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Research Article

Controllability of Impulsive Damped Fractional Order Systems Involving State Dependent Delay

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Abstract: In this article, the concept of controllability on fractional order impulsive systems involving state dependent delay and damping behavior is analysed by utilizing Caputo fractional derivative. The main motivation is to derive the sufficient conditions for the controllability of the considered systems. Based on the Laplace transform and inverse Laplace transform, the solution of fractional-order dynamical systems are obtained. The results are established by utilizing basic ideas of fractional calculus, Mittag-Leffler function and Banach fixed point theorem. Finally, an application is provided to illustrate the derived result.

***Keywords*:** fractional damped system, impulsive system, state dependent delay, fixed point theorem, controllability

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1. Introduction

Differential equations involving fractional derivatives is more precise in describing fractional order models of particular systems with derivatives of non-integer order. Fractional derivatives capture memory effects in systems. In control theory, memory or hereditary effects are often essential to model systems with long-term dependencies, delays, or non-local interactions. These systems can be found in areas such as viscoelasticity, finance, biology, and many more. Caputo fractional derivative incorporates initial conditions naturally which is suitable for solving fractional differential equations with initial values and it is well-suited for modeling real-world phenomena with memory effects. This derivative is often used to describe systems where the initial condition is meaningful and represents a physical state at time $t = 0$. This is especially relevant in areas like viscoelasticity, where fractional calculus plays a crucial role. The controllability problem is to demonstrate that a control function moves the system from intial state to final state. The applications of fractional differential equations have been discussed in [1–4]. Controllability is the primary qualitative feature of dynamical systems. Because of its relevance, in [5–7] many authors have expanded their results of linear and nonlinear systems beyond integer order to fractional systems.

Apart from these works, delay differential equation arises in predeictions and analysis for life sciences including immunology, population dynamics and neural networks. Damping behaviour is when an oscillatory system is affected which limits oscillations. Examples include resistance in electrical oscillators, scattering in optical oscillators and so on.

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Damping effects became an active area of research due to this reason that it describes practical problems like viscous drag in mechanical systems, resistance in electronic oscillators, and absorption and scattering of light in optical oscillators. Fractional derivative terms have been presented in order to develop the models of viscoelastic materials. As reported in [8–13], damping behaviour with delay involving control and state is a factor in controllability issues of linear and nonlinear systems.

Research on impulsive dynamical systems are characterized by rapid events that cause sudden changes and affect the dynamics of emerging processes in the state of the system. The abrupt changes in dynamical system are seen in harvests, thresholds and frequency-modulated systems as impulses. The monograph by Bainov and Simeonov [14] contains the fundamental understanding of impulsive differential equations. Study on controllability criteria for impulsive fractional systems with damping behaviour have been reported in [15]. Schauder [16] fixed point approach has been utilized for deriving the controllability results for impulsive system with distributed delay. Changes in the state of a physical system at a given time depend its past history by state variable rather than the state variable derivative. The past dependence on a variable is state dependent delay (SDD). The study of interest in SDD type systems are enormous in recent years. According to [17], the theory of existence yields a fractional system with resolvent operators and SDD. Existence theory of integro-differential and SDD in fractional order have been studied in [18, 19]. Moreover, second-order systems for controllability results with SDD have been established in [20, 21]. Necessary criteria for the existence of solutions for impulsive fractional system involving SDD has been discussed in [22].

Moreover, approximate controllability can be directed to any small area around the intendent state. Numerous authors have emphasised how approximate controllability systems are increasingly common and adequate in fractional differential system, impulsive effects and SDD in [23–25]. The conditions for approximate controllability problem has been established utilizing semigroup theory, fixed point approch with SDD in [26] for nonlinear system. Based on the above analysis, it is valuable to study the controllability concept for fractional-order system with impulsive effects and SDD involving damping behaviour. Eventhough many authors have emphasized how controllability systems are increasingly common and adequate in applications the controllability criteria of fractional-order systems featuring damping behaviour, impulsive effects and SDD using fixed point techniques has not yet been analysed.

The study includes the contributions, which are specified as follows:

- Most of the prior investigation on fractional systems have been discussed with single order. So, it is important to pay consideration to the study of multi-order fractional impulsive systems.
- Compared with several previous analyses, controllability of multi-order fractional impulsive system involving SDD is firstly presented for designing more general fractional-order model.
- Caputo fractional derivative and Banach contraction principle are utilized to derive the sufficient conditions for the controllability of nonlinear multi-order fractional impulsive system involving SDD, it can be expressed in terms of Mittag-Leffler function.

Structure of the article is as follows: Review of basic definitions and lemma is provided in Section 2. Controllability results is derived for damped impulsive system with SDD via contraction principle in Section 3. In Section 4, demonstration for the illustrated result is given.

2. Problem formulation and preliminaries

Consider the damped nonlinear impulsive fractional-order system with SDD

$$
{}_{0}^{C}D_{t}^{\gamma_1}y(t)-A{}_{0}^{C}D_{t}^{\gamma_2}y(t)=\mathscr{C}u(t)+h(t,\,y_{\hat{\rho}(t,\,y_t)}),\quad t\in[0,\,T]=\mathscr{I}',\tag{1}
$$

$$
y(0)=y_0,\; y'(0)=y_1,\tag{2}
$$

$$
\Delta y(t)=I_j(y(t_j)),
$$

$$
\Delta y'(t)=J_j(y'(t_j)),\quad t=t_j,\quad j=1,\,2,\,\ldots,\,k,\tag{3}
$$

where y denotes the state variable in Banach space X. ${}_{0}^{C}D_{t}^{\gamma_2}$, $\mathscr{A}{}_{0}^{C}D_{t}^{\gamma_1}$ indicates fractional order derivatives of $\gamma_2(0<\gamma_2\leq 1)$ and $\gamma_1(1<\gamma_1\leq 2)$ in caputo sense. $A,\mathscr{C}$ are the known constant matrices. $u(t)\in L_2(\mathscr{I}',\,\mathscr{U})$ is control vector. $\mathscr{C}\colon\mathscr{U}\to X$ is a continuous bounded operator. $\mathbb{PC}=\{y\colon(-\infty,\,\mathscr{I}']\to X\}$ is piecewise continuous such that $y(t_j)=y(t_j^-)$ and $y(t_j^+)$ exist for $j=1,\,2,\,\ldots,\,k$. Except for some t_j, the norm $\|y\|_{\mathbb{PC}}=sup_{t\in\mathscr{I}'}|y(t)|<\infty$ is continuous. $\Delta y(t_j)=y(t_j^+)-y(t_j^-)$ indicates the right and left bounds of $y(t)$ where $\lim_{\delta\to 0^+}y(t_j+\delta)=y(t_j^+)$ and $\lim_{\delta\to 0^-}y(t_j+\delta)=y(t_j^-)$. Similarly $\Delta y'(t_j)$ is definied. The function $h,\,\hat{\rho},\,I_j,\,J_j$ are appropriated function to be mentioned.

Now, we provide several earlier definitions and lemmas which support to obtain the main results.

Definition 1 [2, 27] For a function $h\colon R^+\to R$, the fractional order derivative γ_1 $(0\leq p_1\leq\gamma_1<p_1+1)$ is defined in caputo sense as

$$
{}_{0}^{C}D_{t}^{\gamma_1}h(t)=\frac{1}{\Gamma(p_1-\gamma_1+1)}\int_0^t\frac{h^{(p_1+1)}(\theta)}{(t-\theta)^{\gamma_1-p_1}}d\theta.
$$

The Laplace transform (LT) of ${}_{0}^{C}D_{t}^{\gamma_1}$ is

$$
\mathscr{L}\{{}_{0}^{C}D_{t}^{\gamma_1}h(t)\}(s)=s^{\gamma_1}h(s)-\sum_{i=0}^{m-1}h^{(i)}(t)s^{\gamma_1-1-i}.
$$

Definition 2 [2, 27] For $\gamma_1>0$, the Mittag-Leffler function (MLF) $E_{\gamma_1}(z)$ is

$$
E_{\gamma_1}(Z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\gamma_1 j+1)},\;\gamma_1>0,\;Z\in\mathbb{C}.
$$

The two-parameter $E_{\gamma_1,\,\gamma_2}(Z)$(MLF) with $\gamma_1,\,\gamma_2>0$ is

$$
E_{\gamma_1,\,\gamma_2}(Z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\gamma_1 j+\gamma_2)},\;\gamma_1>0,\;Z\in\mathbb{C}.
$$

The LT of $E_{\gamma_1,\,\gamma_2}(Z)$ is

$$\mathscr{L}\{t^{\gamma_1-1}E_{\gamma_1,\gamma_2}(\pm at^{\gamma_1})\}(s)=\frac{s^{\gamma_1-\gamma_2}}{s^{\gamma_1}\mp a}.$$

For $\gamma_2=1$,

$$\mathscr{L}\{E_{\gamma_1}(\pm at^{\alpha})\}(s)=\frac{s^{\gamma_1-\gamma_2}}{s^{\gamma_1}\mp a}.$$

Using the ideas and notations developed in [28], the abstract space $(\mathfrak{B},\|\cdot\|_{\mathfrak{B}})$ is a seminorm linear space. y_s represents $y_s(\theta)=y(s+\theta)\in\mathfrak{B}$ in the function y_s: $(-\infty,0]\to X$ and the following axioms hold:

• If y: $(-\infty,T]\to X$, is a continuous function such that $y_0\in\mathfrak{B}$, $\forall\, t\in[0,T)$,

(i) $y_t\in\mathfrak{B}$;

(ii) $\|y(t)\|\leq\mathscr{R}_1\,\|y_t\|_{\mathfrak{B}}$;

(iii) $\|y_t\|_{\mathfrak{B}}\leq\mathscr{R}_2(t)\|y_0\|_{\mathfrak{B}}+\mathscr{R}_3(t)\sup\{\|y(s)\|\colon 0\leq s\leq T\}$,

holds and $\mathscr{R}_1>0$ is a constant, $\mathscr{R}_2$, $\mathscr{R}_3$: $[0,\infty)\to[0,\infty)$ such that $\mathscr{R}_3$ is continuous and $\mathscr{R}_2$ is locally bounded.

Lemma 1 ([29]) For $y_0=\varphi$ and $y(\cdot)|_{\mathscr{I}'}\in\mathbb{PC}$ such that y: $(-\infty,T]\to X$ be a function. Then

$$\|y_s\|_{\mathfrak{B}}\leq\left(\mathscr{M}_T+\mathscr{J}_0^{\varphi}\right)\|\varphi\|_{\mathfrak{B}}+\mathscr{R}_T\sup\{\|y(\theta)\|;\,\theta\in[0,\max\{0,s\}]\},\ s\in Z\left(\hat{\rho}^-\right)\cup\mathscr{I}'.$$

Consider the following fractional order system as treated in [6]

$$ {}_0^CD_t^{\gamma_1}y(t)-A\,{}_0^CD_t^{\gamma_2}y(t)=h(t,y),\quad t\in[0,T]=\mathscr{I}',$$

$$y(0)=y_0,\ y'(0)=y_1.$$

with $0<\gamma_2\leq1<\gamma_1\leq2y$ denotes state variable, $A\in\mathbb{R}^{n\times n}$ and h: $J\to\mathbb{R}^n$ is a continuous function. Applying the LT to establish the solution of the system mentioned above,

$$s^{\gamma_1}y(s)-s^{\gamma_1-1}y(0)-s^{\gamma_1-2}y'(0)-As^{\gamma_2}y(s)+As^{\gamma_2-1}y(0)=H(s).$$

Applying the inverse LT to both sides of the previous formula,

$$\begin{aligned}\mathscr{L}^{-1}\{Y(s)\}(t)=&\mathscr{L}^{-1}\left\{s^{\gamma_1-\gamma_2-1}\left(s^{\gamma_1-\gamma_2}I-A\right)^{-1}\right\}(t)y_0-A\mathscr{L}^{-1}\left\{s^{-1}\left(s^{\gamma_1-\gamma_2}I-A\right)^{-1}\right\}(t)y_0\\&+\mathscr{L}^{-1}\left\{s^{\gamma_1-\gamma_2-2}\left(s^{\gamma_1-\gamma_2}I-A\right)^{-1}\right\}(t)y_0'+\mathscr{L}^{-1}\left\{H(s)\times s^{-\gamma_2}\left(s^{\gamma_1-\gamma_2}I-A\right)^{-1}\right\}(t).\end{aligned}$$

Finally, by replacing the Laplace transformation of the ML function and the Laplace convolution operator, we obtain the system solution as

$$y(t) = E_{\gamma_1-\gamma_2}\left(At^{\gamma_1-\gamma_2}\right)y_0 - At^{\gamma_1-\gamma_2}E_{\gamma_1-\gamma_2,\,\gamma_1-\gamma_2+1}\left(At^{\gamma_1-\gamma_2}\right)y_0 + tE_{\gamma_1-\gamma_2,\,2}\left(At^{\gamma_1-\gamma_2}\right)y_0'$$

$$+\int_0^t (t-s)^{\gamma_1-1}E_{\gamma_1-\gamma_2,\,\gamma_1}\left(A(t-s)^{\gamma_1-\gamma_2}\right)h(s)\mathrm{d}s.$$

Definition 3 A function y: $(-\infty,\ T] \to X$ be the solution of the impulsive damped system $(1)-(3)$, if $y(0) = y_0$, $y^{'}(0) = y_1$ and $y_{\hat{\rho}(s,\,y_s)} \in \mathfrak{B}$ for every $y(\cdot)|_{\mathscr{I}'} \in \mathbb{PC}$ then

$$y(t) = E_{\gamma_1-\gamma_2}(AT^{\gamma_1-\gamma_2})y_0 - AT^{\gamma_1-\gamma_2}E_{\gamma_1-\gamma_2,\,\gamma_1-\gamma_2+1}(AT^{\gamma_1-\gamma_2})y_0 + TE_{\gamma_1-\gamma_2,\,2}(AT^{\gamma_1-\gamma_2})y_1$$

$$+\sum_{j=1}^{k}E_{\gamma_1-\gamma_2}(A(T-t_j)^{\gamma_1-\gamma_2})I_j(y(t_j)) - \sum_{j=1}^{k}A(T-t_j)^{\gamma_1-\gamma_2}E_{\gamma_1-\gamma_2,\,\gamma_1-\gamma_2+1}$$

$$\times(A(T-t_j)^{\gamma_1-\gamma_2})I_j(y(t_j)) + \sum_{j=1}^{k}(T-t_j)E_{\gamma_1-\gamma_2,\,2}(A(T-t_j)^{\gamma_1-\gamma_2})J_j(y'(t_j))$$

$$+\int_0^t (T-s)^{\gamma_1-1}E_{\gamma_1-\gamma_2,\,\gamma_1}(A(T-s)^{\gamma_1-\gamma_2})h(s,\,y_{\hat{\rho}(s,\,y_s)})ds$$

$$+\int_0^t (T-s)^{\gamma_1-1}E_{\gamma_1-\gamma_2,\,\gamma_1}(A(T-s)^{\gamma_1-\gamma_2})\mathscr{C}u(s)ds, \quad t \in \mathscr{I}^{'}.$$

3. Main results

In this part, assume the following hypothesis to demonstrate the controllability result for systems (1)-(3).

(H_1) L_h: $[0,\,\infty) \to (0,\,\infty)$ be a continuous function and an integrable function α: $\mathscr{I}' \to [0,\,\infty)$ exist such that

$$\|h(t,\,\psi)\| \le \alpha(t)L_h(\|\psi\|_{\mathfrak{B}}), \quad \liminf_{\omega\to\infty}\frac{L_h(\omega)}{\omega} = \tilde{\omega} \le \infty.$$

(H_2) The functions h: $\mathscr{I}' \times \mathfrak{B} \to X$, I_j, J_j: $\mathfrak{B} \to X$ are continuous and $\exists$ constants L_h, ϕ_j, σ_j such that

$$\|h(t,\,y_1) - h(t,\,y_2)\| \le L_h\|y_1 - y_2\|^2, \quad \|I_j(y_1) - I_j(y_2)\|^2 \le \phi_j\|y_1 - y_2\|^2,$$

$$\|J_j(y_1) - J_j(y_2)\|^2 \le \sigma_j\|y_1 - y_2\|^2.$$

(H_3) The maps I_j, J_j are continuous and the continuous non-decreasing functions β_j, $\tilde{\gamma}_j$: $[0,\,\infty) \to (0,\,\infty)$, j = 1, 2, . . ., k exist such that

$$\|I_j(y)\|^2 \leq \beta_j(\|y\|^2), \quad \liminf_{r\to\infty} \frac{\beta_j(r)}{r} = \Upsilon_j \leq \infty,$$

$$\|J_j(y)\|^2 \leq \tilde{\gamma}_j(\|y\|^2), \quad \liminf_{r\to\infty} \frac{\tilde{\gamma}_j(r)}{r} = \zeta_j \leq \infty.$$

(H_4) Let $Z(\hat{\rho}^-) = \hat{\rho}(s, \varphi) \in \mathscr{I}' \times \mathfrak{B}$. A continuous and bounded function $\mathscr{J}^{\varphi}: Z(\hat{\rho}^-) \to (0, \infty)$ is well definied in $t \to \varphi_t$ from $Z(\hat{\rho}^-)$ into $\mathfrak{B}$ such that $\|\varphi\|_{\mathfrak{B}} \leq \mathscr{J}^{\varphi}(t)\|\varphi\|_{\mathfrak{B}}$ $\forall\, t \in Z(\hat{\rho}^-)$.

(H_5) The linear operator W is defined by

$$Wu = \int_0^t (T-s)^{\gamma_1 - 1} E_{\gamma_1-\gamma_2,\, \gamma_1}(A(T-s)^{\gamma_1-\gamma_2})\mathscr{C}u(s)ds$$

has a bounded invertible operator W^{-1} exists in $L_2(\mathscr{I}', \mathscr{U})/kerW$ such that $\|W^{-1}\| \leq l$ and $\mathscr{C}$: $\mathscr{U} \to X$ is bounded, continuous $\exists$ a constant M such that

$$M = \|(T-s)^{\gamma_1-1}[E_{\gamma_1-\gamma_2,\, \gamma_1}(A(T-s)^{\gamma_1-\gamma_2})]\mathscr{C}\|^2.$$

For brevity,

$$\mathfrak{C}_1 = sup_{t\in\mathscr{I}'}\|E_{\gamma_1-\gamma_2}(AT^{\gamma_1-\gamma_2})\|^2,$$

$$\mathfrak{C}_2 = sup_{t\in\mathscr{I}'}\|AT^{\gamma_1-\gamma_2}E_{\gamma_1-\gamma_2,\, \gamma_1-\gamma_2+1}(AT^{\gamma_1-\gamma_2})\|^2,$$

$$\mathfrak{C}_3 = sup_{t\in\mathscr{I}'}\|TE_{\gamma_1-\gamma_2,\, 2}(AT^{\gamma_1-\gamma_2})\|^2,$$

$$\mathfrak{C}_4 = \|(T-s)^{\gamma_1-1}E_{\gamma_1-\gamma_2,\, \gamma_1}(A(T-s)^{\gamma_1-\gamma_2})\|^2$$

Determining the control function

$$\mathscr{C}^*[(T-s)^{\gamma_1-1}E_{\gamma_1-\gamma_2,\, \gamma_1}(A(T-t)^{\gamma_1-\gamma_2})]^*W^{-1}\hat{y} = u(t),$$

where

$$
\begin{aligned}
\hat{y} =& y_T - E_{\gamma_1-\gamma_2}(AT^{\gamma_1-\gamma_2})y_0 + AT^{\gamma_1-\gamma_2}E_{\gamma_1-\gamma_2,\, \gamma_1-\gamma_2+1}(AT^{\gamma_1-\gamma_2})y_0 \\
& - TE_{\gamma_1-\gamma_2,\, 2}(AT^{\gamma_1-\gamma_2})y_1 - \sum_{j=1}^{k} E_{\gamma_1-\gamma_2}(A(T-t_j)^{\gamma_1-\gamma_2})I_j(y(t_j)) \\
& + \sum_{j=1}^{k} A(T-t_j)^{\gamma_1-\gamma_2}E_{\gamma_1-\gamma_2,\, \gamma_1-\gamma_2+1}(A(T-t_j)^{\gamma_1-\gamma_2})I_j(y(t_j)) \\
& - \sum_{j=1}^{k}(T-t_j)E_{\gamma_1-\gamma_2,\, 2}(A(T-t_j)^{\gamma_1-\gamma_2})J_j(y'(t_j)) \\
& - \int_0^T (T-s)^{\gamma_1-1}E_{\gamma_1-\gamma_2,\, \gamma_1}(A(T-s)^{\gamma_1-\gamma_2})h(s,\, y_{\hat{\rho}(s,\, y_s)})ds.
\end{aligned}
$$

$$
\begin{aligned}
\|u(t)\|^2 \le& M^2l^2T\Bigg(\|y_T\|^2 + \mathfrak{C}_1\|y_0\|^2 + \mathfrak{C}_2\|y_0\|^2 + \mathfrak{C}_3\|y_1\|^2 \\
& + \mathfrak{C}_1\sum_{j=1}^{k}\beta_j(r)\|y(s)\|^2 + \mathfrak{C}_2\sum_{j=1}^{k}\beta_j(r)\|y(s)\|^2 + \mathfrak{C}_3\sum_{j=1}^{k}\tilde{\gamma}_j(r)\|y(s)\|^2 \\
& + \mathfrak{C}_4\frac{T^{2\gamma_1-1}}{2\gamma_1-1}L_h[(\mathscr{M}_T + \mathscr{J}_0^{\varphi})\|\varphi\|_{\mathfrak{B}} + \mathscr{R}_T r]\left[\int_0^T(\alpha(s))ds\right]\Bigg)
\end{aligned}
$$

Theorem 1 The nonlinear system (1)-(3) is controllable on $\mathscr{I}'$ if

$$
1 \ge \left([(\mathfrak{C}_1+\mathfrak{C}_2)\sum_{j=1}^{k}\phi_j + \mathfrak{C}_3\sum_{j=1}^{k}\sigma_j] + \mathfrak{C}_4\frac{T^{2\gamma_1-1}}{2\gamma_1-1}L_h(\tilde{\omega})\right)[1+M^2l^2]
$$

provided that the hypothesis (H_1)-(H_5) are true.

Proof. Define an operator Φ as,

$$
\begin{aligned}
(\Phi y)(t) =& E_{\gamma_1-\gamma_2}(AT^{\gamma_1-\gamma_2})y_0 - AT^{\gamma_1-\gamma_2}E_{\gamma_1-\gamma_2,\,\gamma_1-\gamma_2+1}(AT^{\gamma_1-\gamma_2})y_0 + TE_{\gamma_1-\gamma_2,\,2}(AT^{\gamma_1-\gamma_2})y_1 \\
&+ \sum_{j=1}^{k} E_{\gamma_1-\gamma_2}(A(T-t_j)^{\gamma_1-\gamma_2})I_j(y(t_j)) - \sum_{j=1}^{k} A(T-t_j)^{\gamma_1-\gamma_2}E_{\gamma_1-\gamma_2,\,\gamma_1-\gamma_2+1} \\
&\times (A(T-t_j)^{\gamma_1-\gamma_2})I_j(y(t_j)) + \sum_{j=1}^{k}(T-t_j)E_{\gamma_1-\gamma_2,\,2}(A(T-t_j)^{\gamma_1-\gamma_2})J_j(y'(t_j)) \\
&+ \int_0^t (T-s)^{\gamma_1-1}E_{\gamma_1-\gamma_2,\,\gamma_1}(A(T-s)^{\gamma_1-\gamma_2})h(s,\, y_{\hat{\rho}(s,\,y_s)})ds \\
&+ \int_0^t (T-s)^{\gamma_1-1}E_{\gamma_1-\gamma_2,\,\gamma_1}(A(T-s)^{\gamma_1-\gamma_2})\mathscr{C}u(s)ds.
\end{aligned}
$$

Using the concept of Banach contraction mapping principle, it has been proved that Φ has a fixed point and the system (1)-(3) is controllable on $\mathscr{I}'$.

The set $\mathfrak{B}_r = \{x \in \mathfrak{B}\colon \|y\|_\infty \le r\}$, where $\mathfrak{B}_r$ is closed, bounded and convex set in $\mathfrak{B}$ $\forall$ r, then by Lemma 1,

$$
\left\|y_{\hat{\rho}(t,\,x_t)}\right\|_{\mathfrak{B}} \le \left(\mathscr{M}_{\mathscr{T}} + \mathscr{J}_0^{\varphi}\right)\|\varphi\|_{\mathfrak{B}} + \mathscr{R}_{\mathscr{T}}(r)
$$

Divide the proof into two parts:

Step 1: $\Phi\mathfrak{B}_r \subset \mathfrak{B}_r$.

Assume $\Phi\mathfrak{B}_r \subset \mathfrak{B}_r$ is not valid, then $y \in \mathfrak{B}_r$ for every $r \ge 0$ for $t \in \mathscr{I}'$. Then

$$
\begin{aligned}
r \le& \|\Phi y(t)\|^2 \\
\le& \|E_{\gamma_1-\gamma_2}(AT^{\gamma_1-\gamma_2})y_0\|^2 + \|AT^{\gamma_1-\gamma_2}E_{\gamma_1-\gamma_2,\,\gamma_1-\gamma_2+1}(AT^{\gamma_1-\gamma_2})y_0\|^2 \\
&+ \|TE_{\gamma_1-\gamma_2,\,2}(AT^{\gamma_1-\gamma_2})y_1\|^2 + \|\sum_{j=1}^{k} E_{\gamma_1-\gamma_2}(A(T-t_j)^{\gamma_1-\gamma_2})I_j(y(t_j))\|^2 \\
&+ \|\sum_{j=1}^{k} A(T-t_j)^{\gamma_1-\gamma_2}E_{\gamma_1-\gamma_2,\,\gamma_1-\gamma_2+1}(A(T-t_j)^{\gamma_1-\gamma_2})I_j(y(t_j))\|^2 \\
&+ \|\sum_{j=1}^{k}(T-t_j)E_{\gamma_1-\gamma_2,\,2}(A(T-t_j)^{\gamma_1-\gamma_2})J_j(y'(t_j))\|^2
\end{aligned}
$$

$$+\left\|\int_0^t (T-s)^{\gamma_1-1}E_{\gamma_1-\gamma_2,\,\gamma_1}(A(T-s)^{\gamma_1-\gamma_2})h(s,\,y_{\hat{\rho}(s,\,y_s)})ds\right\|^2$$

$$+\left\|\int_0^t (T-s)^{\gamma_1-1}E_{\gamma_1-\gamma_2,\,\gamma_1}(A(T-s)^{\gamma_1-\gamma_2})\mathscr{C}u(s)ds\right\|^2$$

$$r \le [\mathfrak{C}_1+\mathfrak{C}_2]\left[\|y_0\|^2+\sum_{j=1}^{k}\beta_j(r)\|y(s)\|^2\right][1+M^2l^2T]$$

$$+\mathfrak{C}_3\left[\|y_1\|^2+\sum_{j=1}^{k}\tilde{\gamma}_j(r)\|y(s)\|^2\right][1+M^2l^2T]$$

$$+\mathfrak{C}_4\frac{T^{2\gamma_1-1}}{2\gamma_2-1}[L_h][(\mathscr{M}_T+\mathscr{J}_0^{\varphi})\|\varphi\|_{\mathfrak{B}}+\mathscr{R}_T r]\left(\int_0^T(\alpha(s))ds\right)\Bigg]$$

$$\times[1+M^2l^2T]+M^2l^2T(E\|y_T\|^2)$$

and hence

$$1 \le \left(\sum_{j=1}^{k}[\Upsilon_j+\zeta_j]+\frac{T^{2\gamma_1-1}}{2\gamma_1-1}\tilde{\omega}[\int_0^T(\alpha(s))ds]\right)[1+M^2l^2T]$$

which is contrary to the assumption. Hence Φ maps $\mathfrak{B}_r$ into itself.

Step 2: Φ is a contraction mapping.

Let $y_1,\,y_2\in\mathfrak{B}_r$,

$$\|\Phi y_1(t)-\Phi y_2(t)\|^2 \le \mathfrak{C}_1\Big(\sum_{j=1}^{k}\phi_j\Big)\|y_1(t)-y_2(t)\|^2+\mathfrak{C}_2\Big(\sum_{j=1}^{k}\phi_j\Big)\|y_1(t)-y_2(t)\|^2$$

$$+\mathfrak{C}_3\Big(\sum_{j=1}^{k}\sigma_j\Big)\|y_1(t)-y_2(t)\|^2+\mathfrak{C}_4\frac{T^{2\gamma_1-1}}{2\gamma_1-1}[L_h]\int_0^T\|[y_{1_{\hat{\rho}(s,\,y_s)}}-y_{2_{\hat{\rho}(s,\,y_s)}}]\|^2ds$$

$$+M^2l^2\Bigg[\mathfrak{C}_1\Big(\sum_{j=1}^{h}\phi_j\Big)\|y_1(t)-y_2(t)\|^2+\mathfrak{C}_2\Big(\sum_{j=1}^{k}\phi_j\Big)\|y_1(t)-y_2(t)\|^2$$

$$+\mathfrak{C}_3\Big(\sum_{j=1}^{k}\sigma_j\Big)\|y_1(t)-y_2(t)\|^2+\mathfrak{C}_4\frac{T^{2\gamma_1-1}}{2\gamma_1-1}[L_h]\int_0^T\|[y_{1_{\hat{\rho}(s,\,y_s)}}-y_{2_{\hat{\rho}(s,\,y_s)}}]\|^2ds\Bigg]$$

$$\leq \left([(\mathfrak{C}_1 + \mathfrak{C}_2) \sum_{j=1}^{n} \phi_j + \mathfrak{C}_3 \sum_{j=1}^{k} \sigma_j] + \frac{T^{2\gamma_1 - 1}}{2\gamma_1 - 1} L_h(\tilde{\omega}) \right)$$

$$\times [1 + M^2 l^2] \sup_{0 \leq s \leq T} \|y_1(s) - y_2(s)\|^2$$

Therefore,

$$\left([(\mathfrak{C}_1 + \mathfrak{C}_2) \sum_{j=1}^{n} \phi_j + \mathfrak{C}_3 \sum_{j=1}^{k} \sigma_j] + \mathfrak{C}_4 \frac{T^{2\gamma_1 - 1}}{2\gamma_1 - 1} L_h(\tilde{\omega}) \right) [1 + M^2 l^2] \leq 1.$$

This implies that Φ has a fixed point. Hence, the nonlinear system (1)-(3) is controllable on $\mathscr{I}'$. □

4. Application

Impulsive damped fractional-order system with SDD of the form

$$\begin{cases} {}^C D_t^{\gamma_1} y(t, z) + \lambda {}^C D_t^{\gamma_2} y(t, z) = \mathscr{C} u(t, z) + k^2 \frac{\partial^2}{\partial z^2} y(t, z) \\ \qquad + \int_{-\infty}^{t} g(s - t) y(s - \hat{\rho}_1(t) \hat{\rho}_2(\|y(t)\|), z) ds, \quad t \in \mathscr{I}' = [0, T], \\ y(0, z) = y_0(z), y'(0, z) = y_1(z), \\ y(t, 0) = y(t, \pi) = 0, \\ \Delta y(t_j, z) = \int_{-\infty}^{t_j} q(t_j - s) y(s, z) dz, \quad j = 1, 2, \ldots, k, \\ \Delta y'(t_j, z) = \int_{-\infty}^{t_j} \tilde{q}(t_j - s) y(s, z) dz, \quad j = 1, 2, \ldots, k. \end{cases} \tag{4}$$

Here, Caputo derivatives ${}_0^C D_t^{\gamma_1}$, ${}_0^C D_t^{\gamma_2}$ are of order $0 < \gamma_2 \leq 1$, $1 < \gamma_1 \leq 2$, h: $\mathscr{I}' \times \mathfrak{B} \to X$ and a: $R \to R$ is continuous then,

$$h(t, \psi)(z) = \int_{-\infty}^{0} g(s) \psi(s, z) dz.$$

And, $\hat{\rho}$: $\mathscr{I}' \times \mathfrak{B} \to X$, then $\hat{\rho}_i$: $[0, \infty) \to [0, \infty)$, $i = 1, 2$.

$$\hat{\rho}(t, \psi)(z) = t - \hat{\rho}_1(t) \hat{\rho}_2(\|\psi(0, z)\|).$$

For $z \in [0, \pi]$, $\mathscr{C}u(t, z)$: $U \subset \mathscr{I}' \to X$ is bounded linear operator and $\mathscr{C}u(t, z)$: $[0, T] \times [0, \pi] \to X$ is continuous. Defining the operator W as,

$$(Wu)(\xi) = \sum_{n=1}^{\infty} \int_0^{\pi} \frac{1}{n} sinns(\mathscr{C}(s, \xi), z_n)z_n ds, \, \xi \in [0, \pi].$$

Also, I_j, J_j: $\mathfrak{B} \to X$ and $q, \tilde{q} > 0$ for $j = 1, 2, \ldots, k$,

$$I_j(\psi)(z) = \int_{-\infty}^{t_j} q(t_j - s)y(s, z)dz,$$

$$J_j(\psi)(z) = \int_{-\infty}^{t_j} \tilde{q}(t_j - s)y(s, z)dz.$$

Furthermore, $\|h\| \leq L_h$, $\|I_j\| \leq L_{I_j}$, $\|J_j\| \leq L_{J_j}$ are bounded linear operators.

Hence, the impulsive damped fractional order system with SDD (1)-(3) is represented in the abstract form (4). Therefore, the system (1)-(3) is controllable on $\mathscr{I}'$ as (4) satisfies the conditions of Theorem 1.

5. Conclusion

The controllability of fractional order damped system with impulsive effects and SDD have been examined in this article. Under specific asumptions, sufficient conditions of the considered nonlinear system have been stated using fixed point techniques. An example has been included to validate the illustrated result. Further, the result can be extended to stochastic effects.

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Conflict of interest

The authors declare that they have no competing interests.

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