_2)) \in \mathbb{C}^3 : (t_1,t_2) \in \mathbb{C}^2, |t_1| > M_i^{-1}\}$, for $i = 1, ..., k$.

Remark 2. The series $\ell_i(t_1, t_2)$ satisfy that

$$p_{12}(\ell_i(t_1,t_2)) / p_{11}(\ell_i(t_1,t_2)) = t_1,$$

for $i = 1, ..., k$. Therefore, from (2), we get

$$\varphi_{1i}(t_1,t_2) = \frac{p_{22}(\ell_i(t_1,t_2))}{p_{21}(\ell_i(t_1,t_2))}\cdot\frac{p_{12}(\ell_i(t_1,t_2))}{p_{11}(\ell_i(t_1,t_2))} = \frac{p_{22}(\ell_i(t_1,t_2))}{p_{21}(\ell_i(s,t))}t_1 = p_2(\ell_i(t_1,t_2))t_1,$$

and

$$r_{i1}(t_1,t_2) = t_1\varphi_{1i}(t_1^{-1},t_2) = p_2(\ell_i(t_1^{-1},t_2)).$$

Similarly, $\varphi_{2i}(t_1, t_2) = p_3(\ell_i(t_1, t_2))t_1$ and

$$r_{i2}(t_1,t_2) = t_1\varphi_{2i}(t_1^{-1},t_2) = p_3(\ell_i(t_1^{-1},t_2)).$$

Example 2. Let the surface V be parametrically defined by

$$P(s_1,s_2) = (s_1^2 / s_2^2, (s_1^2 - s_2^2 - 5s_1 + 1)/(s_2^2 - s_1), (s_1^2 + s_2^2 - s_2 - 5)/(s_2^2 + 3)).$$

Let us determine the branches of V from P. For that, we first compute the solutions of

$$p_{12}(s_1, s_2) - t_1 p_{11}(s_1, s_2) = 0,$$

around $t_1 = 0$. We get

$$\ell(t_1,t_2) = (t_2 t_1^{-1/2}, t_2) \in \mathbb{C} \ll t_1, t_2 \gg^2 .$$

For $\ell(t_1,t_2) \in \mathbb{C} \ll t_1, t_2 \gg^2$, we compute the corresponding infinity branch of V,

$$B = \{(t_1, r_1(t_1,t_2), r_2(t_1,t_2)) \in \mathbb{C}^3 : (t_1,t_2) \in \mathbb{C}^2, |t_1| > M_i^{-1}\},$$

where $r_j(t_1,t_2) = p_{(j+1)}(\ell(t_1^{-1}, t_2)), j = 1, 2,$ is given as Puiseux series. We get

$$\begin{aligned} r_1(t_1,t_2) &= -t_2 t_1^{1/2} + 5 - t_2^2 + 6t_2 t_1^{-1/2} - t_2^{-1} t_1^{-1/2} - t_2^3 t_1^{-1/2} + 6t_2^2 t_1^{-1} - t_1^{-} - t_2^4 t_1^{-1} + 6t_2^3 t_1^{-3/2} - t_2 t_1^{-3/2} - t_2^5 t_1^{-3/2} + \cdots, \\ r_2(t_1,t_2) &= t_2^2 t_1 / (t_2^2+3) + t_2^2 / (t_2^2+3) - t_2 / (t_2^2+3) - 5/(t_2^2+3). \end{aligned}$$

Remark 3. If a surface V has degree d and it cannot be approached by a new surface of degree less than d, we say that V is a perfect surface. In addition, we say that $\overline{V}$ is a g-asymptote if it is a perfect surface that approaches V at branch at the infinity. In a future work, we will analyze if the approaching surfaces computed in this paper are perfect surfaces.

Example 3. Let the surface V be parametrically defined by

$$P(s_1,s_2) = (s_1^3, s_1^2(s_2^2+1), s_1^2 s_2).$$

As in Example 2, we obtain one infinity branch

$$B = \{(t_1, r_1(t_1,t_2), r_2(t_1,t_2)) \in \mathbb{C}^3 : (t_1,t_2) \in \mathbb{C}^2, |t_1| > M\}, \quad \text{where}$$

$$r_1(t_1,t_2) = t_1^{2/3}(t_2^2+1), \qquad r_2(t_1,t_2) = t_1^{2/3} t_2.$$

Note that $P(t_1,t_2) = (t_1^3, r_1(t_1^3,t_2), r_2(t_1^3,t_2))$. Thus, we could deduce that V cannot be approached by any surface of degree less than the degree of V. Thus, we would say that V is a perfect surface (see Remark 3).

3.2 *Asymptotic behavior of parametric surfaces*

Once we know how to determine the infinity branches, we could compute surfaces that have the same asymptotic behavior as the given surface at each of them by simply removing the terms with negative exponent in the variable t_1 from r_1 and r_2, and to compute a new surface having the same asymptotic behavior that the input surface for each of them.

More precisely, in Example 2, we observe that once we have the infinity branch, we may determine a surface having the same asymptotic behavior as the input surface at each of them by simply removing the terms with negative exponent in the variable t_1 from r_1 and r_2. In that case, we get

$$\tilde{r}_1(t_1,t_2) = -t_2 t_1^{1/2} + 5 - t_2^2,$$

$$\tilde{r}_2(t_1,t_2) = t_2^2 t_1 / (t_2^2+3) + t_2^2 / (t_2^2+3) - t_2 / (t_2^2+3) - 5/(t_2^2+3).$$

Observe that, for this particular example, $(t_1^2, \tilde{r}_1(t_1^2,t_2), \tilde{r}_2(t_1^2,t_2)) \in \mathbb{R}(t_1,t_2)$ is, in fact, a rational parametrization that defines a surface $\overline{V}$ that has the same asymptotic behavior as V at B.

Theorem 2. Let P be a given parametrization of a surface V such that $h(s_1, s_2)$ divides p_{12} and $\gcd(h, p_{i2}) = 1$, $i = 2, 3$.

Then, V and $\overline{V}$ have the same asymptotic behavior, where $\overline{V}$ is defined by the local parametrization

$$Q(t) = (t_1, p_2(\ell_{10}(t_2), \ell_{20}(t_2)), p_3(\ell_{10}(t_2), \ell_{20}(t_2))) \in (\mathbb{C} \ll t_2 \gg)[t_1],$$

and $h(\ell_{10}(t_2), \ell_{20}(t_2)) = 0$. Under these conditions, V is a cylinder over the x axis which implicit equation can be computed using eliminating techniques and more precisely, by eliminating the variables s_1, s_2 from the system

$$p_{21}(s_1, s_2) - yp_{22}(s_1, s_2) = p_{31}(s_1, s_2) - zp_{32}(s_1, s_2) = h(s_1, s_2) = 0.$$

Proof. First, we observe that since

$$p_{12}(\ell(t_1, t_2)) - t_1 p_{11}(\ell(t_1, t_2)) = 0$$

around $t_1 = 0$, we have that

$$\ell(t_1, t_2) = [(\ell_{10}(t_2) + \ell_{11}(t_2)t_1^{n_{11}} + \ell_{12}(t_2)t_1^{n_{12}} + \cdots, \ell_{20}(t_2) + \ell_{21}(t_2)t_1^{n_{21}} + \ell_{22}(t_2)t_1^{n_{22}} + \cdots),$$

for $0 < n_{r1} < n_{r2} < \ldots$, $r = 1, 2$, and where $h(\ell_{10}(t_2), \ell_{20}(t_2)) = 0$ and $\ell_{rj}(t_2) \in \mathbb{C} \ll t_2 \gg, j = 0, 1, \ldots, r = 1, 2.$ Taking into account that $\gcd(h, p_{i2}) = 1$, $i = 2, 3$, we have that

$$p_i(s_1, s_2) = p_i(\ell_{10}(t_2), \ell_{20}(t_2)) + \frac{\partial p_i}{\partial s_1}(\ell_{10}(t_2), \ell_{20}(t_2))(s_1 - (\ell_{10}(t_2)) + \frac{\partial p_i}{\partial s_2}(\ell_{10}(t_2), \ell_{20}(t_2))(s_2 - (\ell_{20}(t_2)) + \cdots$$

for $i = 2, 3$. Thus, $p_i(\ell(t_1, t_2)) =$

$$p_i(\ell_{10}(t_2), \ell_{20}(t_2)) + \frac{\partial p_i}{\partial s_1}(\ell_{10}(t_2), \ell_{20}(t_2))(\ell_{11}(t_2)t_1^{n_1} + \ell_{12}(t_2)t_1^{n_2} + \cdots) + \cdots, i = 2, 3,$$

and $r_i(t_1, t_2) = p_i(\ell(t_1^{-1}, t_2)) =$

$$p_i(\ell_{10}(t_2), \ell_{20}(t_2)) + t_1^{-n_{11}}(\frac{\partial p_i}{\partial s_1}(\ell_{10}(t_2), \ell_{20}(t_2))(\ell_{11}(t_2) + \ell_{12}(t_2)t_1^{n_{11}-n_{12}} + \cdots) + \cdots,$$

for $i = 2, 3$. Therefore, since $-n_{11} < 0$, $n_{11} - n_{12} < 0\ldots$, we get that

$$Q(t) = (t_1, p_2(\ell_{10}(t_2), \ell_{20}(t_2)), p_3(\ell_{10}(t_2), \ell_{20}(t_2))) \in (\mathbb{C} \ll t_2 \gg)[t_1],$$

where $h(\ell_{10}(t_2), \ell_{20}(t_2)) = 0$.

Remark 4. One reason similarly for the case that $h(s_1, s_2)$ divides p_{22} and $\gcd(h, p_{i2}) = 1$, $i = 1, 3$. In this case, we obtain that $\overline{V}$ is a cylinder over the y axis.

If $h(s_1, s_2)$ divides p_{32} and $\gcd(h, p_{i2}) = 1$, $i = 1, 2$, we obtain that $\overline{V}$ is a cylinder over the z axis.

We observe that the situation presented in Theorem 2 is very usual when we are considering applied problems or rational surface parametrization obtained from numerical problems from CAGD or CAD.

Example 4. Let the surface V be parametrically defined by

$$\begin{aligned} P(s_1, s_2) &= (p_{11}(s_1, s_2) / p_{12}(s_1, s_2), p_{21}(s_1, s_2) / p_{22}(s_1, s_2), p_{31}(s_1, s_2) / p_{32}(s_1, s_2)) \\ &= (s_1^2 / (s_2^2 + s_1^2 + s_1 + s_2 - 3), (s_1^2 - s_2^2 - 5s_1 + 1) / (s_2^2 - 1), (s_1^2 - s_2 - 1) / (s_2 + 2)). \end{aligned}$$

Observe that we are in the conditions of Theorem 2, and thus we compute $\overline{V}$ a cylinder over the x axis that has the same asymptotic behavior as V. More precisely, by eliminating the variables s_1, s_2 from the system

$$p_{21}(s_1, s_2) - yp_{22}(s_1, s_2) = p_{31}(s_1, s_2) - zp_{32}(s_1, s_2) = h(s_1, s_2) = 0,$$

where $h(s_1, s_2) = s_2^2 + s_1^2 + s_1 + s_2 - 3,$ we get the implicit equation

$$\overline{f}_x(y,z) = -3093 - 5022z - 176z^4 - 9y^4 + 372y^3 + 666y^3z + 34y^4z - 200z^4y - 32z^4y^2 + 59y^4z^2 - 457y^3z^3 - 11y^3z^2 + 9y^4z^4 - 60y^4z^3 + 30y^3z^4 - 1972z^3y - 1382z^3y^2 - 3341yz - 1318yz^2 - 762y^2z - 506y^2z^2 - 1496z^2 - 648y^2 - 1112z^3 - 2241y$$

that defines a cylinder, V_1, over the x axis that has the same asymptotic behavior as V. In Figure 4, we plot V and V_1 together (left), V (center), and V_1 (right).

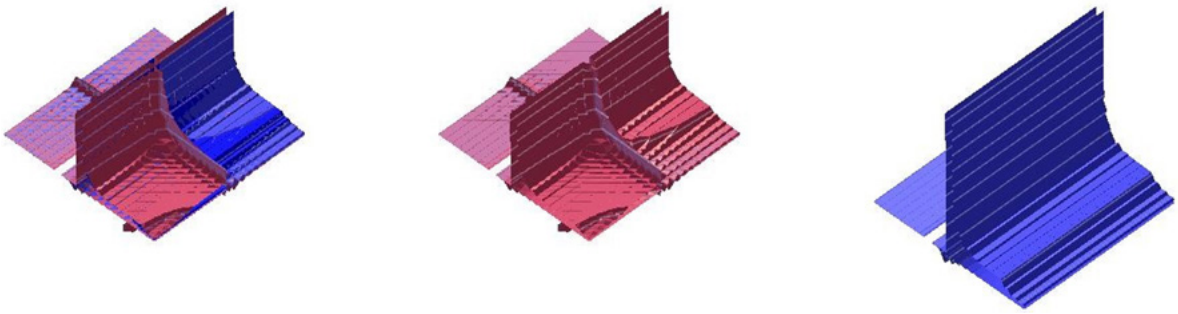

Figure 4. V and V_1 together (left), V (center), and V_1 (right)

Reasoning similarly (see Remark 4), we obtain two new cylinders over the y axis and z axis, respectively, that have the same asymptotic behavior as the input surface. More precisely, we have that by eliminating the variables s_1, s_2 from the system $p_{11}(s_1, s_2) - xp_{12}(s_1, s_2) = p_{31}(s_1, s_2) - zp_{32}(s_1, s_2) = h(s_1, s_2) = 0$, where $h(s_1, s_2) = s_2^2 - 1,$ we get two implicit equations

$$\overline{f}_y^2(x,z) = 9x^2z^2 + 3x^2z - x^2 - 4x + 4 - 18xz + 12z - 18xz^2 + 9z^2$$

and

$$\overline{f}_y^2(x,z) = 9x^2z^2 + 3x^2z - x^2 - 4x + 4 - 18xz + 12z - 18xz^2 + 9z^2$$

that defines two cylinders, V_1 and V_2, over the y axis that has the same asymptotic behavior as V (note that the polynomial h factorizes as $h = (s_2 - 1)(s_2 + 1)$). In Figure 5, we plot V and V_i, $i = 1, 2$ (left), V_1 (center), and V_2 (right).

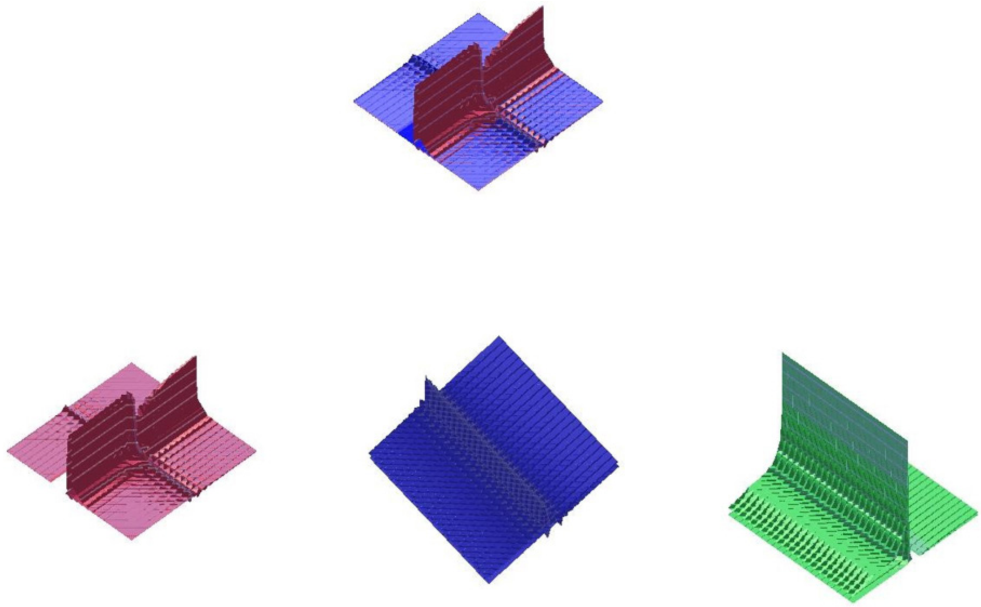

Figure 5. First row: V and V_i, $i = 1, 2$ together. Second row: surfaces V (right), V_1 (center) and V_2 (right)

Similarly, by eliminating the variables s_1, s_2 from the system $p_{21}(s_1, s_2) - yp_{22}(s_1, s_2) = p_{11}(s_1, s_2) - xp_{12}(s_1, s_2) = h(s_1, s_2) = 0$, where $h(s_1, s_2) = s_2 + 2$, we get the implicit equation

$$\overline{f}_z(x, y) = 44x^2 - 9x^2y^2 + 6x^2y + 18xy^2 - 28x + 15xy - 9 - 18y - 9y^2$$

that defines a cylinder over the z axis that has the same asymptotic behavior as V. In Figure 6, we plot the surface V and V_1 together (left), surface V (center), and surface V_1 (right).

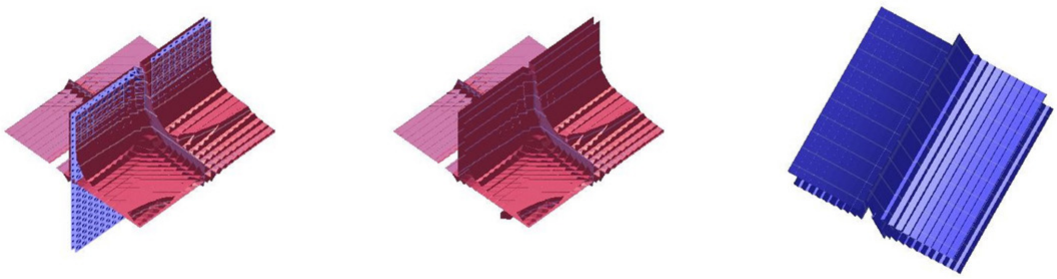

Figure 6. Surface V and V_1 together (left), surface V (center) and surface V_1 (right)

4. Conclusion

In this work, using some previous concepts introduced in [16] as branches at infinity, we obtain a method to compute infinity branches and surfaces having the same asymptotic behavior as an input surface that is parametrically defined. The results are a great novelty and represent an important advance for the analysis of surfaces and the study of their applications. In fact, since infinity branches reflect the status of a given surface at the infinity, in future work, our idea is to use these entities to deal with some important problems, such as plotting surfaces, problems of modeling or blending, high-dimensional interpolation, rational approximation of non-rational curves and surfaces, etc. (see [5-9]).

As in the case of curves, some important questions that should be answered remain open (see [11, 17]). More precisely, we would be interested in formally introducing the notion of perfect surface as well as some properties and effective algorithms for computing generalized asymptotes from implicit and parametrically defined surfaces.

Furthermore, as we stated previously, one should deeply study the computation of φ_i, $i = 1, 2$. In addition, we note that in the approach we present in this paper, once we have computed the branches, we determine surfaces that have the same asymptotic behavior as the original surface by simply removing the terms with negative exponent from r_1 and r_2 (w.r.t. t_1). This question should be carefully analyzed since if we remove these terms in the variable t_1, we could be removing terms in the variable t_2 that could be necessary for the approximation between surfaces.

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Conflict of interest

All authors declare that they have no conflicts of interest.

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