Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Controllability of Impulsive Nonlinear Fractional Dynamical System with Delays in State and Control

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Received: 21 March 2023; **Revised:** 25 November 2023; **Accepted:** 6 December 2023

Abstract: In this paper, the controllability of nonlinear impulsive fractional dynamical system with delays in state and control is analyzed by using a delayed Mittag Leffler (*M-L*) function. Controllability Grammian matrix is used to establish the controllability of linear system. The sufficient conditions of the considered nonlinear system are derived by utilizing the fixed point techniques. Finally, an example is provided for the illustration of the obtained result.

***Keywords*:** controllability, impulsive system, delayed Mittag Leffler function, fractional system, fixed point theorem

MSC: 93B05, 34A08

1. Introduction

In order to divide larger dynamical systems to smaller and more precise systems the concept of differential calculus was introduced, as it was necessary to monitor the changes happening in the system at certain times [1, 2]. As an expansion of the ordinary differential calculus, the fractional calculus was introduced to study the integral and derivative parts of a function. Later on, due to many developments in the field, it has materialized into a very dominant resource to consider all kinds of problems in various fields of science, including machine learning, control systems, biology and finance. A major expansion in fractional calculus was fractional differential equations (*FDE*) rather than the integer differential equations, as it generalised the derivatives of integer order systems to arbitrary order systems for easy computation [3, 4]. In recent days *FDE* is been widely used in various fields like mechanics, neural networking, ecology, optics and image processing [5–11].

Fractional derivatives are extensively used to study and model control systems and to scrutinize the qualitative behaviours of system such as the observability, stability, stabilizability and controllability [12–14]. *FDE* has been the most popular topic for research, to predict the controllability behaviours by scrutinizing and constructing a control system [15–17]. A controllable system is when a state function observed can be transformed into another desired state function by representing a most appropriate solution for the system, at a definite time duration, with an adequate input function [18, 19]. The controllability etiquette of dynamical linear and nonlinear systems with several initial and boundary conditions was a subject of interest for many research works [20–22]. In Hilbert spaces, approximate controllability of an abstract neutral and analytic resolvent integro differential inclusions was discussed by Vijayakumar [23, 24]. Sheng et al. discussed

DOI: https://doi.org/10.37256/cm.5320242717

about controllability of nonlinear system using a *M-L* Kernel [25]. For a class of second order evolution inclusion systems without compactness the controllability analysis was studied by the authors in [26]. Delays are primarily certain disturbances happening either within the system or around the surroundings that may cause adverse effects in the solution representation of the system, hence it becomes unavoidable to calculate the solution representations using delay terms [27–29]. There is always a complexity in solving the control systems with delay terms, to overcome this a delayed *M-L* matrix function was found as an extension of the classical *M-L* function which gives an explicit formula as a solution to the delayed *FDE* [30–32]. In [33, 34], the control concept has been analyzed by utilizing the delayed *M-L* function of Grammian matrix solution representations.

The impulsive conditions taken in a system are considered to be shorter input functions whose time derivative is very brief compared to the time variations of the system. They are taken to indicate brief interference such as shock, natural calamities in the system [35–37]. Vijayakumar et al. discussed the controllability analysis without measures of compactness for a second order Cauchy problem with nonlocal conditions [38]. Debbouche et al. [39] discussed the nonlinear system's controllability with distributed delays and impulsive conditions. In [40], the authors have investigated the controllability of fractional semilinear system with control delay. Li et al. [41] verified the relative controllability behaviour of a fractional pure delay system using delayed *M-L* functions. Nawaz et al. [42] have explored the controllability of fractional systems with both state and control delays. Encouraged by the above mentioned results, the paper reports the controllability of a nonlinear system with a delay in state, impulsive conditions and a delay in control, by using a delayed *M-L* function to represent the solution. Consider a nonlinear impulsive control system with state delay and control delay as follows:

$$
\begin{aligned}
{}^{C}\mathscr{D}^{\rho}_{0+}z(t) = {} & \mathscr{A}z(t-p) + \mathscr{B}\tilde{u}(t) + \mathscr{C}\tilde{u}(t-p) \\
& + \tilde{f}(t,\, z(t-p),\, \tilde{u}(t),\, \tilde{u}(t-p)),\; z(t) \in [0,\, t_1],\; p > 0,
\end{aligned} \tag{1}
$$

$$
z(t) = \tilde{\varphi}(t),\; t \in [-p,\, 0], \tag{2}
$$

$$
\tilde{u}(t) = \tilde{\Psi}(t),\; t \in [-p,\, 0], \tag{3}
$$

$$
z(t_i^+) - z(t_i^-) = \mathscr{J}(z(t_i)) = \mathscr{J}_i, \tag{4}
$$

where ${}^{C}\mathscr{D}^{\rho}_{0+}$ is the fractional derivative of Caputo, of order ρ and $0 < \rho < 1$, $z(t) \in \mathfrak{R}^n$ is the state vector, where $z(t) : [-p,\, t_1] \to \mathfrak{R}^n$ is a continuous differentiable function on $[0,\, t_1]$ with t_1 is the integral multiple of p. $\tilde{u}(t) \in \mathfrak{R}^m$ is the control vector, $\mathscr{A} \in \mathfrak{R}^{n\times n}$, $\mathscr{B}$, $\mathscr{C} \in \mathfrak{R}^{n\times m}$ are some positive definite matrices, $\tilde{f} : \mathscr{J} \times \mathfrak{R}^n \times \mathfrak{R}^m \times \mathfrak{R}^m \in \mathfrak{R}^n$ is a continuous differential function with state delay and control delay, $p > 0$ denotes time delay, $\tilde{\varphi}(t)$ denotes the initial function of state such that $\tilde{\varphi} \in C([-p,\, 0],\, \mathfrak{R}^n)$ and $\tilde{\Psi}(t)$ is the initial function of control. Define a piecewise continuous function $C_{PC} : (\mathscr{J},\, \mathfrak{R}^n) \to \mathfrak{R}^n$, that is continuous on intervals $0 < t_i < t_{i+1}$ and here $z(t_i^+)$ and $z(t_i^-)$ denotes the right and left limits of $z(t)$ where $t = t_i$.

The structure of the paper is as follows: Section 2 states some important definitions and lemmas used to prove the results. Section 3 represents the linear system with both delays and impulsive terms. Section 4 provides the proof of the nonlinear system with some conditions using fixed point techniques. Section 5 illustrates a numerical example to signify the acquired results.

2. Preliminaries

Definition 1 [19] Consider a function $\tilde{f}: [0, \infty) \rightarrow \mathfrak{R}$, then the Caputo derivative of a fractional differential equation with order $(0 < \rho < 1)$ is given by

$$ {}^{C}\mathscr{D}^{\rho}_{0+} z(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{z'(\nu)}{(t-\nu)^{\rho}} d\nu, \quad t > 0. $$

Definition 2 [19] The integral term of the fractional derivative function $\tilde{f}: [0, \infty) \rightarrow \mathfrak{R}$ with order $0 < \rho < 1$ is given by

$$ \mathscr{I}^{\rho}_{0+} \tilde{f}(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-\nu)^{\rho-1} \tilde{f}(\nu) d\nu. $$

Definition 3 [42] Consider a matrix $\varepsilon_p^{\mathscr{A}.\rho}: \mathfrak{R} \rightarrow \mathfrak{R}^{n\times n}$ which can be expressed as a delayed *M-L* matrix function of one parameter as follows:

$$ \varepsilon_p^{\mathscr{A} t^{\rho}} = \begin{cases} \Theta, & -\infty < t < -p, \\ \mathscr{I}_n, & -p \le t \le 0, \\ \mathscr{I}_n + \mathscr{A}\dfrac{(t)^{\rho}}{\Gamma\rho+1} + \mathscr{A}^2 \dfrac{(t-p)^{2\rho}}{\Gamma 2\rho+1} + \cdot + \mathscr{A}^q \dfrac{(t-(q-1)p)^{q\rho}}{\Gamma q\rho+1)}, & (q-1)p \le t \le qp,\ q \in \mathscr{N}, \end{cases} $$

here Θ and $\mathscr{I}_n$ represents the zero matrix and identity matrix respectively.

Definition 4 [22] Let $t \in ((q-1)p, qp]$, $q = 1, 2, \ldots, n$, the impulsive delayed *M-L* matrix function $z_{p,\, \rho}(.) : \mathfrak{R} \rightarrow \mathfrak{R}^n$ is given by

$$ z_{p,\rho}(t) = \sum_{0<t_i<t} \varepsilon^{\mathscr{A}(t-t_i-p)^{\rho}} \mathscr{J}_i, $$

and for $t \in (qp,\ (q+1)p]$, $t_i = qp$, $q = 1, 2, \ldots, n-1$ it becomes,

$$ z_{p,\, \rho}(t) = \sum_{k=1}^{q} \varepsilon^{\mathscr{A}(t-(k+1)p)^{\rho}} \mathscr{J}_k. $$

The following lemma can be used to formulate the solution of the considered system (1)-(4).

Lemma 1 [42] Let $\tilde{f}: [0, t] \rightarrow \mathfrak{R}^n$ be a continuous function with vector values, then a solution vector $z \in C([-p, t], \mathfrak{R}^n)$ of (1)-(4) can be given by

$$
\begin{aligned}
z(t) =& \varepsilon_p^{\mathscr{A} t^{\rho}} \tilde{\varphi}(-p) + \int_{-p}^{0} \varepsilon_p^{\mathscr{A}(t-p-r)^{\rho}} \tilde{\varphi}'(r) dr + \int_{0}^{t} \varepsilon_p^{\mathscr{A}(t-p-r)^{\rho}} \mathscr{B}\tilde{u}(r) dr \\
&+ \int_{0}^{t} \varepsilon_p^{\mathscr{A}(t-p-r)^{\rho}} \mathscr{C}\tilde{u}(r-p) dr + \int_{0}^{t} \varepsilon_p^{\mathscr{A}(t-p-r)^{\rho}} \tilde{f}(r,\, z(r),\, \tilde{u}(r)) dr + \sum_{0<t_i<t} \varepsilon_p^{\mathscr{A}(t-t_i-p)^{\rho}} \mathscr{J}_i.
\end{aligned}
$$

Lemma 2 [22] For any impulsive term $\varepsilon_p^{\mathscr{A} t^{\rho}}$, the delayed *M-L* function can be taken as follows:
Let $z \in ((q-1)p,\, qp]$, $q \in \mathscr{N}$ and $t_i \in (0,\, t)$ is an impulsive point that is fixed arbitrarily, then

$$
{}^{C}\mathscr{D}_{0+}^{\rho}(\varepsilon_p^{\mathscr{A}(\cdot-t_i-p)^{\rho}} \mathscr{J}_i) = \mathscr{A}\varepsilon_p^{\mathscr{A}(t-t_i-2p)^{\rho}} \mathscr{J}_i.
$$

Lemma 3 From Lemma 1 and Lemma 2, the solution of the nonlinear system (1)-(4) can be composed as follows:

$$
\begin{aligned}
z(t) =& \varepsilon_p^{\mathscr{A} t^{\rho}} \tilde{\varphi}(-p) + \int_{-p}^{0} \varepsilon_p^{\mathscr{A}(t-p-r)^{\rho}} \tilde{\varphi}'(r) dr + \int_{0}^{t-p} \varepsilon_p^{\mathscr{A}(t-p-r)^{\rho}} \mathscr{B}\tilde{u}(r) dr \\
&+ \int_{t-p}^{t} \varepsilon_p^{\mathscr{A}(t-p-r)^{\rho}} \mathscr{C}\tilde{u}(r) dr + \int_{-p}^{0} \varepsilon_p^{\mathscr{A}(t-2p-r)^{\rho}} \mathscr{C}\tilde{\Psi}(r) dr \\
&+ \int_{0}^{t} \varepsilon_p^{\mathscr{A}(t-p-r)^{\rho}} \tilde{f}(r,\, z(r),\, \tilde{u}(r),\, \tilde{u}(r-p) dr) + \sum_{0<t_i<t} \mathscr{A}\varepsilon_p^{\mathscr{A}(t-t_i-2p)^{\rho}} \mathscr{J}_i. \qquad (5)
\end{aligned}
$$

Lemma 4 [42] Consider a beta function, $\beta(c,\, d) = \int_0^1 r^{c-1}(1-r)^{d-1} dr$, where c and d are positive real numbers, then

$$
\beta(c,\, d) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}.
$$

Also for $qp \le t \le (q+1)p$, $q \in \mathscr{N}$,

$$
\int_{qp}^{t} (t-r)^{-\rho} (r-qp)^{((q+1)\rho-1)} dr = (t-qp)^{q\rho} \beta[1-\rho,\, q\rho].
$$

Lemma 5 [41] For a delayed *M-L* matrix, say $\varepsilon_p^{\mathscr{A}.\rho} : \mathfrak{R} \to \mathfrak{R}^{n\times n}$ and

$$
{}^{C}\mathscr{D}_{0+}^{\rho}(\varepsilon_p^{\mathscr{A} t^{\rho}}) = \mathscr{A}\varepsilon_p^{\mathscr{A}(t-p)^{\rho}} \qquad (6)
$$

with initial conditions $\varepsilon_p^{\mathscr{A} t^{\rho}} = \mathscr{I}$, $-p \le t \le 0$, then $\varepsilon_p^{\mathscr{A} t^{\rho}}$ is a solution of $({}^{C}\mathscr{D}_{0+}^{\rho} x)(t) = \mathscr{A} x(t-p)$.

Proof. For any arbitrary $-\infty < t \le -p$,

$$\varepsilon_p^{\mathscr{A} t^\rho} = \varepsilon_p^{\mathscr{A}(t-p)^\rho} = \Theta.$$

Then (1) holds for $t \in (-p,\ 0]$,

$$\varepsilon_p^{\mathscr{A} t^\rho} = \mathscr{I}_n,$$

$$\varepsilon_p^{\mathscr{A}(t-p)^\rho} = \Theta,$$

$$\Longrightarrow {}^C\mathscr{D}_{0^+}^{\rho} \mathscr{I}_n = \Theta = \mathscr{A}\Theta.$$

Using the method of mathematical induction to obtain the results.

Case 1 When $q = 1,\ 0 \le t \le p$,

$$z(t) = \varepsilon_p^{\mathscr{A} t^\rho} = \mathscr{I}_n + \frac{\mathscr{A}(t)^\rho}{\Gamma(\rho+1)},$$

$$z'(t) = 0 + \frac{1}{\Gamma(\rho+1)}(\mathscr{A}(t)^{\rho-1}\rho).$$

Then by Caputo's fractional expression of $\varepsilon_p^{\mathscr{A}\cdot^\rho}$,

$$({}^C\mathscr{D}_{0^+}^{\rho} \varepsilon_p^{\mathscr{A} r^\rho})t = \frac{1}{\Gamma(1-\rho)} \cdot \frac{1}{\Gamma(\rho+1)}(\rho\mathscr{A}) \int_0^t (t-r)^{-\rho}(r)^{\rho-1} dr = \mathscr{A}.$$

Case 2 When $q = 2,\ p \le t \le 2p$,

$$z(t) = \varepsilon_p^{\mathscr{A} t^\rho} = \mathscr{I}_n + \frac{\mathscr{A}(t)^\rho}{\Gamma(\rho+1)} + \frac{\mathscr{A}^2(t-p)^{2\rho}}{\Gamma(2\rho+1)},$$

$$z'(t) = 0 + \frac{1}{\Gamma(\rho+1)}(\mathscr{A}(t)^{\rho-1}\rho) + \frac{1}{\Gamma(2\rho+1)}(\mathscr{A}^2(t-p)^{2\rho-1}2\rho).$$

Similarly by Caputo's fractional expression $\varepsilon_p^{\mathscr{A}\cdot^\rho}$,

$$({}^C\mathscr{D}^{\rho}_{0^+}\varepsilon_p^{\mathscr{A}r^\rho})t = \frac{1}{\Gamma(1-\rho)}\cdot\frac{1}{\Gamma(\rho+1)}(\rho\mathscr{A})\int_0^t (t-r)^{-\rho}(r)^{\alpha-1}dr$$

$$+\frac{1}{\Gamma(1-\rho)}\cdot\frac{1}{\Gamma(2\rho+1)}(2\rho\mathscr{A}^2)\int_p^t (t-r)^{-\rho}(r-p)^{2\rho-1}dr$$

$$= \mathscr{A} + \frac{\mathscr{A}^2(t-p)^\rho}{\Gamma(\rho+1)}.$$

Case 3 When $q = 3,\ 2p \leq t \leq 3p$, by continuing the same process,

$$({}^C\mathscr{D}^{\rho}_{0^+}\varepsilon_p^{\mathscr{A}r^\rho})t = \mathscr{A} + \frac{\mathscr{A}^2(t-p)^\rho}{\Gamma(\rho+1)} + \frac{\mathscr{A}^3(t-2p)^{2\rho}}{\Gamma(2\rho+1)}.$$

Case 4 When $q = k,\ (k-1)p \leq t \leq kp$ and $k \in \mathscr{N}$,

$$z(t) = \varepsilon_p^{\mathscr{A}t^\rho} = \mathscr{I}_n + \frac{\mathscr{A}(t)^\rho}{\Gamma(\rho+1)} + \frac{\mathscr{A}^2(t-p)^{2\rho}}{\Gamma(2\rho+1)} + \ldots + \frac{\mathscr{A}^k(t-kp)^{(k\rho)}}{\Gamma(k\rho+1)},$$

$$z'(t) = 0 + \frac{1}{\Gamma(\rho+1)}(\mathscr{A}(t)^{\rho-1}\rho) + \frac{1}{\Gamma(2\rho+1)}(\mathscr{A}^2(t-p)^{2\rho-1}2\rho) + \ldots$$

$$+\frac{1}{\Gamma(k+1)\rho+1}(\mathscr{A}^{(k+1)}(t-kp)^{(k+1)\rho-1})(k+1)\rho.$$

Now by Caputo's fractional expression of $\varepsilon_p^{\mathscr{A}.\rho}$,

$$({}^C\mathscr{D}^{\rho}_{0^+}\varepsilon_p^{\mathscr{A}r^\rho})(t) = \mathscr{A} + \frac{\mathscr{A}^2(t-p)^\rho}{\Gamma(\rho+1)} + \frac{\mathscr{A}^3(t-2p)^{2\rho}}{\Gamma(2\rho+1)} + \ldots + \frac{\mathscr{A}^k(t-(k_1)p)^{(k-1)\rho}}{\Gamma((k-1)\rho+1)}.$$

Hence this shows that the lemma can be satisfied for any values of q say $(q+1)p \leq t \leq (q-1)p$ and $(q+1) \in \mathscr{N}$. The proof is completed which leads to the conclusion

$${}^C\mathscr{D}^{\rho}_{0^+}(\varepsilon_p^{\mathscr{A}(t-p-r)^\rho}) = \mathscr{A}\varepsilon_p^{\mathscr{A}(t-2p-r)^\rho}. \tag{7}$$

□

Definition 5 [15] The dynamical system is controllable on $\mathscr{J}$ if there exists $z_0,\ z_1 \in \Re^n$ with a control $\bar{u}(t)$ has a solution at $z(t)$ that satisfies the condition $z(0) = z_0$ and $z(t) = z_1$.

3. Linear system

Consider the linear dynamical system as follows:

$$^{C}\mathscr{D}_{0^+}^{\rho} z(t) = \mathscr{A} z(t-p) + \mathscr{B}\tilde{u}(t) + \mathscr{C}\tilde{u}(t-p), \qquad z(t) \in [0,\ t_1],\ p > 0, \tag{8}$$

$$z(t) = \tilde{\varphi}, \qquad t \in [-p,\ 0], \tag{9}$$

$$\tilde{u}(t) = \tilde{\Psi}(t), \qquad t \in [-p,\ 0], \tag{10}$$

$$z(t_i^+) - z(t_i^-) = \mathscr{J}(z(t_i)) = \mathscr{J}_i. \tag{11}$$

where $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $z(t)$, $\tilde{u}(t)$, $\tilde{\varphi}$ and $\tilde{\Psi}(t)$ are defined as same as in Section 1.

Theorem 1 The system (8)-(11) is controllable on $[0,\ t_1]$ iff the Grammian controllability matrix

$$G(t) = \int_0^b (\varepsilon_p^{\mathscr{A}(b-p-r)^{\rho}} \mathscr{B})(\varepsilon_p^{\mathscr{A}(b-p-r)^{\rho}} \mathscr{B})^* dr$$

$$+ \int_0^{b-p} (\varepsilon_p^{\mathscr{A}(b-2p-r)^{\rho}} \mathscr{C})(\varepsilon_p^{\mathscr{A}(b-2p-r)^{\rho}} \mathscr{C})^* dr, \tag{12}$$

is positive definite for some $b > 0$.

Proof. The positive definite Grammian matrix is not singular hence the inverse can be defined as $G^{-1}(t)$. The control function is taken as follows:

$$\tilde{u}(t) = \mathscr{B}^* \mathscr{C}^* \varepsilon_p^{\mathscr{A}^*(b-p-r)^{\rho}} G^{-1}[z_1 - \varepsilon_p^{\mathscr{A}t^{\rho}} \tilde{\varphi}(-p) - \int_{-p}^{0} \varepsilon_p^{\mathscr{A}(b-p-r)^{\rho}} \tilde{\Psi}(r) dr z_0], \tag{13}$$

here $*$ indicates the transpose of each matrix. The system gets steered from z_0 to z_1 by the control $\tilde{u}(t)$.

The solution of the linear system is

$$z(t) = \varepsilon_p^{\mathscr{A}t^{\rho}} \tilde{\varphi}(-p) + \int_{-p}^{0} \varepsilon_p^{\mathscr{A}(t-p-r)^{\rho}} \tilde{\varphi}'(r) dr + \int_0^{t-p} \varepsilon_p^{\mathscr{A}(t-p-r)^{\rho}} \mathscr{B}\tilde{u}(r) dr$$

$$+ \int_{t-p}^{t} \varepsilon_p^{\mathscr{A}(t-p-r)^{\rho}} \mathscr{C}\tilde{u}(r) dr + \int_{-p}^{0} \varepsilon_p^{\mathscr{A}(t-2p-r)^{\rho}} \mathscr{C}\tilde{\Psi}(r) dr + \sum_{0<t_i<t} \mathscr{A} \varepsilon_p^{\mathscr{A}(t-t_i-2p)^{\rho}} \mathscr{J}_i. \tag{14}$$

Substitute $t = b$ in (14),

$$z(b) = \varepsilon_p^{\mathscr{A} b^\rho} \tilde{\varphi}(-p) + \int_{-p}^{0} \varepsilon_p^{\mathscr{A}(b-p-r)^\rho} \tilde{\varphi}'(r)dr + \int_{0}^{b-p} \varepsilon_p^{\mathscr{A}(b-p-r)^\rho} \mathscr{B}\tilde{u}(r)dr$$

$$+ \int_{b-p}^{b} \varepsilon_p^{\mathscr{A}(b-p-r)^\rho} \mathscr{C}\tilde{\Psi}(r)dr. \tag{15}$$

Substituting $\tilde{u}(t)$ in (15),

$$z(b) = \varepsilon_p^{\mathscr{A} b^\rho} \tilde{\varphi}(-p) + \int_{-p}^{0} \varepsilon_p^{\mathscr{A}(b-p-r)^\rho} \tilde{\varphi}'(r)dr + \int_{0}^{b-p} \varepsilon_p^{\mathscr{A}(b-p-r)^\rho} \mathscr{B}\mathscr{B}^*\mathscr{C}^* \varepsilon_p^{\mathscr{A}^*(b-p-r)^\rho}$$

$$\times G^{-1}[z_1 - \varepsilon_p^{\mathscr{A} t^\rho} \tilde{\varphi}(-p) - \int_{-p}^{0} \varepsilon_p^{\mathscr{A}(b-2p-r)^\rho} \tilde{\Psi}(r)drz_0],$$

$$\Longrightarrow z(b) = z_1. \tag{16}$$

Next, to prove that $G(t)$ is positive definite. Consider $y \neq 0$ such that $y^* G y = 0$.

$$y^* \int_0^b (\varepsilon_p^{\mathscr{A}(b-p-r)^\rho} \mathscr{B}) \int_0^b (\varepsilon_p^{\mathscr{A}(b-p-r)^\rho} \mathscr{B})^* dr + \int_0^{b-p} (\varepsilon_p^{\mathscr{A}(b-2p-r)^\rho} \mathscr{C})(\varepsilon_p^{\mathscr{A}(b-2p-r)^\rho} \mathscr{C})^* dr.y = 0$$

on $[0, b]$. Then

$$z(b) = 0 = \varepsilon_p^{\mathscr{A} b^\rho} \tilde{\varphi}(-p) + \int_{-p}^{0} \varepsilon_p^{\mathscr{A}(b-p-r)^\rho} \tilde{\varphi}'(r)dr + \int_{0}^{b-p} \varepsilon_p^{\mathscr{A}(b-p-r)^\rho} \mathscr{B}\tilde{u}(r)dr$$

$$+ \int_{b-p}^{b} \varepsilon_p^{\mathscr{A}(b-p-r)^\rho} \mathscr{C}\tilde{u}(r)dr,$$

$$\Longrightarrow 0 = y + \int_0^b \varepsilon_p^{\mathscr{A}(b-p-r)^\rho} (\mathscr{B} + \mathscr{C})\tilde{u}(r)dr,$$

$$\Longrightarrow 0 = y^* y + \int_0^b y^* \varepsilon_p^{\mathscr{A}(b-p-r)^\rho} (\mathscr{B} + \mathscr{C})\tilde{u}(r)dr.$$

But the second integral becomes zero. Hence,

$$y^* y = 0,$$

$$\Longrightarrow y = 0,$$

which is a contradiction. Hence the system (8)-(11) is controllable on $\mathscr{J}$. □

4. Nonlinear system

Consider a Banach space X of the real continuous valued functions $\mathfrak{R}^n \times \mathfrak{R}^m \times \mathfrak{R}^m$ defined on $\mathscr{J}$, that satisfies the norm

$$\|(z,\ \tilde{u})\| = \|z\| + \|\tilde{u}\|,$$

and here

$$\|z\| = sup|z(t)| : t \in \mathscr{J},\ \|\tilde{u}\| = sup|\tilde{u}(t)| : t \in \mathscr{J},$$

where $X = \mathscr{C}_{n(\mathscr{J})} \times \mathscr{C}_{m(\mathscr{J})} \times \mathscr{C}_{m(\mathscr{J})}$ denotes the Banach space of $\mathfrak{R}^n$ real valued continuous functions on the interval $\mathscr{J}$ with the supremum norm for each $(\mu,\ \nu) \in X$ where μ and ν are some variable that belongs to X. Now assign a value $h = (z,\ \tilde{u}) \in \mathfrak{R}^n \times \mathfrak{R}^m$, then at $t \in [0,\ p]$, with control $\tilde{u}(t) = \tilde{\varphi}(t)$ at the interval $-p \leq t \leq 0$, it becomes $|h| = |z| + |\tilde{u}|$.

Theorem 2 The nonlinear impulsive delayed system (1)-(4) is controllable on $\mathscr{J}$, if a continuous function $\tilde{f}$ satisfies

$$\lim_{|h|\to\infty} \frac{|\tilde{f}(t,\ h)|}{|h|} = 0, \tag{17}$$

for some h, uniformly in $t \in \mathscr{J}$. Also, $\|(z,\ \tilde{u})\| = \|z\| + \|\tilde{u}\|$, where z is the state vector and $\tilde{u}$ is the delayed control which are admissable.

Proof. Define an operator $\mathscr{W} : X \to X$ by $\mathscr{W}(z,\ \tilde{u}) = (\mu,\ \nu),\ t \in [0,\ p]$.
Consider the Grammian matrix

$$G(t) = \int_0^b (\varepsilon_p^{\mathscr{A}(b-p-r)^\rho} \mathscr{B})(\varepsilon_p^{\mathscr{A}(b-p-r)^\rho} \mathscr{B})^* dr$$

$$+ \int_0^{b-p} (\varepsilon_p^{\mathscr{A}(b-2p-r)^\rho} \mathscr{C})(\varepsilon_p^{\mathscr{A}(b-2p-r)^\rho} \mathscr{C})^* dr,$$

and let

$$\nu(t) = \mathscr{B}^* \varepsilon_p^{\mathscr{A}^*(t-p-r)^\rho} G^{-1}[z_1 - \varepsilon_p^{\mathscr{A}t^\rho} \tilde{\varphi}(-p) - \int_{-p}^{0} \varepsilon_p^{\mathscr{A}(t-p-r)^\rho} \tilde{\varphi}'(r)dr]$$

$$- \int_0^t \varepsilon_p^{\mathscr{A}(t-p-r)^\rho} \tilde{f}(r,\, z(r),\, \tilde{u}(r),\, \tilde{u}(r-p))dr,$$

$$\mu(t) = \varepsilon_p^{\mathscr{A}t^\rho} \tilde{\varphi}(-p) + \int_{-p}^{0} \varepsilon_p^{\mathscr{A}(t-p-r)^\rho} \tilde{\varphi}'(r)dr + \int_0^t \varepsilon_p^{\mathscr{A}(t-p-r)^\rho} \mathscr{B}\tilde{u}(r)dr$$

$$+ \int_{-p}^{0} \varepsilon_p^{\mathscr{A}(t-2p-r)^\rho} \mathscr{C}\tilde{\Psi}(r)dr + \int_0^t \varepsilon_p^{\mathscr{A}(t-p-r)^\rho} \tilde{f}(r,\, z(r),\, \tilde{u}(r),\, \tilde{u}(r-p)dr)$$

$$+ \sum_{0\,<\,t_i\,<t} \mathscr{A} \varepsilon_p^{\mathscr{A}(t-t_i-2p)^\rho} \mathscr{J}_i.$$

Consider the following assumptions for brevity,

$$\hat{a}_1 = sup\|\varepsilon_p^{\mathscr{A}(t-p-r)^\rho} \mathscr{B}\|,\;\; \hat{a}_2 = sup\|\varepsilon_p^{\mathscr{A}t^\rho} \tilde{\varphi}(-p) + \int_{-p}^{0} \varepsilon_p^{\mathscr{A}(t-p-r)^\rho} \tilde{\varphi}'(r)dr\|,$$

$$sup|\tilde{f}| = sup|\tilde{f}(r,\, z(r-p),\, \tilde{u}(r),\, \tilde{u}(r-p)dr| : r \in \mathscr{J},$$

$$\hat{a}_3 = sup\|\int_{-p}^{0} \varepsilon_p^{\mathscr{A}(t-2p-r)^\rho} \mathscr{C}\tilde{\Psi}(r)dr\|,\;\; \hat{a}_4 = \sum_{0<t_i<t} \|\mathscr{A} \varepsilon_p^{\mathscr{A}(t-t_i-2p)^\rho} \mathscr{J}_i\|,\;\; \hat{a}_5 = sup\|\varepsilon_p^{\mathscr{A}(t-p-r)^\rho}\|,$$

$$\hat{d}_1 = 4\hat{a}\hat{a}_1\|G^{-1}\||z_1| + \hat{a}_2,\;\; \hat{c} = 4\hat{a}\hat{a}_5,\;\; \hat{d}_2 = 4\hat{a}(\hat{a}_3 + \hat{a}_4),$$

$$\hat{d} = max\{\hat{d}_1 + \hat{d}_2\},$$

where $\hat{a},\ \hat{c},\ \hat{d}$ are some constants. Then

$$|\nu(t)| \leq \hat{a}_1\|G^{-1}\|[|z_1| + \hat{a}_2] + [\hat{a}_5 sup|\tilde{f}|]$$

$$\leq [\hat{a}_1\|G^{-1}\||z_1| + \hat{a}_2] + [\hat{a}_5 sup|\tilde{f}|]$$

$$\leq \frac{\hat{d}_1}{4\hat{a}} + \frac{\hat{c}}{4\hat{a}} sup|\tilde{f}|$$

$$\leq \frac{1}{4\hat{a}}(\hat{d}_1 + \hat{c}) sup|\tilde{f}|, \tag{18}$$

and

$$
\begin{aligned}
|\mu(t)| &\leq \hat{a_2}+\hat{a_1}+\hat{a_3}+\hat{a_5} sup|\tilde{f}|+\hat{a_4} \\
&\leq \hat{a_3}+[\hat{a_1}+\hat{a_2}+\frac{\hat{c}}{4\hat{a}} sup|\tilde{f}|]+\hat{a_4} \\
&\leq \hat{a_3}+\hat{a_4}+[\frac{\hat{d}_1}{4\hat{a}}+\frac{\hat{c}}{4\hat{a}} sup|\tilde{f}|] \\
&\leq \frac{\hat{d}_2}{4\hat{a}}+[\frac{\hat{d}_1}{4\hat{a}}+\frac{\hat{c}}{4\hat{a}} sup|\tilde{f}|] \\
&\leq \frac{\hat{d}}{4\hat{a}}+\frac{\hat{c}}{4\hat{a}} sup|\tilde{f}|.
\end{aligned} \tag{19}
$$

From (18) and (19), the function $\tilde{f}$ fulfils the following conditions:

For a constant r_1 such that if $|h| \leq r_1$ and $t \in [0,\ p] = \mathscr{J}$, then $\hat{c}|\tilde{f}(t,\ h)|+\hat{d} \leq r_1$, where $\hat{c}$ and $\hat{d}$ are constants. Suppose for a constant r_2 such that $r_1 < r_2$, then $\hat{c}|\tilde{f}(t,\ h)|+\hat{d} \leq r_2$.

For all $r \in \mathscr{J}$, suppose that $\|z\| \leq \frac{r_1}{2}$ and $\|\tilde{u}\| \leq \frac{r_1}{2}$ gives $|h| \leq \frac{r_1}{2}+\frac{r_1}{2}$ which implies $|h| \leq r_1$.

$$
\hat{c}\ sup|\tilde{f}|+\hat{d} \leq r,
$$

$$
|\tilde{u}(r)| \leq \frac{r_1}{4\hat{a}},
$$

$$
\Longrightarrow \|t\| \leq \frac{r_1}{2}, \quad \forall\, t \in \mathscr{J}.
$$

Hence the considered operator $\mathscr{W}$ maps $X(r)$ into itself with the condition

$$
X(r) = \{(z,\ \tilde{u}) \in X : \|\tilde{u}\| \leq \frac{r}{2},\ \|z\| \leq \frac{r}{2}\}.
$$

Next, to prove that the operator $\mathscr{W}$ has a point fixed in $X(r)$. By the Arzela Ascoli's theorem if the function $\tilde{f}$ is continuous then $\mathscr{W}$ is continuous which implies $\mathscr{W}$ is completely continuous. Also, the fixed point theorem of Schauder's suggests that if X is closed, bounded and convex then $(z,\ \tilde{u}) \in X(r)$ such that $\mathscr{W}(z,\ \tilde{u}) = (\mu,\ \nu)$, so

$$
\begin{aligned}
z(t) =& \varepsilon_p^{\mathscr{A} t^\rho} \tilde{\varphi}(-p)+\int_{-p}^{0} \varepsilon_p^{\mathscr{A}(t-p-r)^\rho} \tilde{\varphi}'(r)dr+\int_{0}^{t-p} \varepsilon_p^{\mathscr{A}(t-p-r)^\rho} \mathscr{B}\tilde{u}(r)dr+\int_{-p}^{0} \varepsilon_p^{\mathscr{A}(t-2p-r)^\rho} \mathscr{C}\tilde{\Psi}(r)dr \\
&+\int_{0}^{t} \varepsilon_p^{\mathscr{A}(t-p-r)^\rho} \tilde{f}(r,z(r-p),\tilde{u}(r),\tilde{u}(r-p)dr)+\sum_{0<t_i<t} \mathscr{A}\,\varepsilon_p^{\mathscr{A}(t-t_i-2p)^\rho} \mathscr{J}_i,
\end{aligned} \tag{20}
$$

which provides the solution of the considered system and it can be computed that $z(t) = z_1$. Hence the system (1)-(4) is controllable on $\mathscr{J}$. $\square$

5. Example

The concluded result from the previous section is illustrated with a numerical example for an impulsive system with state and control delays, given by

$$ {}^{C}\mathscr{D}_{0^+}^{0.2} z(t) = \mathscr{A} z(t-0.3) + \mathscr{B}\tilde{u}(t) + \mathscr{C}\tilde{u}(t-0.3) $$

$$ + \tilde{f}(t,\ z(t-0.3),\ \tilde{u}(t),\ \tilde{u}(t-0.3)),\ z(t) \in [0,\ t_1],\ p > 0, \tag{21} $$

$$ z(t) = \tilde{\varphi}(t),\ t \in [-0.3,\ 0], \tag{22} $$

$$ z(t_i^+) - z(t_i^-) = \mathscr{J}(z(t_i)) = \mathscr{J}_i, \tag{23} $$

Now for the system (21)-(23), let $\rho = 0.2,\ t = 1.2,\ n = 2,\ p = 0.3,\ t = t_i$.

$$ \mathscr{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathscr{B} = \begin{pmatrix} 0.09 & 0 \\ 0 & 0.09 \end{pmatrix}, \quad \mathscr{C} = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, $$

$$ \tilde{f}(t,h) = \begin{pmatrix} \dfrac{1}{1 + z_2^2(t-p) + \tilde{u}^2(t) + \tilde{u}^2(t-p)} \\ \dfrac{1}{1 + z_1^2(t-p) + \tilde{u}^2(t) + \tilde{u}^2(t-p)} \end{pmatrix}, \text{ where } z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}. $$

Considering the delayed fractional grammian matrix of the system with impulsive terms,

$$ G(t) = \int_0^t (\varepsilon_p^{\mathscr{A}(t-p-r)^\rho}) \mathscr{B}\mathscr{B}^* (\varepsilon_p^{\mathscr{A}^*(t-p-r)^\rho}) dr + \int_0^{t-p} (\varepsilon_p^{\mathscr{A}(t-2p-r)^\rho}) \mathscr{C}\mathscr{C}^* (\varepsilon_p^{\mathscr{A}^*(t-2p-r)^\rho}) dr $$

$$ + \sum_{0 < t_i < t} \mathscr{A} \varepsilon_p^{\mathscr{A}(t-t_i-2p)^\rho}. $$

From Definition 3, the solution can be obtained,

$$G(1.2)=\int_0^{1.2}\varepsilon_{0.3}^{\mathscr{A}(1.2-0.3-r)^{0.2}}\mathscr{B}^2\varepsilon_{0.3}^{\mathscr{A}^*(1.2-0.3-r)^{0.2}}dr+\int_0^{0.9}\varepsilon_{0.3}^{\mathscr{A}(1.2-0.6-r)^{0.2}})\mathscr{C}^2\varepsilon_{0.3}^{\mathscr{A}^*(1.2-0.6-r)^{0.2}}dr$$

$$+\mathscr{A}\varepsilon_{0.3}^{\mathscr{A}(-0.6)^{0.2}},$$

$$G(1.2)=G_{B1}+G_{B2}+G_{B3}+G_{C1}+G_{C2}+G_{C3}+G_{Im}.$$

By the delayed *M-L* derivative,

$$G_{B1}=\int_0^{0.4}(\mathscr{I}+\mathscr{A}\frac{(1.2-r)^{0.2}}{\Gamma(1.2)}+\mathscr{A}^2\frac{(0.9-r)^{0.4}}{\Gamma(1.4)})\mathscr{B}^2(\mathscr{I}+\mathscr{A}^T\frac{(1.2-r)^{0.2}}{\Gamma(1.2)}+\mathscr{A}^{T^2}\frac{(0.9-r)^{0.4}}{\Gamma(1.4)})dr,$$

$$G_{B2}=\int_{0.4}^{0.8}(\mathscr{I}+\mathscr{A}\frac{(1.2-r)^{0.2}}{\Gamma(1.2)})\mathscr{B}^2(\mathscr{I}+\mathscr{A}^T\frac{(1.2-r)^{0.2}}{\Gamma(1.2)})dr,$$

$$G_{B3}=\int_{0.8}^{1.2}(\mathscr{I})\mathscr{B}^2(\mathscr{I})dr,$$

$$G_{C1}=\int_0^{0.3}(\mathscr{I}+\mathscr{A}\frac{(1.2-r)^{0.2}}{\Gamma(1.2)}+\mathscr{A}^2\frac{(0.6-r)^{0.4}}{\Gamma(1.4)})\mathscr{C}^2(\mathscr{I}+\mathscr{A}^T\frac{(1.2-r)^{0.2}}{\Gamma(1.2)}+\mathscr{A}^{T^2}\frac{(0.6-r)^{0.4}}{\Gamma(1.4)})dr,$$

$$G_{C2}=\int_{0.3}^{0.6}(\mathscr{I}+\mathscr{A}\frac{(1.2-r)^{0.2}}{\Gamma(1.2)})\mathscr{C}^2(\mathscr{I}+\mathscr{A}^T\frac{(1.2-r)^{0.2}}{\Gamma(1.2)})dr,$$

$$G_{C3}=\int_{0.6}^{0.9}(\mathscr{I})\mathscr{C}^2(\mathscr{I})dr,$$

$$G_{Im}=\mathscr{A}(\mathscr{I}+\mathscr{A}\frac{(-0.6)^{0.2}}{\Gamma(1.2)}+\mathscr{A}^2\frac{(-0.9)^{0.4}}{\Gamma(1.4)}).$$

By simple computations and calculations the acquired Grammian matrix is

$$G(t)=\begin{pmatrix}1.432 & 1.3412\\ 0.3412 & 1.096\end{pmatrix}.$$

Hence the obtained matrix is a nonsingular matrix and its inverse $G^{-1}(t)$ exists. Also, the linear system is controllable and $\tilde{f}(t,\ h)$ satisfies the assumption of Theorem 2. Hence, the considered system (21)-(23) based on Theorem 2 is controllable on $\mathscr{J}$.

Disclosure

The controllability results of both linear and nonlinear fractional dynamical systems with impulsive conditions and delays in state and control have been studied by using delayed *M-L* functions and fixed point techniques under certain criteria. A numerical example is provided to validate the results obtained. Further, the proposed result can be extended to system with multiple delay and distributed delay, which are likely to give useful results.

Funding

The work of G.Arthi was supported by the Science and Engineering Research Board (SERB) POWER Grant (No.: SPG/2022/001970) funded by the Government of India.

Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings.

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