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Research Article

A New First Finite Class of Classical Orthogonal Polynomials Operational Matrices: An Application for Solving Fractional Differential Equations

H. M. Ahmed

Department of Mathematics, Faculty of Technology and Education, Helwan University, Cairo, Egypt
Email: hanyahmed@techedu.helwan.edu.eg

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Abstract: In this paper, new operational matrices (OMs) of ordinary and fractional derivatives (FDs) of a first finite class of classical orthogonal polynomials (FFCOP) are introduced. Also, two algorithms are proposed for using the tau and collocation spectral methods (SPMs) to get new approximate solutions to the given fractional differential equations (FDEs). These algorithms convert the given FDEs subject to initial/boundary conditions (I/BCs) into linear or nonlinear systems of algebraic equations that can be solved using appropriate solvers. To demonstrate the robustness, efficiency, and accuracy of the proposed spectral solutions, several illustrative examples are presented. The obtained results show that the proposed algorithms exhibit higher accuracy compared to existing techniques in the literature. Furthermore, an error analysis is provided.

Keywords: tau method, collocation method, fractional-order differential equations, operational matrix, classical orthogonal polynomials, error bounds for initial value, initial boundary value problems

MSC: 65M70; 34A08; 33C45; 65M15

1. Introduction

At the present time, the families of classical orthogonal polynomials (COPs) play a significant role in the fundamental mathematical mechanisms used in modern physics, engineering, and mathematical algorithms [1-4]. Three new classes of finite families of hypergeometric orthogonal polynomials (OPs) were introduced by Masjed-Jamei [5, 6]. An important set of attributes for these classes has been described in [7-9]. There are extensive studies in which the COPs play a main role in the numerical techniques for solving many types of differential equations (DEs) (see, for instance, [10-12]).

Collocation, tau, and Galerkin - three well-known collocation spectral methods (SPMs) - have suggested a numerical solution expressed in terms of specific trial-basis functions, which are typically OPs. These methods produce superior error characteristics and quick convergence. Based on these SPMs, several authors have come up with different algorithms to solve ordinary, partial, and fractional differential equations (FDEs) [13-17].

Orthogonal polynomial-based SPMs are a popular choice for solving FDEs. Recent papers have proposed various

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SPMs using Laguerre, Hermite, Chebyshev, and Jacobi polynomials for solving linear, nonlinear, and variable-coefficient FDEs [18-21]. These methods have demonstrated high accuracy and efficiency. Overall, these recent papers highlight the continued usefulness of these methods in solving FDEs.

Many studies have shown how useful it is to use operational matrices (OMs) of the derivatives of OPs to solve ordinary and FDEs numerically [22-29]. The main goal of this paper is to come up with new numerical spectral solutions for some types of FDE models. To achieve this goal, we construct OMs of ordinary and fractional derivatives (FDs) of the finite class of classical orthogonal polynomials (FFCOP) and utilize them with the tau and collocation SPMs to obtain numerical solutions to these models. Additionally, FFCOP still has a few applications in the field of numerical approaches.

The paper is structured as follows: Section 2 provides the most important definitions of fractional calculus theory, along with some important properties and useful identities involving FFCOP. In Section 3, two OMs of ordinary and fractional-order derivatives of FFCOP in the Caputo sense are made. In Section 4, these OMs are utilized to create two numerical algorithms for solving two models of linear and nonlinear FDEs using the tau and collocation methods, respectively. The convergence and error analysis of the suggested FFCOP expansion are investigated in Section 5. In Section 6, numerical examples and discussions are given to show how the two suggested algorithms can be used and how effective and accurate the proposed spectral solutions are. Finally, Section 7 provides some conclusions.

2. Preliminaries and notation

This section provides the fundamental definitions and properties of fractional differentiation that are utilized in this article. Additionally, we present some basic properties and useful identities involving FFCOP.

2.1 *Some basic definitions of fractional differentiation*

While there are many mathematical definitions of FDs [30], in this section, we provide some well-known definitions and properties of fractional calculus.

Definition 1. The Riemann-Liouville fractional integral operator I^{α} of order α on the usual Lebesgue space $L_1[0, 1]$ is defined as

$$(I^{\alpha} f)(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0, x > 0, \\ f(x), & \alpha = 0. \end{cases}$$

For μ, $\nu \geq 0$, and $\gamma > -1$, the following properties are satisfied:

$$\begin{aligned} I^{\mu} I^{\nu} &= I^{\mu+\nu}, \\ I^{\mu} I^{\nu} &= I^{\nu} I^{\mu}, \\ I^{\mu} x^{\gamma} &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)} x^{\gamma+\mu}. \end{aligned}$$

Definition 2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by

$$(D_*^{\alpha} f)(x) = \left(\frac{d}{dt}\right)^m (I^{m-\alpha} f)(x), m-1 < \alpha \leq m, m \in \mathbb{N}.$$

Definition 3. The fractional differential operator in Caputo sense is defined as

$$(D^{\alpha} f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \alpha > 0, x > 0,$$

where $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$.

The following property is satisfied by the operator D^{α},

$$D^{\alpha} x^k = \begin{cases} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} x^{k-\alpha}, & k \in \mathbb{N}, k \geq \lceil \alpha \rceil, \\ 0, & k \in \mathbb{N}, k < \lceil \alpha \rceil, \end{cases} \tag{1}$$

where the symbol $\lceil \alpha \rceil$ refers to the smallest integer greater than or equal to α. For more information on the fundamentals of FDs and integrals, we recommend that interested readers refer to Atangana's work [31, 32].

Remark 1. It is noteworthy here to indicate that the commonly used definition of a fractional derivative is the Caputo derivative. This is because it is the best fractional operator for modeling real-world problems [33-35]. One of the main advantages of this definition is that it permits the I/BCs to be considered when forming the problem [31, 32]. As well, applying this definition to a constant gives zero. Moreover, one can convert higher fractional-order differential systems into lower ones using this definition [36]. For more information on the comparison between Caputo and Riemann-Liouville operators, we recommend that interested readers refer to [32, 37].

2.2 *Some properties of FFCOP*

FFCOP $\{M_i^{(p,q)}(x) : i = 0, 1, 2, ..., N, p > 2N+1, q > -1\}$ was introduced by Masjed-Jamei [5]. These polynomials satisfy the following DE

$$(x^2 + x)y''(x) + ((2-p)x + q)y'(x) - n(n+1-p)y(x) = 0, \tag{2}$$

and they are finitely orthogonal w.r.t. the weight function $w(x) = x^q(1 + x)^{-(p+q)}$, $x \in [0, \infty)$, when $p > 2N + 1$, $q > -1$ and satisfying the orthogonality relation

$$\int_0^{\infty} w(x) M_n^{(p,q)}(x) M_m^{(p,q)}(x)\, dx = h_n \delta_{nm}, \tag{3}$$

where $h_n = \frac{n!\Gamma(p-n)\Gamma(q+n+1)}{(p-2n-1)\Gamma(p+q-n)}$. An explicit polynomial solution for equation (2) is

$$M_n^{(p,q)}(x) = \sum_{k=0}^{n} C_{n,k}^{(p,q)} x^k, \ n \geq 0, \tag{4}$$

where

$$C_{n,k}^{(p,q)} = (-1)^{n+k} n! \binom{p-n-1}{k} \binom{q+n}{n-k}, \tag{5}$$

which can be written in the following hypergeometric representation

$$M_n^{(p,q)}(x) = (-1)^n n! \binom{q+n}{n} {}_2F_1\left(-n, n+1-p; q+1, -x\right). \tag{6}$$

Another representation of (4) is Rodrigues formula

$$M_n^{(p,q)}(x) = (-1)^n \frac{(1+x)^{p+q}}{x^q} D^n[x^{q+n}(1+x)^{n-p-q}],\ D^n \equiv \frac{d^n}{dx^n},\ n = 0, 1, 2, \ldots, \tag{7}$$

and by using this formula [5, p.186-187], Masjed-Jamei gave and proved the inversion formula

$$x^m = \sum_{n=0}^{m} d_{m,n}^{(p,q)} M_n^{(p,q)}(x), \tag{8}$$

where

$$d_{m,n}^{(p,q)} = \frac{(p-2n-1)\Gamma(p-n-m-1)\Gamma(q+m+1)}{\Gamma(p-n)\Gamma(q+n+1)} \binom{m}{n}. \tag{9}$$

Also, in the same paper, he shows that:

$$\begin{aligned} &M_{n+1}^{(p,q)}(x) = \frac{1}{(p-n-1)(p-2n)} \left[(\alpha_n x + \beta_n) M_n^{(p,q)}(x) - \gamma_n M_{n-1}^{(p,q)}(x)\right], n = 1, 2, \ldots, \\ &M_0^{(p,q)}(x) = 1,\ M_1^{(p,q)}(x) = (p-2)x - (q+1), \end{aligned} \tag{10}$$

and

$$DM_n^{(p,q)}(x) = n(p-n-1) M_{n-1}^{(p-2,q+1)}(x), \tag{11}$$

where $\alpha_n = (p-2n-2)_3$, $\beta_n = (p-2n-1)(2n(n+1) - p(q+2n+1))$, $\gamma_n = n(p-2n-2)(p+q-n)(q+n)$.

3. Operational matrices of derivatives of $M_n^{(p,q)}(x)$

Assume that $y(x) \in L_w^2[0, 1]$, and suppose it can be written as follows in terms of $M_k^{(p,q)}(x)$:

$$y(x) \approx y_N(x) = \sum_{k=0}^{N} a_k M_k^{(p,q)}(x) = \mathbf{A}^T \Phi(x),\ N < \frac{p-1}{2}, \tag{12}$$

where

$$\mathbf{A}^T = [a_0, a_1, \ldots, a_N], \tag{13}$$

and

$$\Phi(x) = [M_0^{(p,q)}(x), M_1^{(p,q)}(x), \ldots, M_N^{(p,q)}(x)]^T, \tag{14}$$

and the coefficient a_k $(0 \le k \le N)$ is given by [5, p.185]:

$$a_k = \frac{1}{h_k} \int_0^{\infty} w(x) y(x) M_k^{(p,q)}(x)\, dx,\ k = 0, 1, 2, \ldots, N. \tag{15}$$

Theorem 1. The following is an expression for $M_n(p, q)$'s first derivative:

$$DM_n^{(p,q)}(x) = \sum_{k=0}^{n-1} \Theta_{n,k} M_k^{(p,q)}(x), \tag{16}$$

where

$$\Theta_{n,k} = \frac{(-1)^{n-k}\, n!(p-2k-1)(p-n-1)!}{k!(p-k-1)!(q+k)!(p+q-n-1)!}[(n+q)!(p+q-n-1)! - (q+k)!(p+q-k-1)!]. \tag{17}$$

Therefore, the form of the derivative of $\Phi(x)$ of order r is as follows:

$$D^r\,\Phi(x) = \mathbf{M}^{(r)}\Phi(x) = \left(\mathbf{M}^{(1)}\right)^r \Phi(x),\ r = 1, 2, 3, ..., \tag{18}$$

where the elements of $\mathbf{M}^{(1)} = (m_{ij}^{(1)})_{0\le i,j\le N}$ have the form

$$m_{ij}^{(1)} = \begin{cases} \Theta_{i,j}, & i > j, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Proof. Differentiating formula (4) with respect to x gives

$$DM_n^{(p,q)}(x) = \sum_{k=0}^{n-1}(k+1)C_{n,k+1}^{(p,q)}\,x^k,\ n \ge 1. \tag{20}$$

Substituting (8) into (20), expanding and collecting similar terms, we obtain

$$DM_n^{(p,q)}(x) = \sum_{k=0}^{n-1}\Theta_{n,k}\,M_k^{(p,q)}(x),$$

where

$$\Theta_{n,k} = \sum_{i=0}^{n-k-1}(i+k+1)C_{n,i+k+1}^{(p,q)}\,d_{i+k,k}^{(p,q)}, \tag{21}$$

then

$$\Theta_{n,n-k-1} = \sum_{i=0}^{k}(i+n-k)C_{n,i+n-k}^{(p,q)}\,d_{i+n-k-1,n-k-1}^{(p,q)} = \sum_{i=0}^{k}(n-i)C_{n,n-i}^{(p,q)}\,d_{n-i-1,n-k-1}^{(p,q)}. \tag{22}$$

Substitution of (5) and (9) into (22) gives

$$\Theta_{n,n-k-1} = \frac{n!(n+q)!(p-n-1)!(p+2k-2n+1)}{(n-k-1)!(p+k-n)!(n+q-k-1)!}S_k(n), \tag{23}$$

where

$$S_k(n) = \sum_{i=0}^{k}\frac{(-1)^i(p+k-2n+i)!}{i!(k-i)!(p-2n+i-1)!(n+q-i)}.$$

Let $A_k(n) = S_k(n) + (-1)^k$, then by employing Zeilberger's algorithm, we obtain:

$$(k-n-q+1)A_{k+1}(n) - (p+q+k-n+1)A_k(n) = 0, \tag{24}$$

with the initial conditions $A_0(n) = \dfrac{p+q-n}{n+q}$. This recurrence relation has the exact solution

$$A_k(n) = \frac{(-1)^k (p+q+k-n)!(n+q-k-1)!}{(p+q-n-1)!(n+q)!}, \tag{25}$$

then

$$S_k(n) = (-1)^{k+1}\left[1 - \frac{(n+q-k-1)!(p+q+k-n)!}{(p+q-n-1)!(n+q)!}\right]. \tag{26}$$

Substitution of (26) into (23) – after some rather manipulation – yields (17). In view of formula (16), we get

$$D\Phi(x) = \mathbf{M}^{(1)}\Phi(x), \tag{27}$$

and then formula (18) is a direct consequence of (27) and this completes the proof of the theorem.

According to Theorem 1, for $N = 5$, the OMs $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ for the case of $p = 12$, $q = 0$ are the following two (6×6) matrices:

$$\mathbf{M}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 \\ -\frac{108}{5} & \frac{72}{5} & 0 & 0 & 0 & 0 \\ \frac{328}{5} & -\frac{252}{5} & 14 & 0 & 0 & 0 \\ -\frac{1316}{5} & \frac{1044}{5} & -70 & 10 & 0 & 0 \\ \frac{9220}{7} & -\frac{7380}{7} & 370 & -\frac{450}{7} & \frac{30}{7} & 0 \end{pmatrix}_{6\times 6},$$

and

$$\mathbf{M}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 144 & 0 & 0 & 0 & 0 & 0 \\ -\frac{4032}{5} & \frac{1008}{5} & 0 & 0 & 0 & 0 \\ 4256 & -1512 & 140 & 0 & 0 & 0 \\ -23880 & \frac{66240}{7} & -1200 & \frac{300}{7} & 0 & 0 \end{pmatrix}_{6\times 6},$$

respectively.

Theorem 2. Let $\alpha > 0$ and $s = \lceil \alpha \rceil$, then

$$D^{\alpha}\Phi(x) \simeq \mathbf{D}^{(\alpha)}\Phi(x), \tag{28}$$

where $\mathbf{D}^{(\alpha)} = (d_{\alpha}(i, j))$ is a matrix of order $(N + 1) \times (N + 1)$ which can be expressed explicitly as

$$\begin{pmatrix} 0 & \dots & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & \dots & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ \Theta_{s,0}^{(\alpha)} & \dots & \dots & \Theta_{s,s}^{(\alpha)} & 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & \dots & \dots & 0 \\ \Theta_{i,0}^{(\alpha)} & \dots & \dots & \dots & \dots & \Theta_{i,i}^{(\alpha)} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots \\ \Theta_{N,0}^{(\alpha)} & \dots & \dots & \dots & \dots & \Theta_{N,i}^{(\alpha)} & \dots & \dots & \Theta_{N,N}^{(\alpha)} \end{pmatrix}. \tag{29}$$

where

$$d_\alpha(i,j) = \begin{cases} \Theta_{i,j}^{(\alpha)}, & i \ge s, i \ge j, \\ 0, & \text{otherwise}, \end{cases} \tag{30}$$

and

$$\Theta_{i,j}^{(\alpha)} = \frac{(-1)^{i+s} i!\, s!(p-2j-1)\Gamma(1+q+s-\alpha)\Gamma(-1-j+p-s+\alpha)}{j!\,\Gamma(p-j)\Gamma(q+j+1)\Gamma(1-j+s-\alpha)} \binom{p-i-1}{s}\binom{i+q}{i-s}$$
$$\times\, {}_4F_3\left(\begin{matrix} -i+s, 1, 1+i-p+s, q+s-\alpha+1 \\ 1+q+s, 1-j+s-\alpha, 2+j-p+s-\alpha \end{matrix}; 1\right). \tag{31}$$

Proof. In view of (1) and (4),

$$D^\alpha M_i^{(p,q)}(x) = \sum_{k=s}^{i} C_{i,k}^{(p,q)} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} x^{k-\alpha}, \, i \ge s. \tag{32}$$

Now, using (15) with $y(x) = x^{k-\alpha}$, one can obtain

$$x^{k-\alpha} \simeq \sum_{j=0}^{N} d_{k-\alpha,j}^{(p,q)} M_j^{(p,q)}(x), \tag{33}$$

where

$$d_{\alpha,j}^{(p,q)} = \frac{(p-2j-1)\Gamma(p-j-\alpha-1)\Gamma(q+\alpha+1)}{\Gamma(p-j)\Gamma(q+j+1)} \binom{\alpha}{j}. \tag{34}$$

Substituting (33) into (32), collecting comparable terms, and using some manipulation leads to

$$D^\alpha M_i^{(p,q)}(x) \simeq \sum_{j=0}^{N} \Theta_{i,j}^{(\alpha)} M_j^{(p,q)}(x), \tag{35}$$

where $\Theta_{i,j}^{(\alpha)}$ is defined as in (31). Thus, equation (35) takes the form:

$$D^\alpha M_i^{(p,q)}(x) \simeq [\Theta_{i,0}^{(\alpha)}, \Theta_{i,1}^{(\alpha)}, \dots, \Theta_{i,N}^{(\alpha)}]\Phi(x), \, i = s, s+1, \dots, N. \tag{36}$$

In view of (1), we have

$$D^\alpha M_i^{(p,q)}(x) \simeq [0, 0, \dots, 0]\Phi(x), i = 0, 1, \dots, s-1. \tag{37}$$

The desired outcome is obtained when two equations (36) and (37) are combined.

According to Theorem 2, for $N = 5$, the OMs $\mathbf{D}^{(1/2)}$ and $\mathbf{D}^{(3/2)}$ for the case of $p = 12$, $q = 1$ are the following two (4×4) matrices:

$$\mathbf{D}^{(1/2)} = \sqrt{\pi}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{692835}{262144} & \frac{328185}{524288} & 0 & 0 & 0 & 0 \\ -\frac{1990989}{262144} & \frac{1007721}{524288} & \frac{375375}{524288} & 0 & 0 & 0 \\ \frac{478335}{16384} & -\frac{274131}{32768} & \frac{41195}{32768} & \frac{86505}{131072} & 0 & 0 \\ -\frac{4675671}{32768} & \frac{2704779}{65536} & -\frac{562275}{65536} & \frac{89775}{262144} & \frac{66675}{131072} & 0 \\ \frac{27769005}{32768} & -\frac{16033545}{65536} & \frac{3539025}{65536} & -\frac{1387125}{262144} & -\frac{92385}{131072} & \frac{63679}{262144} \end{pmatrix}_{6\times 6},$$

and

$$\mathbf{D}^{(3/2)} = \sqrt{\pi}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1247103}{32768} & \frac{590733}{65536} & -\frac{45045}{65536} & 0 & 0 & 0 \\ -\frac{1072071}{4096} & -\frac{81081}{8192} & \frac{105105}{8192} & -\frac{31185}{32768} & 0 & 0 \\ \frac{3378375}{2048} & -\frac{135135}{2048} & -\frac{121275}{2048} & \frac{193725}{16384} & -\frac{6405}{8192} & 0 \\ -\frac{89131185}{8192} & \frac{12309165}{16384} & \frac{4984875}{16384} & -\frac{5214375}{65536} & \frac{242325}{32768} & -\frac{19803}{65536} \end{pmatrix}_{6\times 6},$$

respectively.

4. Spectral solutions for fractional-order differential equation through FFCOP operational matrices

The tau and collocation approaches will be used in this section to build two numerical algorithms for solving general linear and nonlinear multi-term FDEs (38) and (47), respectively, subject to I/BCs, which are represented in the two forms (39) and (40), respectively, and for which the FFCOP tau method (FFCOPTM) and the FFCOP collocation method (FFCOPCM) were proposed to solve them numerically.

4.1 *FFCOPTM for handling linear multi-order FDEs*

In this part, the following linear FDEs are taken into account:

$$D^{\mu} y(x) = \sum_{i=1}^{m} \zeta_i D^{\nu_i} y(x) + \zeta_{m+1} y(x) + g(x), x \in (0,1), \mu \geq 1, \tag{38}$$

subject to the ICs

$$y^{(i)}(0) = \alpha_i, i = 0, 1, ..., n-1, n \geq 1, \tag{39}$$

or the BCs

$$y^{(i)}(0) = \alpha_i, y^{(j)}(\ell) = \beta_j,\ i = 0,1,...,n_1, j = 0,1,...,n_2, n_1 + n_2 = n-2, n \geq 2, \tag{40}$$

where $\zeta_k (k = 1,...,m+1), \alpha_i$ and β_j, for specified values of i and j, are real constants and $n - 1 < \mu \leq n$, $0 < \nu_1 < \nu_2 < ... < \nu_m < \mu$. In addition, $D^{\mu}y(x)$ denotes the Caputo FD of order μ for $y(x)$, and $g(x)$ is a given source function. To solve FDE (38) subject to the conditions (39) or (40), we approximate $y(x)$ and $g(x)$ by FFCOP as

$$y(x) \simeq \mathbf{A}^T \Phi(x) \text{ and } g(x) \simeq \mathbf{G}^T \Phi(x), \tag{41}$$

where $\mathbf{G} = [g_0,...,g_N]^T$ and $\mathbf{A} = [a_0,...,a_N]^T$. Also, the derivative $D^{\gamma} y(x)$ can be approximated as

$$D^{\gamma} y(x) = \begin{cases} \mathbf{A}^T \mathbf{M}^{(\gamma)} \Phi(x), \gamma \text{ is an integer}, \\ \\ \mathbf{A}^T \mathbf{D}^{(\gamma)} \Phi(x), \gamma \text{ is a fraction}. \end{cases} \tag{42}$$

In virtue of these approximations, the residual can be written in the form

$$R_N(x) = [\mathbf{A}^T \mathbf{D}^{(\mu)} - \mathbf{A}^T \sum_{i=1}^{m} \zeta_i \mathbf{D}^{(\nu_i)} - \zeta_{m+1} \mathbf{A}^T - \mathbf{G}^T] \Phi(x). \tag{43}$$

The following $(N - n + 1)$ linear equations in the unknown vector $\mathbf{A}$ can be obtained by applying the tau method,

$$(R_N(x), M_i^{(p,q)}(x))_w = \int_0^{\infty} w(x) R_N(x) M_i^{(p,q)}(x) dx = 0, i = 0,1,..., N-n. \tag{44}$$

In addition, conditions (39) or (40), give

$$y^{(i)}(0) = \mathbf{A}^T \mathbf{M}^{(i)} \Phi(0) = \alpha_i, i = 0,1,..., n-1, \tag{45}$$

or

$$y^{(i)}(0) = \mathbf{A}^T \mathbf{M}^{(i)} \Phi(0) = \alpha_i, y^{(j)}(\ell) = \mathbf{A}^T \mathbf{M}^{(j)} \Phi(\ell) = \beta_j,\ i = 0,1,...,n_1, j = 0,1,...,n_2, \tag{46}$$

respectively. In order to find the unknown coefficients of the vector $\mathbf{A}$, two equations (43) and (45) or two equations (43) and (46) give a system of $(N + 1)$ linear algebraic equations that can be solved using a suitable solver. As a result, $y_N(x)$ is obtained as a solution of (38) with the conditions (39) or (40).

4.2 *FFCOPCM for handling nonlinear multi-order FDEs*

In this section, we wish to solve numerically the nonlinear FDE

$$y^{(\mu)}(x) = F(x, y(x), y^{(\nu_1)}(x), ..., y^{(\nu_m)}(x)), x \in (0,1), \mu \geq 1, \tag{47}$$

subject to the conditions (39) or (40). To do that, we first approximate $y(x), g(x), D^{\mu} y(x)$, and $D^{\nu_i} y(x)(i = 1, ..., m)$ as we show in Section 4.1. Then, substitute these approximations in equations (39), (40), and (47). In view of these substitutions, we obtain

$$\mathbf{A}^T \mathbf{D}^{(\mu)} \Phi(x) \simeq F(x, \mathbf{A}^T \Phi(x), \mathbf{A}^T \mathbf{D}^{(\nu_1)} \Phi(x), ..., \mathbf{A}^T \mathbf{D}^{(\nu_m)} \Phi(x)), \tag{48}$$

in addition to equations (45) and (46). To find the numerical solution $y_N(x)$ of (47) subject to conditions (39) or (40), we employ the collocation method. The $(N - n+1)$ zeros of $M_{N-n+1}^{(p,q)}(x)$ on $(0, \infty)$ are employed as collocation points and collect equation (48) at these points to get:

$$\mathbf{A}^T \mathbf{D}^{(\mu)} \Phi(x_i) - F(x_i, \mathbf{A}^T \Phi(x_i), \mathbf{A}^T \mathbf{D}^{(\nu_1)} \Phi(x_i), \ldots, \mathbf{A}^T \mathbf{D}^{(\nu_m)} \Phi(x_i)) = 0, i = 0,1,\ldots, N-n. \tag{49}$$

These equations, together with equation (45) or (46) generate $(N + 1)$ nonlinear algebraic system whose solution is found using Newton's iterative method.

5. Convergence and error analysis

In this section, we give a thorough investigation of the proposed FFCOP expansion's convergence analysis. It is proved that the FFCOP expansion of a function $f(x)$ with bounded nth derivatives converges uniformly to $f(x)$. Additionally, an upper bound for the truncated expansion's error is provided. The following lemma is needed:

Lemma 1. The following estimate holds for $M_i^{(p,q)}(x)$ for all $i = 0, 1, 2, ..., N$, where $N < \frac{p-1}{2}$ and $q > -1$,

$$\left| M_i^{(p,q)}(x) \right| \leq \frac{\Gamma(p+q)}{\Gamma(p+q-i)}, \; x \in [0, 1]. \tag{50}$$

Proof. From (4) and (5), we have

$$\left| M_i^{(p,q)}(x) \right| \leq i! \sum_{k=0}^{i} \binom{p-i-1}{k} \binom{q+i}{i-k}, x \in [0, 1]. \tag{51}$$

The inequality (51), after some rather manipulation, takes the form

$$\left| M_i^{(p,q)}(x) \right| \leq i! \binom{q+i}{i} {}_2F_1\left(-i, 1+i-p, \, q+1; 1\right). \tag{52}$$

The application of Chu-Vandermonde identity leads to writing (52) in the form of (50) and the proof is completed.

Theorem 3. A function $y(x) \in L_w^2(0, \infty)$, with $\left| y^{(n)}(x) \right| \leq L_n$, can be represented by an infinite sum of FFCOP basis functions given in equation (12). Furthermore, the series converges uniformly to $y(x)$, and the expansion coefficients in equation (12) satisfy the following inequality:

$$|a_i| < \frac{L_n \left(\frac{q}{p} \right)^n}{n!} \frac{2^i}{i!} \frac{\Gamma(p+q-i)}{\Gamma(p+q)}, \; i \geq n, 0 < q < p, \tag{53}$$

Proof. From equation (15), we have

$$a_i = \frac{1}{h_i}\int_0^\infty w(x)\,y(x)\,M_i^{(p,q)}(x)\,dx. \tag{54}$$

The function $y(x)$ can be written as

$$y(x) = \sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}x^k + \frac{y^{(n)}(\zeta_x)}{n!}x^n,\ \zeta_x \in (0,x). \tag{55}$$

Substitution of (55), using the orthogonality relation and taking into consideration the given condition $i \ge n$,

$$\int_0^\infty w(x)\,x^k\,M_i^{(p,q)}(x)\,dx = 0,\ k = 0,1,\dots,n-1,$$

and $\left|y^{(n)}(x)\right| \le L_n$, enable one to obtain

$$|a_i| < \frac{L_n}{n!\,h_i}\int_0^\infty w(x)\,x^n\left|M_i^{(p,q)}(x)\right|dx. \tag{56}$$

Now, using formula (4) and

$$\int_0^\infty w(x)\,x^{k+n}\,dx = \frac{\Gamma(p-k-n-1)\Gamma(q+k+n+1)}{\Gamma(p+q)},$$

enable one to write relation (56) in the form

$$|a_i| < \frac{L_n}{n!\,h_i}\sum_{k=0}^{i}\left|C_{i,k}^{(p,q)}\right|\frac{\Gamma(p-k-n-1)\Gamma(q+k+n+1)}{\Gamma(p+q)}. \tag{57}$$

Substituting (5) into (57) and using the asymptotic result in [38, p.233]:

$$\Gamma(ax+b) \sim \sqrt{2\pi}\,e^{-ax}(ax)^{ax+b-1/2},\ x \gg 0,\ a > 0, \tag{58}$$

and after some algebraic computations, we obtain (53) and this completes the theorem's proof.

Theorem 4. Assume that $y(x)$ meets the conditions of Theorem 3, then the inequality shown below is valid

$$\|y - y_N\|_\infty \preceq \frac{\left(\frac{q}{p}\right)^n}{n!\sqrt{2\pi}}\frac{1}{(N+1)^{1/2}(N-2e+1)}\left(\frac{2e}{N+1}\right)^N, \tag{59}$$

for $0 < q < p$ and $N > 2e - 1$.

Proof. Applying Theorem 3 and Lemma 1 lead to

$$\|y - y_N\|_\infty \le \max_{x\in[0,1]}\sum_{i=N+1}^{\infty}\left|a_i\,M_i^{(p,q)}(x)\right| \le \frac{L_n\left(\frac{q}{p}\right)^n}{n!}\sum_{i=N+1}^{\infty}\frac{2^i}{i!}.$$

Using that

$$i! \sim \sqrt{2\pi i}\left(\frac{i}{e}\right)^i, \tag{60}$$

we obtain

$$\|y-y_N\|_\infty \le \frac{L_n\left(\frac{q}{p}\right)^n}{n!\sqrt{2\pi}}\sum_{i=N+1}^{\infty} i^{-1/2}\left(\frac{2e}{i}\right)^i < \frac{L_n\left(\frac{q}{p}\right)^n}{n!\sqrt{2\pi}}\frac{1}{(N+1)^{1/2}}R_N, \tag{61}$$

where $R_N = \sum_{i=N+1}^{\infty}\left(\frac{2e}{N+1}\right)^i$, which is convergent for $N > 2e - 1$, so

$$R_N = \frac{2e}{(N-2e+1)}\left(\frac{2e}{N+1}\right)^N,$$

and Theorem 4 is now proved.

Theorem 5. Assume that $y(x)$ meets the conditions of Theorem 3, then the inequality shown below is valid:

$$\|y-y_N\|_w \preceq \frac{\left(\frac{q}{p}\right)^n q^{1/4}}{n!(N+1)^{1/4}}\frac{\left(\frac{4eq}{(N+1)p}\right)^{(N+1)/2}}{\sqrt{1-\frac{4eq}{(N+1)p}}}, \tag{62}$$

for $0 < q < p$ and $N > 4e - 1$.

Proof. Using equation (3) gives

$$\|y-y_N\|_w^2 = \sum_{i=N+1}^{\infty} a_i^2 h_i. \tag{63}$$

The following inequality results from applying Theorem 3, then identity (63):

$$\|y-y_N\|_w^2 < \frac{L_n^2 p^{-2n} q^{2n}}{(n!)^2\Gamma(p+q)}\sum_{i=N+1}^{\infty}\frac{4^i\Gamma(p-i)\Gamma(q+i+1)}{i!}.$$

Applying two relations (58) and (60) and after some rather manipulation, the following inequality also holds:

$$\|y-y_N\|_w^2 < \frac{L_n^2\sqrt{q}\left(\frac{q}{p}\right)^{2n}}{\sqrt{N+1}(n!)^2}\sum_{i=N+1}^{\infty}\left(\frac{4eq}{(N+1)p}\right)^i. \tag{64}$$

Under the given two conditions $0 < q < p$ and $N > 4e - 1$, we get

$$\|y-y_N\|_w^2 < \frac{L_n^2\sqrt{q}\left(\frac{q}{p}\right)^{2n}}{\sqrt{N+1}(n!)^2}\frac{\left(\frac{4eq}{(N+1)p}\right)^{N+1}}{1-\frac{4eq}{(N+1)p}}. \tag{65}$$

Finally, the inequality (62) is obtained. This proves Theorem 5.

Theorem 6. Under Theorem 3's presumptions, it results in:

$$\lim_{N\to\infty}\|y-y_N\|_\infty=0 \quad\text{and}\quad \lim_{N\to\infty}\|y-y_N\|_w=0. \tag{66}$$

Proof. The proof follows from (59) and (62).

6. Numerical results

In this section, we carry out several test problems to demonstrate the effectiveness of the proposed methods presented in this paper. The calculations were performed on a computer running Mathematica 12 with an Intel(R) Core(TM) i9-10850 CPU at 3.60 GHz, 3600 MHz, 10 cores, and 20 logical processors.

Problem 1. The initial value problem for the inhomogeneous Bagely-Torvik equation is, see [39]

$$D^2 y(x)+D^{3/2}y(x)+y(x)=x+1,\ y(0)=1,\ y^{'}(0)=1,\ x\in[0,1], \tag{67}$$

where $y(x) = x + 1$. By applying the proposed method FFCOPTM with $N = 2$, $p > 5$, and $q > -1$, we have

$$y_N(x)=\sum_{i=0}^{2}a_iM_i^{(p,q)}(x)=\mathbf{A}^T\Phi(x),\, g(x)\simeq\sum_{i=0}^{2}g_i\,M_i^{(p,q)}(x)=\mathbf{G}^T\Phi(x), \tag{68}$$

where

$$\mathbf{A}=\begin{pmatrix}a_0\\ a_1\\ a_2\end{pmatrix} \text{and } G=\begin{pmatrix}\dfrac{p+q-1}{p-2}\\ \dfrac{1}{p-2}\\ 0\end{pmatrix}.$$

In view of two Theorems 1 and 2, we have

$$\mathbf{M}^{(2)}=\begin{pmatrix}0&0&0\\0&0&0\\2(p-4)(p-3)&0&0\end{pmatrix},\ \mathbf{D}^{(3/2)}=\begin{pmatrix}0&0&0\\0&0&0\\ \Theta_{2,0}^{(3/2)}&\Theta_{2,1}^{(3/2)}&\Theta_{2,2}^{(3/2)}\end{pmatrix},$$

then equation (44) gives

$$a_0+(2(p-4)(p-3)+\Theta_{2,0}^{(3/2)})a_2-\frac{p+q-1}{p-2}=0, \tag{69}$$

and equation (45) gives

$$\left.\begin{aligned}a_0-(1+q)a_1+(1+q)(2+q)a_2-1=0,\\ (p-2)a_1-2(p-3)(2+q)a_2-1=0.\end{aligned}\right\} \tag{70}$$

Finally, the solution of three equations (69) and (70) has the form

$$a_0 = \frac{p+q-1}{p-2}, \, a_1 = \frac{1}{p-2}, \, a_2 = 0,$$

then, $y_2(x) = \mathbf{A}^T \Phi(x) = y(x)$.

Problem 2. In this problem, we have the boundary Bagely-Torvik equation (see [40]),

$$D^2 y(x) + D^{3/2} y(x) + y(x) = x^2 + 2 + 4\sqrt{\frac{x}{\pi}}, \, y(0) = 0, \, y(1) = 1, \, x \in [0, 1], \tag{71}$$

where $y(x) = x^2$. The application of FFCOPTM with $N = 2$, $p > 5$, and $q > -1$, gives the system of equations

$$\left.\begin{aligned} &a_0 + (2(p-4)(p-3) + \Theta_{2,0}^{(3/2)})a_2 - g_0 = 0, n \\ &a_0 - (1+q)a_1 + (1+q)(2+q)a_2 = 0, \\ &a_0 + (p-q-3)a_1 + (q^2 - 2pq + 9q + p(p-11) + 26)a_2 - 1 = 0, \end{aligned}\right\} \tag{72}$$

where $g_0 = \frac{14 + 2(p-5)p + q(q+3)}{(p-3)(p-2)} + \frac{4\Gamma(p-3/2)\Gamma(q+3/2)}{\sqrt{\pi}\Gamma(p-1)\Gamma(q+1)}$. The solution of system (72) has the form

$$a_0 = \frac{(q+1)(q+2)}{(p-2)(p-3)}, \, a_1 = \frac{2(q+2)}{(p-2)(p-4)}, \, a_2 = \frac{1}{(p-3)(p-4)},$$

then, $y_2(x) = x^2 = y(x)$.

Problem 3. The solution to the initial value problem [41]

$$\begin{aligned} &D^{5/2} y(x) + D^{1/2} y(x) = 0, \, x \in (0,1), \\ &y(0) = 1, \, y'(0) = 0, \, y''(0) = -1, \end{aligned} \tag{73}$$

is $y(x) = \cos x$. In Table 1, we provide the FFCOPTM's maximum absolute error (MAE) for some values of p, q, and N, where $N < (p-1)/2$ and $0 < q < p$. Additionally, Figure 1 displays the absolute error for various selections of p, q, and N to show how the solutions have converged.

Table 1. MAE for different choices of p, q, and N for Problem 3

N	p	q	MAE	p	q	MAE	p	q	MAE
3			1.1×10^{-4}			7.6×10^{-5}			4.9×10^{-5}
5			8.7×10^{-7}			3.5×10^{-7}			5.7×10^{-8}
7			6.7×10^{-9}			3.3×10^{-9}			5.9×10^{-10}
9	39	3	3.9×10^{-11}	50	10	1.5×10^{-11}	100	20	6.2×10^{-12}
11			3.6×10^{-13}			2.4×10^{-13}			1.5×10^{-14}
13			2.0×10^{-15}			6.7×10^{-16}			2.3×10^{-16}
15			5.4×10^{-16}			5.5×10^{-16}			2.1×10^{-18}

Table 2 provides a comparison between the errors obtained from the numerical scheme and those obtained using FFCOPTM for different values of N, using $p = 100$, $q = 20$.

Table 2. Comparison between different errors obtained by S6CTM [41] and FFCOPTM for Problem 3 using $p = 100$, $q = 20$

N	FFCOPTM	S6CTM [41]
3	4.9×10^{-5}	2.5×10^{-5}
5	5.7×10^{-8}	5.8×10^{-8}
7	5.9×10^{-10}	7.3×10^{-10}
9	6.2×10^{-12}	9.2×10^{-12}
11	1.5×10^{-14}	1.4×10^{-14}
13	2.3×10^{-16}	2.2×10^{-16}
15	2.1×10^{-18}	2.2×10^{-18}

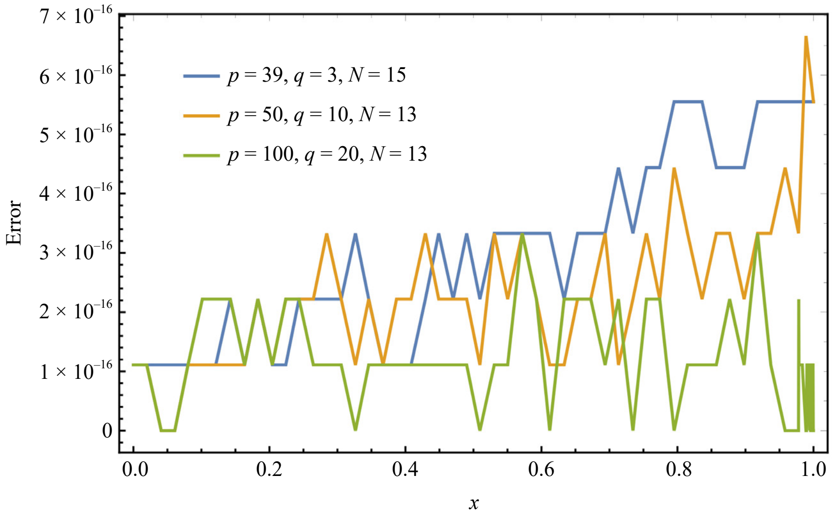

Figure 1. Errors in Problem 3 at $N = 13$ and 15 are compared

Problem 4. The solution to the initial value problem [29, 42]

$$D^3 y(x) + D^{5/2} y(x) + y^2(x) = x^4,\ y(0) = y'(0) = 0,\ y''(0) = 2, \tag{74}$$

is $y(x) = x^2$. By applying the proposed method FFCOPCM with $N = 3$, $p > 7$ and $q > -1$, we have

$$y_N(x) = \sum_{i=0}^{3} a_i M_i^{(p,q)}(x) = \mathbf{A}^T \Phi(x), \tag{75}$$

where

$$\mathbf{A} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

In view of two Theorems 1 and 2, we have

$$\mathbf{M}^{(3)}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6(p-6)_3 & 0 & 0 & 0 \end{pmatrix}, \mathbf{D}^{(5/2)}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Theta_{3,0}^{(5/2)} & \Theta_{3,1}^{(5/2)} & \Theta_{3,2}^{(5/2)} & \Theta_{3,3}^{(5/2)} \end{pmatrix},$$

then equation (49) can be written as

$$\mathbf{A}^T\mathbf{M}^{(3)}\Phi(x_1)+\mathbf{A}^T\mathbf{D}^{(5/2)}\Phi(x_1)+\left[\mathbf{A}^T\Phi(x_1)\right]^2-x_1^4=0, \tag{76}$$

where $x_1 = 1$ is the root of $M_1^{(p,q)}(x)$. Also, equation (45) gives

$$\left.\begin{aligned} &a_0-6a_1+42a_2-336a_3=0,\\ &3a_1-35a_2+48a_3=0,\\ &20a_2-144a_3-1=0. \end{aligned}\right\} \tag{77}$$

The nonlinear system of equations (76) and (77) can be solved exactly, resulting in the following solution:

$$a_0=\frac{7}{5}, a_1=\frac{7}{12}, a_2=\frac{1}{20}, a_3=0,$$

then $y^3(x) = x^2$, which is the exact solution of (74).

Problem 5. The solution of initial value problem [43]

$$\begin{aligned} &D^\varsigma y(x)+D^\eta y(x).D^\theta y(x)+y^2(x)=x^6+\frac{6x^{3-\zeta}}{\Gamma(4-\zeta)}+\frac{36x^{3-\eta-\theta}}{\Gamma(4-\eta)\Gamma(4-\theta)},\\ &\varsigma\in(2,3),\eta\in(1,2),\theta\in(0,1),\\ &y(0)=y'(0)=y''(0)=0, \end{aligned} \tag{79}$$

is $y(x) = x^3$. Table 3 displays the MAEs obtained using FFCOPCM at $\zeta = 2.5$, $\eta = 1.5$, and $\theta = 0.9$ for various values of p, q, and N, with the constraint that $N < (p - 1)/2$ and $0 < q < p$. It also shows MAEs for four different choices of N, ζ, η, θ, and $p = 300$, $q = 200$. As demonstrated in Table 4, we observe that as ζ, η, and θ approach their integer values, the solution of the FDE approaches that of the integer-order DE, resulting in more accurate approximate solutions. Furthermore, Table 5 provides a comparison between the errors obtained from the numerical scheme presented in [43] and those obtained using FFCOPCM for different values of N, using $p = 300$, $q = 200$, $\zeta = 2.5$, $\eta = 1.5$, and $\theta = 0.9$.

Table 3. MAE for $\zeta = 2.5$, $\eta = 1.5$, $\theta = 0.9$, and different choices of p, q, and N for Problem 5

N	p	q	MAE	N	p	q	MAE	N	p	q	MAE
	8	5	2.19×10^{-3}	3			4.74×10^{-5}	3			2.83×10^{-6}
3	12	9	9.55×10^{-5}	5	100	50	7.31×10^{-6}	7	300	200	6.95×10^{-7}
	15	12	1.36×10^{-5}	6			7.45×10^{-7}	12			4.76×10^{-9}
	20	14	7.33×10^{-6}	3			1.43×10^{-5}	3			3.90×10^{-6}
3	30	30	5.56×10^{-6}	6	200	100	1.77×10^{-6}	12	400	200	5.74×10^{-8}
	40	40	4.43×10^{-6}	9			3.97×10^{-8}	15			4.08×10^{-9}

Table 4. MAE for $p = 300$, $q = 200$, and different choices of ζ, η, θ, and N for Problem 5

N	ζ	η	θ	MAE	N	ζ	η	θ	MAE
3				1.54×10^{-11}	3				3.70×10^{-6}
4	$2 + 10^{-6}$	$1 + 10^{-6}$	10^{-6}	3.95×10^{-12}	10	2.75	1.75	0.75	6.94×10^{-7}
7				4.82×10^{-13}	12				3.01×10^{-7}
9				8.83×10^{-14}	14				1.57×10^{-14}
3				1.67×10^{-7}	3				1.67×10^{-9}
11	2.99	1.99	0.99	8.14×10^{-8}	7	2.9999	1.9999	0.9999	1.35×10^{-9}
14				3.03×10^{-8}	11				8.46×10^{-10}
16				4.30×10^{-9}	13				2.99×10^{-10}

Table 5. Comparison between the method in [43] and FFCOPCM for Problem 5 using $\zeta = 2.5$, $\eta = 1.5$, $\theta = 0.9$, $p = 300$, and $q = 200$

N	FFCOPCM	N	[43]
3	1.13×10^{-6}	4	1.27×10^{-3}
7	5.43×10^{-8}	8	3.47×10^{-4}
12	2.63×10^{-9}	16	8.98×10^{-5}

Problem 6. The solution of a multi-term linear FDE [41]

$$(10+x)^2 D^{5/2} y(x) + \frac{5}{2}(10+x) D^{3/2} y(x) + \frac{1}{2} D^{1/2} y(x) = \frac{x^{3/2}}{100\sqrt{\pi}},$$
$$x \in (0,1),\ y(0) = \ln(10),\ y(1) = \ln(11),\ y'(0) = \frac{1}{10}, \tag{80}$$

is $y(x) = \ln(x + 10)$. In Table 6, we list the MAEs of FFCOPCM for different values of p, q, and N, such that $N < (p - 1)/2$ and $0 < q < p$. Additionally, Figure 2 shows the Log-errors for different choices of p, q, and N to demonstrate the stability of solutions. A comparison between the method in [41] and FFCOPCM using $p = 1000$ and $q = 500$ is given in Table 7.

Table 6. MAE for different choices of p, q, and N for Problem 6

N	p	q	MAE	N	p	q	MAE
4			2.4×10^{-8}	4			3.1×10^{-8}
6			5.4×10^{-11}	6			2.8×10^{-11}
8	200	100	2.7×10^{-13}	8	600	300	6.1×10^{-13}
10			1.3×10^{-14}	10			2.7×10^{-14}
12			7.8×10^{-16}	12			6.8×10^{-16}
14			5.3×10^{-18}	14			5.1×10^{-18}

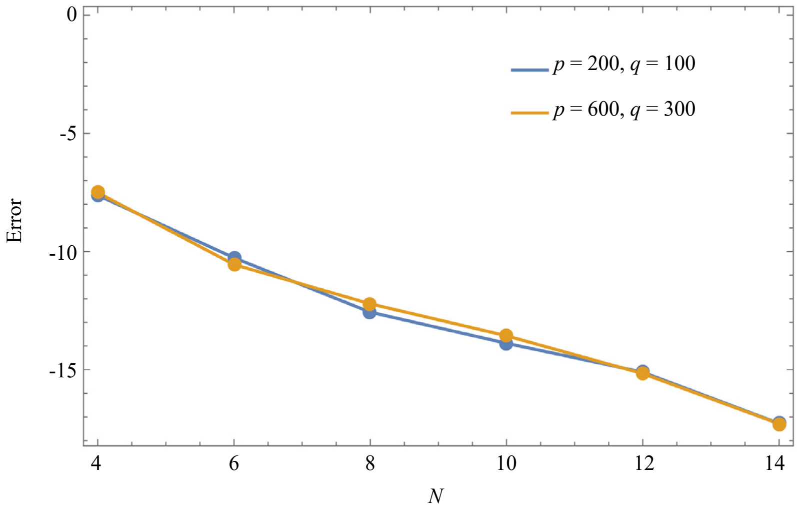

Figure 2. Errors in Problem 6 at $N = 4, 5, ..., 14$, are compared

Table 7. Comparison between the method in [41] and FFCOPCM for Problem 6 using $p = 1{,}000$ and $q = 500$

N	FFCOPCM	S6CTM [41]
4	3.3×10^{-8}	3.7×10^{-7}
6	3.5×10^{-11}	2.1×10^{-8}
8	2.9×10^{-13}	5.8×10^{-9}
10	1.2×10^{-14v}	1.5×10^{-11}
12	2.7×10^{-16}	2.5×10^{-14}
14	4.3×10^{-18}	2.2×10^{-16}
16	3.5×10^{-18}	4.4×10^{-18}

Remark 2. Attempting to show the procedures for applying the two presented algorithms, FFCOPTM and FFCOPCM. In Algorithm 1, the stages for solving the linear FDEs in Problems 1-3 are written using the FFCOPTM notation, whereas in Algorithm 2, the steps for solving the nonlinear FDE in Problems 4-6 are written using the FFCOPCM notation. For doing the necessary computations, the Mathematica program, version 12, is used.

Algorithm 1. FFCOPTM algorithm for Problems 1-3.
Step 1. Given $v_1, v_2, ..., v_m, \mu, n$, and N
Step 2. Define and compute the elements of $(N + 1)(N + 1)$ matrices $\mathbf{M}^{(1)}$, $\mathbf{D}^{(\mu)}$, and $\mathbf{D}^{(vi)}$, $i = 0, 1, ..., m$
Step 3. Find $\mathbf{M}^{(1)}$, $\mathbf{D}^{(\mu)}$, and $\mathbf{D}^{(v_i)}$, $i = 0, 1, ..., m$
Step 4. Evaluate $R_N(x)$ defined in equation (43)
Step 5. List $(R_N(x), M_i^{(p,q)}(x))_w = 0$, $i = 0,1, ..., N - n$ defined in equation (45)
Step 6. Join [Output 5, $\mathbf{A}^T \mathbf{M}^{(i)}\Phi(0) = \alpha_i$, $i = 0, 1,..., n - 1$] (In the case of IC) or Join [Output 5, $\mathbf{A}^T \mathbf{M}^{(i)}\Phi(0) = \alpha_i$, $y^{(j)}(\ell) = \mathbf{A}^T \mathbf{M}^{(j)}\Phi^{(\ell)} = \beta_j$, $i = 0, 1, ..., n_1$, $j = 0, 1, ..., n - n_1 - 2$] (The case of BC)
Step 7. Use Mathematica's built-in numerical solver to obtain the solution to the system of equations in [Output 6]

Algorithm 2. FFCOPCM algorithm for Problems 4-6.

Step 1. Given v_1, v_2, ..., v_m, μ, n, and N

Step 2. Define and compute the elements of $(N + 1)(N + 1)$ matrices $\mathbf{M}^{(1)}$, $\mathbf{D}^{(\mu)}$, and $\mathbf{D}^{(v_i)}$, $i = 0, 1, ..., m$

Step 3. Find $\mathbf{M}^{(1)}$, $\mathbf{D}^{(\mu)}$, and $\mathbf{D}^{(v_i)}$, $i = 0, 1, ..., m$

Step 4. Evaluate $R_N(x)$ defined in equation (43)

Step 5. List $(R_N(x), M_i^{(p, q)}(x))_w = 0$, $i = 0,1, ..., N - n$ defined in equation (45)

Step 6. Join [Output 4, $\mathbf{A}^T \mathbf{M}^{(i)}\Phi(0) = \alpha_i$, $i = 0, 1,..., n - 1$] (The case of IC) or Join [Output 4, $\mathbf{A}^T \mathbf{M}^{(i)}\Phi(0) = \alpha_i$, $y^{(j)}(\ell) = \mathbf{A}^T \mathbf{M}^{(j)}\Phi^{(\ell)} = \beta_j$, $i = 0, 1, ..., n_1$, $j = 0, 1, ..., n - n_1 - 2$] (The case of BC)

Step 7. Use Mathematica's built-in numerical solver to obtain the solution to the system of equations in [Output 6]

7. Conclusion

This study proposes two numerical algorithms for solving linear and nonlinear multi-term FDEs subject to I/BCs. The suggested algorithms are developed by constructing two new OMs for ordinary and FDs of FFCOP. These OMs are utilized to transform both linear and nonlinear FDEs into sets of effectively solvable linear or nonlinear algebraic equations. To the best of our knowledge, the derived FFCOP OMs are novel and have not been previously utilized to solve FDEs. This is the first time that these matrices have been proposed and applied to solve FDEs. The given numerical problems demonstrate the high efficiency and performance of the given algorithms. We are confident that these algorithms can be applied to other types of FDEs.

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Conflict of interest

In relation to the publication of this paper, the author declares that there are no competing interests.

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