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Research Article

Stability of Solutions to a Caginalp Phase-Field Type Equations

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Abstract: This paper is concerned with the study of the asymptotic behavior of a generalization of the Caginalp phase-field model subject to homogeneous Neumann boundary conditions and regular potentials involving two temperatures. This work follows on from a paper in which the well-posedness of the problem, the dissipativity of the system, and the existence of global and exponential attractors were demonstrated. In addition, a study on the semi-infinite cylinder was also carried out. Indeed, if it is true that the existence of a global attractor makes it possible to predict the asymptotic behavior of solutions on a bounded domain, it does not say that these solutions converge. After having shown the existence of the global attractor, it is therefore important to look at the convergence of the solutions over time. There are several methods for determining the asymptotic behavior of the solutions of a differential system. We can mention the one that consists of transforming the given differential equations into integral equations and then applying the classical Picard successive approximation procedure to them. This work is devoted to the study of the convergence of solutions to steady states, adapting a well-known result concerning Lojasiewicz-Simon's inequality.

Keywords: Caginalp phase-field system, two temperatures, well-posedness, dissipativity, global attractor, Lojasiewicz-Simon inequality, convergence to steady states

MSC: 35B41, 80A22, 93D05

1. Introduction

Let us consider the model problem defined by

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \varphi - \Delta \varphi, \tag{1}$$

$$\frac{\partial \varphi}{\partial t} - \Delta \frac{\partial \varphi}{\partial t} - \Delta \varphi = -\frac{\partial u}{\partial t}, \tag{2}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{3}$$

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$$u_{|t=0} = u_0, \varphi_{|t=0} = \varphi_0, \tag{4}$$

arising from the Caginalp phase-field system (see [1]), namely,

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \theta, \tag{5}$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t}, \tag{6}$$

called a non-conservative system, in the sense that, when it has Neumann boundary conditions, the spatial average of u is not conserved.

The variable u denotes the order parameter, and θ is the conducting temperature. More precisely, this model takes into account two temperatures (see [2-6]): the conductive θ and the thermodynamic φ. They are linked by the linearized law

$$\theta = \varphi - \Delta\varphi. \tag{7}$$

For time-independent problems, the difference between these temperatures is proportional to the heat supply; they thus coincide when there is no heat supply. However, for time-dependent problems, they are generally different, even in the absence of heat supply; this is in particular the case for non-simple materials.

The function f is the derivative of the double-well potential F (typically $F(s) = \frac{1}{4}(s^2 - 1)^2$). In addition, for convenience, we set all physical parameters equal to one. This system has been introduced to model phase transition phenomena, such as melting-solidification phenomena, and has been studied extensively from a mathematical point of view (see, e.g., [7-22]).

An important aspect of these equations is the consideration of thermal conductivity. In particular, in this model, the classical Fourier law for heat conduction is considered

$$q = -\nabla\theta, \tag{8}$$

where q is the heat flux.

It is also well known that the Fourier law allows the thermal wave to propagate instantaneously. This fact violates the causality principle. For this reason, several authors have proposed alternative laws for heat flow that overcome this drawback. We can mention, among others, the Maxwell-Cattaneo law, the Green and Naghdi law, and other constitutive laws for heat flux coming from thermomechanics; see, e.g., [20, 23-33].

Furthermore, let us introduce the total Ginzburg-Landau free energy in terms of the conduction temperature θ defined by

$$\psi(u,\theta) = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 + F(u) - u\theta - \frac{1}{2}\theta^2\Big)dx, \tag{9}$$

where Ω is the domain occupied by the system (here, we assume that it is a bounded and smooth domain of $\mathbb{R}^n, n = 1, 2$ or 3, with a boundary $\partial\Omega$) and the enthalpy

$$H = u + \theta = u + \varphi - \Delta\varphi. \tag{10}$$

Taking into account (7), (8), (9), and (10), the derivation of the model (1)-(4) is easily shown (see [6]).

In the paper [6], several results have been proven. These include the existence and uniqueness of the solution obtained, respectively, using the Galerkin approximation scheme and Gronwall's lemma (see [11, 12]), and the regularity of $H^2(\Omega)$ acquired by Agmon, Holder, and Sobolev injections (see [18, 34]). The application of the uniform

Gronwall lemma allowed to show the dissipativity of the system (see [18, 34]). The existence of the global attractor required a semi-group decomposition. This is due to the presence of the $\Delta \frac{\partial \varphi}{\partial t}$ term, which by its strong dissipativity leads to a loss of the regularisation effect (see [21]).

In this work, we are interested in the Caginalp system endowed with homogenous boundary conditions in the framework of the two-temperature theory with a regular potential *f*. Precisely, we focus on the associated stationery problem, namely:

$$-\Delta u + f(u) = \varphi, \tag{11}$$

$$\Delta \varphi = 0, \tag{12}$$

$$\frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{13}$$

which can be considered as the asymptotic case of (1)-(4). Indeed, the existence of the global attractor allows us to predict the asymptotic behavior of the solution in a bounded domain without giving more precision. An important issue is whether any trajectory converges to some steady state as time goes to infinity. It is important to notice that such a question is not a trivial one, as there may be a continuum of steady states. In particular, according to [13], we can prove the convergence of trajectories to the steady state by using an approach based on Lojasiewicz-Simon's inequality and the analyticity of the nonlinear terms. Such an approach, first considered in [35] and then simplified and further developed in [22], has been applied with success to many equations and, in particular, to models in phase separation and transition (see [12, 18, 36-38]).

1.1 *Notation*

We introduce the following Hilbert spaces

$$\Phi = H^1(\Omega) \times H^2(\Omega) \text{ and } W = H^2(\Omega) \times H^3(\Omega). \tag{14}$$

It appears clearly that the average (in space) of the function $u + \varphi$ in the problem (1)-(4) is conserved in time, namely

$$\int_\Omega (u+\varphi)dx = \int_\Omega (u_0 + \varphi_0)dx, \forall t > 0. \tag{15}$$

In fact, integrating (2) over Ω and taking into account (3), we get (15). And then, we introduce the following functional spaces:

$$\Phi_\beta = \left\{ (u,\varphi) \in \Phi; \frac{1}{|\Omega|}\int_\Omega (u+\varphi)dx = \beta \right\}, \beta \geq 0 \tag{16}$$

and

$$\phi_\alpha = \bigcup_{|\beta| \leq \alpha} \Phi_\beta, \alpha \geq 0, \tag{17}$$

which are subspaces of the phase space Φ. We define the quantity $\langle a \rangle$ by:

$$\langle a \rangle := \frac{1}{|\Omega|}\int_\Omega a\,dx, \tag{18}$$

where $|\Omega|$ is the volume of the domain Ω. For a given space H, we denote the norm in H by $\|.\|_H$. Throughout this paper, the inner product and the norm of the $L^2(\Omega)$ space will be denoted by $(.,.)$ and $\|.\|$, respectively.

2. Convergence to an equilibrium

This section is devoted to the study of the convergence of solutions. We prove that solutions converge to steady states when time goes to infinity using Lojasiewicz's inequality and the analyticity of the nonlinearity f. For this, we consider the equilibrium problem corresponding to (1)-(4)

$$-\Delta u + f(u) = \varphi \tag{19}$$

$$\Delta\varphi = 0 \tag{20}$$

$$\frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{21}$$

We also note that $I_0 = \langle u_0 + \varphi_0 \rangle = \langle u + \varphi \rangle$.

But, (20)-(21) implies that φ is constant, i.e., $\varphi = I_0 - \langle u \rangle$. As a result, the corresponding equilibrium problem reads

$$\begin{cases} -\Delta u + f(u) - \varphi = 0, \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ \varphi = I_0 - \langle u \rangle. \end{cases} \tag{22}$$

We now assume that the nonlinearity f is analytic in $\mathbb{R}$.

2.1 *Main result*

Theorem 2.1. We take $(u_0, \varphi_0) \in \Phi_M := \{(u,\varphi) \in H^1(\Omega) \times H^2(\Omega) : |I_0| \leq M\}$. Let (u,φ) be the solution to (1)-(4). Then, there exists a solution $(\overline{u}, \overline{\varphi})$ to (22), such that

$$u(t,x) \to \overline{u}(x) \text{ strongly in } H^1(\Omega),$$

$$\varphi(t,x) \to \overline{\varphi}(x) \text{ strongly in } H^2(\Omega),$$

as time goes to infinity.

The proof of Theorem 2.1 is based on results below. We first give the following definitions.

Definition 2.2. We assume that X is a complete metric space, $T(t)$, a semigroup defined from X to itself, and $\mathcal{F}(.,.)$ a Lyapunov functional. Then, the system $(X, T(t), \mathcal{F})$ is termed gradient system if it satisfies the following conditions:

(i) Let $(u_0, \varphi_0) \in X$. If for all $t > 0, \mathcal{F}(T(t)(u_0,\varphi_0)) = \mathcal{F}(u_0,\varphi_0)$, then (u_0,φ_0) is a fixed point of the semigroup $T(t)$.

(ii) For every $(u_0,\varphi_0) \in X$, it exists $t_0 > 0$, such that the orbit $\cup_{t \geq t_0} T(t)(u_0,\varphi_0)$ is relatively compact in X.

We can define, according to [14], the solving semigroup associated with problem (1)-(3), namely,

$$S(t) : \Phi_{\mathcal{M}} \to \Phi_{\mathcal{M}}, S(t)(u_0,\varphi_0) = (u(t), \varphi(t)),$$

where $(u(t), \varphi(t))$ is the unique solution to problem (1)-(3) with initial data (u_0, φ_0) and

$$\Phi_{\mathcal{M}} = \{(u,\varphi) \in \Phi_{\mathcal{M}} : |I_0| \leq \mathcal{M}\}, \forall \mathcal{M} \geq 0 \quad (\Phi_{\mathcal{M}} \subset \Phi)$$

endowed with the norm $\|(u,\varphi)\|^2_{\Phi_{\mathcal{M}}} = \|u\|^2_{H^1} + \|\varphi\|^2_{H^2}$.

Let us introduce the functional $E:\Phi\to\mathbb{R}$ defined as follows:

$$E(u(t),\varphi(t))=\int_\Omega\Big(\frac{1}{2}|\nabla u(t)|^2+F(u(t))+\frac{1}{2}|\varphi(t)|^2+\frac{1}{2}|\Delta\varphi(t)|^2\Big)dx,$$

where $F(s)=\int_0^s f(t)dt$.

Theorem 2.3. $(\Phi,S(t),E)$ is a gradient system, where $\Phi=H^1(\Omega)\times H^2(\Omega)$ and $S(t)$, $t\geq 0$ is the semigroup associated to our dynamical system.

The proof of Theorem 2.3 leans on the three lemmata below.

Lemma 2.4. The functional E is a Lyapunov function for our problem.

Proof. Indeed, the functional E satisfies

$$\begin{aligned}\frac{d}{dt}E(u,\varphi)&=\Big(-\Delta u+f(u),\frac{\partial u}{\partial t}\Big)+\Big(\varphi-2\Delta\varphi,\frac{\partial\varphi}{\partial t}\Big)+\Big(\Delta\varphi,\Delta\frac{\partial\varphi}{\partial t}\Big)\\&=-\left\|\frac{\partial u}{\partial t}\right\|^2+\Big(\varphi-\Delta\varphi,\frac{\partial u}{\partial t}+\frac{\partial\varphi}{\partial t}\Big)-\Big(\Delta\varphi,\frac{\partial\varphi}{\partial t}\Big)+\Big(\Delta\varphi,\Delta\frac{\partial\varphi}{\partial t}\Big)\\&=-\left\|\frac{\partial u}{\partial t}\right\|^2+\Big(\varphi-\Delta\varphi,\Delta\frac{\partial\varphi}{\partial t}+\Delta\varphi\Big)+\Big(\Delta\varphi,\Delta\frac{\partial\varphi}{\partial t}\Big)\\&=-\left\|\frac{\partial u}{\partial t}\right\|^2-\|\nabla\varphi\|^2-\|\Delta\varphi\|^2\leq 0.\end{aligned}\tag{23}$$

Lemma 2.5. Let $(u_0,\varphi_0)\in\Phi$. If for all $t>0, E(S(t)(u_0,\varphi_0))=E(u_0,\varphi_0)$, then (u_0,φ_0) is a fixed point of the semigroup $S(t)$.

Proof. Let $T>0$ be fixed, such that $E(S(T)(u_0,\varphi_0))=E(u_0,\varphi_0)$. Then, from (23) we deduce that

$$\frac{\partial u}{\partial t}(t)=\Delta\varphi(t)=\Delta(t)=0,\forall t\in(0,T),$$

and according to (2), we get

$$\frac{\partial u}{\partial t}(t)=\nabla\frac{\partial u}{\partial t}(t)=0,\forall t\in(0,T).$$

Consequently, (u_0,φ_0) is a stationary solution.

Lemma 2.6. For every $(u_0,\varphi_0)\in\Phi$, it exists $t_0>0$, such that the orbit $\cup_{t\geq t_0}S(t)(u_0,\varphi_0)$ is relatively compact in Φ.

Proof. We are going to prove that there exists a time $t_0>0$, such that the orbit actually lies in $H^2(\Omega)\times H^3(\Omega)$. To this end, we now perform a priori estimates. We first differentiate (1) with respect to time, obtaining

$$\frac{\partial^2 u}{\partial t^2}-\Delta\frac{\partial u}{\partial t}+f'(u)\frac{\partial u}{\partial t}=\frac{\partial\varphi}{\partial t}-\Delta\frac{\partial\varphi}{\partial t}.$$

Owing to the equation (2), we obtain

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}-\Delta\frac{\partial u}{\partial t}+f'(u)\frac{\partial u}{\partial t}=\Delta\varphi.\tag{24}$$

Multiplying (24) by $\dfrac{\partial^2 u}{\partial t^2}$ and integrating over Ω, one has

$$\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\right)+\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2\leq c\left(\left\|\frac{\partial u}{\partial t}\right\|^2+\|\Delta\varphi\|^2\right).$$

Noting that

$$\frac{d}{dt}\left\{t\left(\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\right)\right\}=t\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\right)+\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2,$$

the previous inequality yields

$$t\left(\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\right)\leq ct\int_0^t\left(\left\|\frac{\partial u}{\partial t}\right\|^2+\|\Delta\varphi\|^2\right)ds+\int_0^t\left(\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\right)ds,$$

which gives, according to the estimates above,

$$\left\|\frac{\partial u}{\partial t}\right\|^2+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\leq C_\eta,\forall t\geq\eta>0. \tag{25}$$

Rewriting (1) as

$$\Delta u=\frac{\partial u}{\partial t}+f(u)-\varphi+\Delta\varphi.$$

And from (25) and previous estimates, we infer

$$\|\Delta u(t)\|\leq\left\|\frac{\partial u}{\partial t}(t)\right\|+\|f(u(t))\|+\|\varphi(t)\|+\|\Delta\varphi(t)\|\leq C_0,\forall t\geq t_1>0.$$

We deduce that

$$\|u(t)\|_{H^2(\Omega)}\leq C_1',\forall t\geq t_1$$

where C_1' depends on initial data and t_1. Now, rewriting (2) as

$$\Delta\varphi=\frac{\partial u}{\partial t}+\frac{\partial\varphi}{\partial t}-\Delta\frac{\partial\varphi}{\partial t}.$$

Taking into account (25) and the estimates performed above, we get

$$\|\nabla\Delta\varphi(t)\|\leq\left\|\nabla\frac{\partial u}{\partial t}\right\|+\left\|\nabla\frac{\partial\varphi}{\partial t}\right\|+\left\|\nabla\Delta\frac{\partial\varphi}{\partial t}\right\|\leq C_1''\leq,\forall t\geq t_1>0,$$

and we deduce that $\|\varphi(t)\|_{H^3(\Omega)}$ is bounded for any $t\geq t_1$. Finally, the orbit $\cup_{t\geq t_1}S(t)(u_0,\varphi_0)$ is relatively compact in Φ.

Proof of the Theorem 2.3. In the light of Definition 2.2 and Lemmata 2.4, 2.5, and 2.6, it appears clearly that $(\Phi, S(t), E)$ is a gradient system. As a result, the ω-limit set $\omega(u_0,\varphi_0)$ consists of equilibria.

Remark 2.7. The equilibria points coincide with critical points of the functional E.

Proof of Theorem 2.1. By virtue of the definition of ω-limit set $\omega(u_0,\varphi_0)$, it exists $(\overline{u},\overline{\varphi})\in\omega(u_0,\varphi_0)$ and a sequence $t_n\to+\infty$, such that

$$u(t_n) \to \overline{u} \text{ strongly in } H^1(\Omega)$$

$$\varphi(t_n) \to \overline{\varphi} \text{ strongly in } H^2(\Omega).$$

Now, from (23), we have

$$E(u(t),\varphi(t)) - E(u,\varphi) \geq 0, \forall t \geq 0.$$

First, we assume that it exists $t \in \mathbb{R}_+$, such that

$$E(u(t),\varphi(t)) - E(u,\varphi) = 0, \forall t \geq t_1.$$

Then, (23) implies that

$$u(t) = \overline{u}, \varphi(t) = \overline{\varphi}, \forall t \geq t_1,$$

which gives the expected result.

Now, assuming that $E(u(t),\varphi(t)) > E(u,\varphi), t \geq 0$. We have from (23) that

$$\frac{d}{dt}\Big(E(u(t),\varphi(t)) - E(\overline{u},\overline{\varphi})\Big) + \left\|\frac{\partial u}{\partial t}\right\|^2 + \|\nabla\varphi\|^2 + \|\Delta\varphi\|^2 = 0.$$

Consequently,

$$\begin{aligned}
&-\frac{d}{dt}\left\{\big(E(u(t),\varphi(t)) - E(\overline{u},\overline{\varphi})\big)^\theta\right\} \\
&= \theta\big(E(u(t),\varphi(t)) - E(\overline{u},\overline{\varphi})\big)^{\theta-1}\left(\left\|\frac{\partial u}{\partial t}(t)\right\|^2 + \|\nabla\varphi(t)\|^2 + \|\Delta\varphi(t)\|^2\right) \\
&\geq \frac{\theta}{4}\big(E(u(t),\varphi(t)) - E(\overline{u},\overline{\varphi})\big)^{\theta-1}\left(\left\|\frac{\partial u}{\partial t}(t)\right\| + \|\nabla\varphi(t)\| + \|\Delta\varphi(t)\|\right)^2,
\end{aligned}$$

from Lojasiewicz-Simon's inequality, it exists T_L, such that $\forall t \geq T_L$, we write

$$-\frac{d}{dt}\big(E(u(t),\varphi(t)) - E(\overline{u},\overline{\varphi})\big)^\theta \geq \frac{\theta}{4c}\left(\left\|\frac{\partial u}{\partial t}(t)\right\| + \|\nabla\varphi(t)\| + \|\Delta\varphi(t)\|\right).$$

Integrating this inequality on $(T_L, +\infty)$, one finds that

$$\frac{\partial u}{\partial t} \in L^1(T_L, +\infty, L^2(\Omega)), \nabla\varphi \in L^1(T_L, +\infty, L^2(\Omega)), \Delta\varphi \in L^1(T_L, +\infty, L^2(\Omega)).$$

Since

$$\left\|\frac{\partial\varphi}{\partial t}\right\|_{H^{-1}} \leq \left\|\Delta\frac{\partial\varphi}{\partial t}\right\|_{H^{-1}} + \|\Delta\varphi\|_{H^{-1}} + \left\|\frac{\partial u}{\partial t}\right\|_{H^{-1}} \leq C\left(\left\|\nabla\frac{\partial\varphi}{\partial t}\right\| + \|\nabla\varphi\| + \left\|\frac{\partial u}{\partial t}\right\|\right).$$

We already proved that $\left\|\nabla\frac{\partial\varphi}{\partial t}\right\|$ is bounded, we finally have that $\frac{\partial\varphi}{\partial t} \in L^1(T_L, +\infty, H^{-1}(\Omega))$, and we conclude that the limit

$$\lim_{t\to\infty}(u(t),\varphi(t)) = (\overline{u},\overline{\varphi})$$

exists in $L^2(\Omega)\times H^{-1}(\Omega)$, and that $(\overline{u},\overline{\varphi})$ is a solution to the stationary problem associated to (1)-(4). Thus, on account of the relative compactness of the orbit, this limit also exists in the space Φ.

Finally, we conclude that

$$u(t)\to\overline{u} \text{ strongly in } H^1(\Omega),$$

$$\varphi(t)\to\overline{\varphi} \text{ strongly in } H^2(\Omega),$$

where $(\overline{u},\overline{\varphi})$ is a solution to (22).

3. Conclusion

We have considered in this paper that the system (1)-(4) in a bounded smooth domain Ω of $\mathbb{R}^3$. This system of equations generalizes the one proposed by Caginalp in [6] in order to model the melting-solidification phenomenon in certain classes of materials. Here, φ corresponds to the thermodynamic temperature, and u is the order parameter or phase-field, which describes the proportion of either of the phases. Following the paper [14], in which we proved the existence of global and exponential attractors, we were interested in the question of the convergence of the solution towards a state of equilibrium as time goes to infinity. Indeed, we were able to demonstrate the convergence to a steady state by using Lojasiewicz-Simon's inequality. As a perspective on this work, it would be interesting to study the asymptotic solution in space and, obviously, to make a numerical study of such a system of equations.

Conflict of interest

There is no conflict of interest in this study.

References

[1] Caginalp G. An analysis of a phase field model of a free boundary. *Archive of Rational Mechanics and Analysis*. 1986; 92: 205-245. Available from: https://doi.org/10.1007/BF00254827.

[2] Chen PJ, Gurtin ME. On a theory of heat conduction involving two temperatures. *Zeitschrift für angewandte Mathematik und Physik ZAMP*. 1968; 19: 614-627. Available from: https://doi.org/10.1007/BF01594969.

[3] Chen PJ, Williams WO. A note on non-simple heat conduction. *Zeitschrift für angewandte Mathematik und Physik ZAMP*. 1968; 19: 969-970. Available from: https://doi.org/10.1007/BF01602278.

[4] Chen PJ, Gurtin ME, Williams WO. On the thermodynamics of non-simple materials with two temperatures. *Zeitschrift für angewandte Mathematik und Physik ZAMP*. 1969; 20: 107-112. Available from: https://doi.org/10.1007/BF01591120.

[5] Bangola BLD. Phase-field system with two temperatures and a nonlinear coupling term. *AIMS Mathematics*. 2018; 3(2): 298-315. Available from: https://doi.org/10.3934/Math.2018.2.298.

[6] Bangola B. Global and exponential attractors for a Caginalp type phase-field problem. *Open Mathematics*. 2013; 11(9): 1651-1676. Available from: https://doi.org/10.2478/s11533-013-0258-0.

[7] Aizicovici S, Feireisl E. Long-time stabilization of solutions to a phase-field model with memory. *Journal of Evolution Equations*. 2001; 1: 69-84. Available from: https://doi.org/10.1007/PL00001365.

[8] Milani A, Han Y. Long-time convergence of solutions to a phase-field system. *Mathematical Methods in Applied*

Sciences. 2001; 24(5): 277-287. Available from: https://doi.org/10.1002/mma.215.

[9] Brochet D, Hilhorst D, Chen X. Finite dimensional exponential attractors for the phase field model. *Applicable Analysis.* 1993; 49(3-4): 197-212. Available from: https://doi.org/10.1080/00036819108840173.

[10] Brokate M, Sprekels J. *Hysteresis and phase transitions.* New York: Springer; 1996. Available from: https://doi.org/10.1007/978-1-4612-4048-8.

[11] Cherfils L, Miranville A. Some results on the asymptotic behavior of the Caginalp system with singular potentials. *Advances in Mathematical Sciences and Applications*. 2007; 17(1): 107-129.

[12] Cherfils L, Miranville A. On the Caginalp system with dynamic boundary conditions and singular potentials. *Applications of Mathematics*. 2009; 54(2): 89-115. Available from: https://doi.org/10.1007/s10492-009-0008-6.

[13] Chill R, Fašangová E, Prüss J. Convergence to steady states of solutions of the Cahn-Hilliard and Caginalp equations with dynamic boundary conditions. *Mathematische Nachrichten*. 2006; 279(13-14): 1448-1462. Available from: https://doi.org/10.1002/mana.200410431.

[14] Gatti S, Miranville A. Asymptotic behavior of a phase-field system with dynamic boundary conditions. In: Favini A, Lorenzi A. (eds.) *Differential equations: Inverse and direct problems*. Cortona: Chapman & Hall; 2006. p.149-170.

[15] Giorgi C, Grasselli M, Pata V. Uniform attractors for a phase-field model with memory and quadratic nonlinearity. *Indiana University Mathematics Journal.* 1999; 48(4): 1395-1445.

[16] Grasseli M, Miranville A, Pata V, Zelik S. Well-posedness and long time behavior of a parabolic-hyperbolic phase-field system with singular potentials. *Mathematische Nachrichten*. 2007; 280(13-14): 1475-1509. Available from: https://doi.org/10.1002/mana.200510560.

[17] Grasselli M, Miranville A, Shimperna G. The Caginalp phase-field system with coupled dynamic boundary conditions and singular potentials. *Discrete and Continuous Dynamical Systems.* 2010; 28(1): 67-98. Available from: http://dx.doi.org/10.3934/dcds.2010.28.67.

[18] Grasselli M, Petzeltová H, Schimperna G. Long time behavior of solutions to the Caginalp system with singular potentials. *Zeitschrift für Analysis und ihre Anwendungen*. 2006; 25(1): 51-72. Available from: http://dx.doi.org/10.4171/ZAA/1277.

[19] Laurençot P. Long-time behaviour for a model of phase-field type. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics.* 1996; 126(1): 167-185. Available from: https://doi.org/10.1017/S0308210500030663.

[20] Miranville A, Quintanilla R. A Caginalp phase-field system based on type III heat conduction with two temperatures. *Quarterly of Applied Mathematics*. 2016; 74(2): 375-398.

[21] Miranville A, Zelik S. Robust exponential attractors for singularly perturbed phase-field type equations. *Electronic Journal of Differential Equations*. 2002; 2002(63): 1-28.

[22] Zhang Z. Asymptotic behavior of solutions to the phase-field equations with Neumann boundary conditions. *Communications on Pure and Applied Analysis*. 2005; 4(3): 683-693. Available from: https://doi.org/10.3934/cpaa.2005.4.683.

[23] Ovono AA, Bangola BD, Ipopa MA. On the Caginalp phase-field system based on type III with two temperatures and nonlinear coupling. *Journal of Applied Analysis and Computation.* 2019; 9(1): 345-362. Available from: https://doi.org/10.11948/2019.345.

[24] Green AE, Naghdi PM. A new thermoviscous theory for fluids. *Journal of Non-Newtonian Fluid Mechanics.* 1995; 56(3): 289-306. Available from: https://doi.org/10.1016/0377-0257(94)01288-S.

[25] Green AE, Naghdi PM. A re-examination of the basic postulates of thermomechanics. *Proceeding of the Royal Society London A*. 1991; 432(1885): 171-194. Available from: https://doi.org/10.1098/rspa.1991.0012.

[26] Green AE, Naghdi PM. On undamped heat waves in an elastic solid. *Journal of Thermal Stresses.* 1992; 15(2): 253-264. Available from: https://doi.org/10.1080/01495739208946136.

[27] Jiang J. Convergence to equilibrium for a parabolic-hyperbolic phase-field model with Cattaneo heat flux law. *Journal of Mathematical Analysis and Applications*. 2008; 341(1): 149-169. Available from: https://doi.org/10.1016/j.jmaa.2007.09.041.

[28] Jiang J. Convergence to equilibrium for a fully hyperbolic phase field model with Cattaneo heat flux law. *Mathematical Methods in the Applied Sciences*. 2009; 32(9): 1156-1182. Available from: https://doi.org/10.1002/mma.1092.

[29] Miranville A, Quintanilla R. Some generalizations of the Caginalp phase-field system. *Applicable Analysis.* 2009;

88(6): 877-894. Available from: https://doi.org/10.1080/00036810903042182.
[30] Novick-Cohen A. A phase field system with memory: Global existence. *Journal of Integral Equations and Applications*. 2002; 14(1): 73-107. Available from: https://doi.org/10.1216/jiea/1031315435.
[31] Quintanilla R. Existence in thermoelasticity without energy dissipation. *Journal of Thermal Stresses*. 2002; 25(2): 195-202. Available from: https://doi.org/10.1080/014957302753384423.
[32] Quintanilla R. End effects in thermoelasticity. *Mathematical Methods in Applied Sciences*. 2001; 24(2): 93-102. Available from: https://doi.org/10.1002/1099-1476(20010125)24:2%3C93::AID-MMA199%3E3.0.CO;2-N.
[33] Quintanilla R, Racke R. Stability in thermoelasticity of type III. *Discrete and Continuous Dynamical Systems B*. 2003; 3(3): 383-400. Available from: https://doi.org/10.3934/dcdsb.2003.3.383.
[34] Grasselli M, Petzoltova H, Shimperna G. Convergence to stationary solutions for a parabolic-hyperbolic phase-field system. *Communications on Pure and Applied Analysis*. 2006; 5(4): 827-838. Available from: https://doi.org/10.3934/cpaa.2006.5.827.
[35] Lojasiewicz S. *Ensembles semi-analytiques*. Bures-sur-Yvette, France: Institut des Hautes Etudes Scientifiques; 1965.
[36] Moulay E. *Stabilité des équations différentielles ordinaires*. Master's thesis. Université de Poitiers; 2007.
[38] Jendoubi MA. A simple unified approach to some convergence theorems of L. Simons. *Journal of Functional Analysis*. 1998; 153(1): 187-202. Available from: https://doi.org/10.1006/jfan.1997.3174.
[39] Simon L. Asymptotics for a class of nonlinear evolution equations, with applications to Geometric problems. *Annals of Mathematics Second Series*. 1983; 118(3): 525-571. Available from: https://doi.org/10.2307/2006981.