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Research Article

# Graceful Labeling of Prime Index Graph of Group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$

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**Abstract:** The prime index graph π($G$) of a finite group $G$ is a special type of undirected simple graph whose vertex set is set of subgroups of $G$, in which two distinct vertices are adjacent if one has prime index in the other. Let $p$ and $q$ be distinct primes. In this paper, we establish that prime index graph of a finite cyclic $p$-group $\mathbb{Z}_{p^n}$, a finite abelian group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ and a finite abelian $p$-group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ always have graceful labeling without any condition on $n$ using the concept of path graph or $p$-layer ladder graph of size $n + 1$.

***Keywords***: ladder graph, path graph, prime index graph, graceful graph and graceful labeling

**MSC:** 05C25, 05C78, 05C50

## 1. Introduction

Let $G$ be a finite group. The notion of prime index graph π($G$) of a group $G$ was introduced by Akbari et.al [1]. The prime index graph π($G$) of a group $G$ is defined as a graph with the set of all subgroups of $G$ as its vertex set and there is an edge between two distinct vertices if index of one vertex is prime in other vertex. They proved that for every group $G$, π($G$) is bipartite and girth of π($G$) is contained in the set {4, ∞}. Graph π($G$) is connected in case $G$ is finite solvable group. Further, Ahanjideh and Iranmanesh [2] extended works of Akbari et.al [1] as graph π($G$) connected of finite simple group $G$ if and only if $G$ is isomorphic to $A_5$, $PSL_2(11)$, $PSL_3(3)$ or $PSL_2(2^{2n})$ where $n \leq 4$. In recent years, many researchers have studied various properties of graphs based on algebraic structure mainly finite groups and showed their utility in characterising finite groups. For example, Singh [3] studied Laplacian spectra of power graph and Sehgal et. al [4] gave a general formula for the degree of a vertex in the power graph of a finite abelian group.

A function $f$ is called graceful labeling of a graph π($G$) if $f: V(\pi(G)) \rightarrow \{0, 1, 2, ..., |E(\pi(G))|\}$ is injective and the induced function $f^*: E(\pi(G)) \rightarrow \{1, 2, ..., |E(\pi(G))|\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph is called graceful graph if it admits a graceful labeling. The concept of graceful labeling was introduced by Rosa [5]. Golomb [6] named such labeling as graceful labeling, which was called earlier as $\beta-$ valuation. In [7], Bu and Cao have discussed gracefulness of complete bipartite graph and its union with path. In [8], Acharya and Gill have investigated graceful labeling for the grid graph $P_n \times P_m$. Sehgal et. al [9], established that power graph of group $Z_2^{k-1} \times Z_4$ has graceful


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labeling. For a dynamic survey on graph labeling, we refer to Gallian [10]. See [11-14] for more details on graph based on finite groups.

In this paper, we identify a new class of graceful graphs, i.e., prime index graph of a finite cyclic $p$-group $\mathbb{Z}_{p^n}$, a finite abelian group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ and a finite abelian $p$-group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ always have graceful labeling without any condition on $n$ using the concept of path graph or $p$-layer ladder graph of size $n + 1$.

## 2. Structure of prime index graph of finite cyclic $p$-group $\mathbb{Z}_{p^n}$

We know that group $\mathbb{Z}_{p^n} = \{x^i \mid x^{p^n} = e, i = 0, 1, \ldots, p^n - 1\}$ is a cyclic group of order $p^n$. From [15], list of subgroups of group $\mathbb{Z}_{p^n}$ is given below.

$H_i = < x^{p^{n-i}} > \cong \mathbb{Z}_{p^i}$, if $i = 0, 1, 2, \ldots, n$.

Hence, we get the list of $n + 1$ subgroups.

On the basis of above list, we state following theorem which is very useful for structure of prime index graph of group $\mathbb{Z}_{p^n}$.

**Theorem 1**. Let $\mathbb{Z}_{p^n}$ be a group whose subgroups list as $H_i$ where $i = 0, 1, 2, \ldots, n$. Then, index of $H_i$ where $i = 0, 1, 2, 3, \ldots, n - 1$ in $H_j$ where $j = 0, 1, 2, \ldots, n$ is prime if and only if $j = i + 1$.

*Proof*. Here, $|H_i| = p^i$ where $i = 0, 1, 2, \ldots, n - 1$, then $H_i$ has prime index in $H_j$ if and only $|H_j| = p^{i+1}$ and $H_i \subset H_j$. So, $|H_j| = p^{i+1}$ and $H_i \subset H_j$ is only possible if and only if $j = i + 1$.

On the basis of above theorem, prime index graph of group $\mathbb{Z}_{p^n}$ is given below.

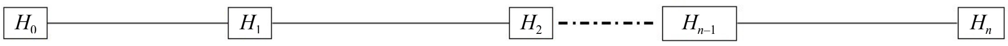


**Figure 1.** Prime index graph of group $\mathbb{Z}_{p^n}$

This graph is known as path graph of size $n + 1$ which is denoted by $P_{n+1}$ with two vertices $H_0$, $H_n$ having degree one and remaining vertices are of degree two. So, total number of edges is $n$.

## 3. Structure of prime index graph of finite cyclic group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$

We know that group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q = \{x^i y^i \mid x^{p^n} = y^q = e, xy = yx, i = 0, 1, \ldots, p^n - 1, j = 0, 1, 2, \ldots, q - 1\}$ is a cyclic group of order $p^n q$. From [15], list of subgroups of group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ is given below.

(i) $H_i = < x^{p^{n-\frac{i-1}{2}}} > \cong \mathbb{Z}_{p^{\frac{i-1}{2}}}$, if $i = 1, 3, 5, \ldots, 2n + 1$.

(ii) $H_i = < x^{p^{n-\frac{i-2}{2}}} y > \cong \mathbb{Z}_{p^{\frac{i-2}{2}}} \mathbb{Z}_q$, if $i = 2, 4, 6, \ldots, 2n + 2$.

Hence, we get the list of $(2n + 2)$ subgroups.

On the basis of above list, we state following theorem which is very useful for structure of prime index graph of group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$.

**Theorem 2.** Let $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ be a group whose subgroups list as $H_i$ where $i = 1, 2, \ldots, 2n + 2$, then we have

(1) Subgroup $H_i$ where $i = 1, 3, 5, \ldots, 2n - 1$ in $H_j$ where $j = 1, 2, \ldots, 2n + 2$ has index prime if and only if $j = i + 1$, $i + 2$.

(2) Subgroup $H_{2n+1}$ in $H_j$ where $j = 1, 2, \ldots, 2n + 2$ has index prime if and only if $j = 2n + 2$.

(3) Subgroup $H_i$ where $i = 2, 4, 6, \ldots, 2n$ in $H_j$ where $j = 1, 2, 3, \ldots, 2n + 2$ has index prime if and only if $j = i + 2$.

*Proof.*

(1) Here, $|H_i| = p^{\frac{i-1}{2}}$ where $i = 1, 3, 5, \ldots, 2n - 1$, then $H_i$ has prime index in $H_j$ if and only if $|H_j| = p^{\frac{i-1}{2}} q$ or $p^{\frac{i+1}{2}}$ and $H_i \subset H_j$. So, $|H_j| = p^{\frac{i-1}{2}} q$ or $p^{\frac{i+1}{2}}$ and $H_i \subset H_j$ is only possible if and only if $j = i + 1$, $i + 2$.

(2) Here, $|H_{2n+1}| = p^n$, then $H_{2n+1}$ has prime index in $H_j$ if and only $|H_j| = p^n q|$ and $H_{2n+1} \subset H_j$. So, $|H_j| = p^n q$ and $H_{2n+1} \subset H_j$. is only possible if and only if $j = 2n + 2$.

(3) Here, $|H_i| = p^{\frac{i-1}{2}}$ where $i = 2, 4, 6, ..., 2n$, then $H_i$ has prime index in $H_j$ if and only $|H_j| = p^{\frac{i}{2}} q$ and $H_i \subset H_j$. So, $|H_j| = p^{\frac{i}{2}} q$ and $H_i \subset H_j$ is only possible if and only if $j = i + 2$.

On the basis of above *theorem*, prime index graph of group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ is given below.

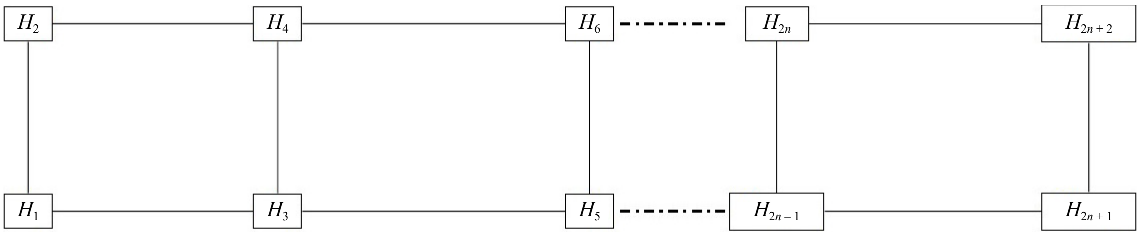


**Figure 2.** Prime index graph of group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$

This graph is known as ladder graph of size $n + 1$ which is denoted by $L_{n+1,1}$ with four vertices $H_1$, $H_2$, $H_{2n+1}$, $H_{2n+2}$ having degree two and remaining vertices have degree three. So, total number of edges is $(3n + 1)$.

## 4. Structure of prime index graph of finite abelian *p*-group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$

We know that group $\mathbb{Z}_p \times \mathbb{Z}_{p^n} = \{x^i y^j \mid x^p = y^{p^n} = e, xy = yx, i = 0,1,\ldots, p-1, j = 0,1,2,\ldots, p^n - 1\}$ is an abelian group of order $p^{n+1}$. From [11, 16, 17], list of subgroups of group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ is given below.

(i) $H_1 = \langle e \rangle \cong \mathbb{Z}_1$.

(ii) $H_{j+1} = \langle xy^{jp^{n-1}} \rangle \cong \mathbb{Z}_p$ where $j = 1, 2, ..., p$.

(iii) $H_{p+2} = \langle y^{p^{n-1}} \rangle \cong \mathbb{Z}_p$.

(iv) $H_{kp+2+k} = \langle x, y^{p^{n-k}} \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_{p^k}$ where $k = 1, 2, ..., n$.

(v) $H_{j+(k-1)p+1+k} = \langle x^j y^{p^{n-k}} \rangle \cong \mathbb{Z}_{p^k}$ with $j = 1, 2, ..., p$ and $k = 2, 3, ..., n$.

Hence, we get the list of $(np + n + 2)$ subgroups.

On the basis of above list, we state following theorem which is very useful for structure of prime index graph of group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$.

**Theorem 3.** Let $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ be group whose subgroup list $H_i$ where $i = 1, 2, ..., np + n + 2$, then we have

(1) Index of $H_1$ in $H_j$ where j = 1, 2, 3, ..., $np + n + 2$ is prime if and only if $j = 2, 3, ..., p + 2$.

(2) Index of $H_i$ where $i = 2, 3, ..., p + 1$ in $H_j$ is prime where $j = 1, 2, 3, ..., np + n + 2$ if and only if $j = p + 3$.

(3) Index of $H_{kp+1+k}$ where $k = 1, 2, ..., n$ in $H_j$ where $j = 1, 2, 3, ..., np + n + 2$ is prime if and only if $j = kp + 2 + k$, $kp + 3 + k, ..., (k + 1)p + k + 2$.

(4) Index of $H_{ip+2+i}$ where $i = 1, 2, ..., n - 1$ in $H_j$ where $j = 1, 2, 3, ..., np + n + 2$ is prime if and only if $j = (i + 1)p + i + 3$.

(5) Index of $H_{i+(k-1)p+k+1}$ where $i = 2, 3, ..., p - 1$ and $k = 2, 3, ..., n$ in $H_j$ where $j = 1, 2, 3, ..., np + n + 2$ is prime if and only if $j = kp + k + 2$.

*Proof.*

(1) Here, $|H_1| = 1$, then $H_1$ has index $p$ in $H_j$ if and only $|H_j| = p$ and $H_1 \subset H_j$. So, $|H_j| = p$ and $H_1 \subset H_j$ is only possible if and only if $j = 2, 3, ..., p + 2$.

(2) As $|H_i| = p$ where $i = 2, 3, ..., p + 1$, then $H_i$ has index $p$ in $H_j$ if and only $|H_j| = p^2$ and $H_i \subset H_j$. So, $|H_j| = p^2$ and $H_i \subset H_j$ is only possible if and only if $j = p + 3$.

(3) Here, $|H_{kp+1+k}| = p^{k+1}$, then $H_i$ has index $p$ in $H_j$ if and only $|H_j| = p^{i+2}$ and $H_{kp+1+k} \subset H_j$. So, $|H_j| = p^{i+2}$ and $H_{kp+1+k} \subset H_j$ is only possible if and only if $j = kp + 2 + k$, $kp + 3 + k,..., (k + 1)p + k + 2$.

(4) Here, $|H_{ip+2+i}| = p^{i+1}$ where $i$ = 1, 2, ..., $n$ −1, then $H_{ip+2+i}$ has index $p$ in $H_j$ if and only $|H_j| = p^{i+2}$ and $H_{ip+2+i} \subset H_j$ . So, $|H_j| = p^{i+2}$ and $H_{ip+2+i} \subset H_j$ is only possible if and only if $j = (i + 1)p + i + 3$.

(5) Here, $|H_{i+(k-1)p+k+1}| = p^k$ where $i$ = 2, 3, ..., $p$ − 1 and $k$ = 2, 3, ..., $n$, then $H_{i+(k-1)p+k+1}$ has index $p$ in $H_j$ if and only $|H_j| = p^{k+1}$ and $H_{i+(k-1)p+k+1} \subset H_j$. So, $|H_j| = p^{k+1}$ and $H_{i+(k-1)p+k+1} \subset H_j$ is only possible if and only if $j = kp + k + 2$.

On the basis of above theorem, prime index graph of group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ is given below.

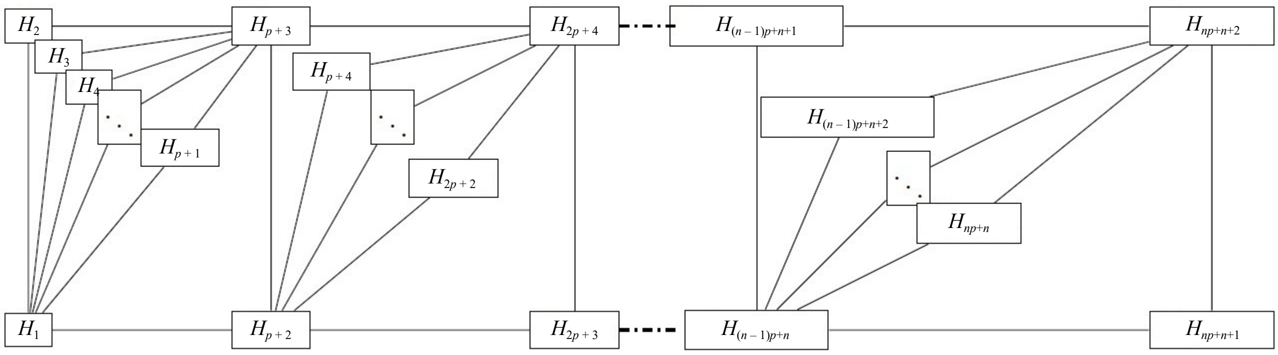


**Figure 3.** Prime index graph of group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$

This graph is known as $p$-layer ladder graph of size $n$ + 1 which is denoted by $L_{n+1,p}$ with two vertices $H_1$, $H_{np+n+2}$ having degree $p$ + 1; $2n$ − 2 vertices $H_{p+2}$, $H_{p+3}$, $H_{2p+3}$, $H_{2p+4}$, ..., $H_{(n-1)p+n}$, $H_{(n-1)p+n+1}$ have degree $p$ + 2 and remaining vertices are of degree two. So, total number of edges is $2pn + n + 1$.

## 5. Graceful labeling

**Theorem 4.** The path graph $P_n$ is graceful for all $n \geq 1$.

Proof. Clearly, the number of edges in this graph is $n$ − 1. Now, we define mapping $f: V(P_n) \rightarrow \{0, 1, 2, ..., n-1\}$ where $V(P_n) = \{v_0, v_1, ..., v_n\}$ such as

$$f(v) = \begin{cases} \frac{i}{2} & \text{if } v = v_i \text{ where } i \text{ is even} \\ n - \frac{i-1}{2} & \text{if } v = v_i \text{ where } i \text{ is odd} \end{cases}$$

Therefore, we get vertex labeled in strictly increasing manners as $f(v_0) < f(v_2) < f(v_4) < \ldots < f(v_{(2\times[\frac{n}{2}])}) < f(v_{(2\times[\frac{n-1}{2}]+1)}) < f(v_{(2\times[\frac{n-1}{2}]-1)}) < \ldots < f(v_3) < f(v_1)$.

Now, we calculate $f^*$ for each edge of the graph $P_n$. $f^*(e) = n - i$ where e = $v_i v_{i+1}$ and $i$ = 0, 1, 2, ..., $n$ − 1. Clearly, the induced edge labels are all distinct positive integers from {1, 2, ..., $n$}. Hence, the theorem.

**Corollary 1.** Prime index graph of finite cyclic $p$-group $\mathbb{Z}_{p^n}$ always has graceful labeling.

**Theorem 5.** The $p$-layer ladder graph $L_{n+1,p}$ is graceful for all $n \geq 1$.

*Proof.* Clearly, the number of edges in this graph is $n$ − 1. Now, we define mapping $f: V(L_{n+1,p}) \rightarrow \{0, 1, 2, ..., 2pn + n + 1\}$ where $V(L_{n+1,p}) = \{v_{11}, v_{12}, ..., v_{1(n+1)}, v_{21}, v_{22}, ..., v_{2(n+1)}, v_{31}, v_{32}, ..., v_{3(p-1)}, ..., v_{(n+2)1}, v_{(n+2)2}, ..., v_{(n+2)(p-1)}\}$ such as

Case 1: $n$ is odd

$$
f(v_{ij}) = \begin{cases} 2pn+n+1 & \text{if } v = v_{11} \\ 2pn+n-1-3\frac{(j-3)}{2} & \text{if } v = v_{1j} \text{ where } j = 3, 5, \ldots, n \\ 3\frac{(j-2)}{2} & \text{if } v = v_{1j} \text{ where } j = 2, 4, \ldots, n+1 \\ 1+3\frac{(j-1)}{2} & \text{if } v = v_{2j} \text{ where } j = 1, 3, \ldots, n \\ 2pn+n-2-3\frac{(j-2)}{2} & \text{if } v = v_{2j} \text{ where } j = 2, 4, \ldots, n-1 \\ 2pn+n-4-3\frac{(n-3)}{2} & \text{if } v = v_{2j} \text{ where } j = n+1 \\ 2nj-\frac{(i-3)}{2} & \text{if } v = v_{ij} \text{ where } j = 1, \ldots, p-1 \text{ and } i = 3, 5, \ldots, n+2 \\ 2pn+n-2nj+\frac{(i-3)}{2} & \text{if } v = v_{ij} \text{ where } j = 1, \ldots, p-1 \text{ and } i = 4, 6, \ldots, n+1 \end{cases}
$$

On the basis of this vertex labeling, we have following observations.

(1) Vertex label of vertices $v_{12}$, $v_{21}$, $v_{14}$, $v_{23}$, ..., $v_{1(n+1)}$, $v_{2n}$ are strictly increasing manners from vertex label 0 to $\frac{3n-1}{2}$.

(2) Vertex label of vertices $v_{(n+2)(1)}$, $v_{(n)(1)}$, ..., $v_{51}$, $v_{31}$, $v_{4(p-1)}$, $v_{6(p-1)}$, ..., $v_{(n+1)(p-1)}$, ..., $v_{(n+2)(p-1)}$, $v_{(n)(p-1)}$, ..., $v_{3(p-1)}$, $v_{41}$, $v_{61}$, ..., $v_{(n+1)(1)}$ are strictly increasing manners from vertex label $\frac{3n+1}{2}$ to $2pn-\frac{n+1}{2}$.

(3) Vertex label of vertices $v_{2(n+1)}$, $v_{2(n-1)}$, $v_{1n}$, ..., $v_{24}$, $v_{15}$, $v_{22}$, $v_{13}$, $v_{11}$ are in strictly increasing manner from $2pn-\frac{n-1}{2}$ to $2pn+n+1$.

Therefore, we get edge labels as follow.

(1) $v_{(n+3-j)(i)}$ with vertex $v_{(2)(n+1-j)}$ and $v_{(1)(n+2-j)}$ where $i = 1, 2, ..., p-1, j = 1, 2, ..., n$ provides edge labeling 1 to $2np-2n$.

(2) Vertex $v_{2n}$ with vertex $v_{(2)(n+1)}$ provides edge labeling as $2np-2n+1$ and vertex $v_{(1)(n+1)}$ to $v_{(2)(n+1)}$ provides edge labeling as $2np-2n+2$.

(3) Vertex $v_{(1)(n+2-j)}$ with vertex $v_{1(n+1-j)}$ and vertex $v_{1(n+1-j)}$ with vertex $v_{2(n+1-j)}$ and vertex $v_{2(n+1-j)}$ with vertex $v_{2(n+-j)}$ where $j = 1, 2, ..., n-1$ provides edges from $2np-2n+3$ to $2np+n-1$.

(4) $v_{12}$ with $v_{11}$ and $v_{11}$ with $v_{21}$ provides edge labeling as $2np+n$ to $2np+n+1$.

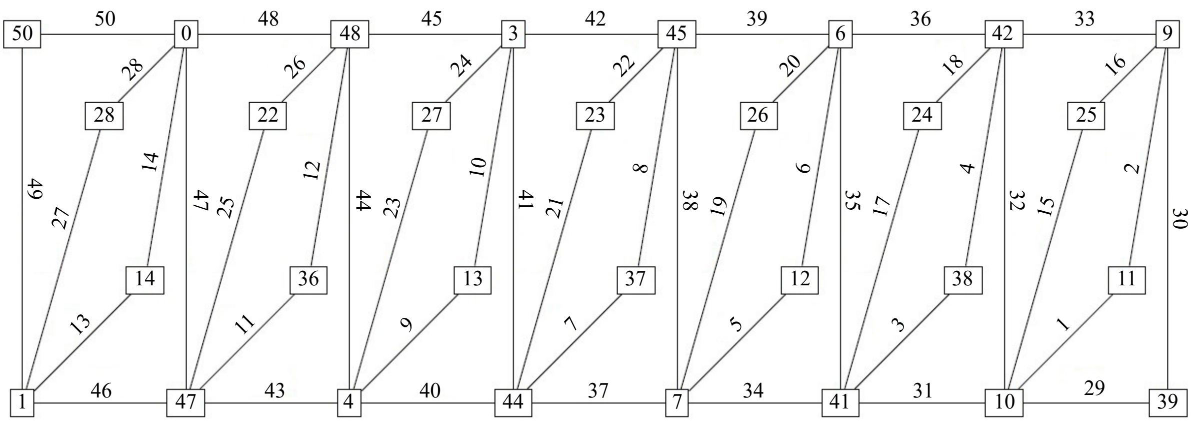


**Figure 4.** Graceful labeling of graph $L_{8,3}$

Case 2: $n$ is even

$$f(v_{ij})=\begin{cases} 2pn+n+1 & \text{if } v=v_{11} \\ 2pn+n-1-3\frac{(j-3)}{2} & \text{if } v=v_{1j} \text{ where } j=3,5,\ldots,n+1 \\ 3\frac{(j-2)}{2} & \text{if } v=v_{1j} \text{ where } j=2,4,\ldots,n \\ 1+3\frac{(j-1)}{2} & \text{if } v=v_{2j} \text{ where } j=1,3,\ldots,n-1 \\ 2pn+n-2-3\frac{(j-2)}{2} & \text{if } v=v_{2j} \text{ where } j=2,4,\ldots,n \\ \frac{3n}{2} & \text{if } v=v_{2j} \text{ where } j=n+1 \\ 2nj-\frac{(i-3)}{2} & \text{if } v=v_{ij} \text{ where } j=1,\ldots,p-1 \text{ and } i=3,5,\ldots,n+1 \\ 2pn+n+1-2nj+\frac{(i-4)}{2} & \text{if } v=v_{ij} \text{ where } j=1,\ldots,p-1 \text{ and } i=4,6,\ldots,n+2 \end{cases}$$

On the basis of this vertex labeling, we have following observations.

(1) Vertex label of vertices $v_{12}$, $v_{21}$, $v_{14}$, $v_{23}$, ..., $v_{1n}$, $v_{2(n-1)}$, $v_{2(n+1)}$ are strictly increasing manners from vertex label 0 to $\frac{3n}{2}$.

(2) Vertex label of vertices $v_{(n+1)(1)}$, $v_{(n-1)(1)}$, ..., $v_{51}$, $v_{31}$, $v_{4(p-1)}$, $v_{6(p-1)}$, ..., $v_{(n+2)(p-1)}$, ..., $v_{(n+1)(p-1)}$, $v_{(n-1)(p-1)}$, ..., $v_{3(p-1)}$, $v_{41}$, $v_{61}$, ..., $v_{(n+2)(1)}$ are strictly increasing manners from vertex label $\frac{3n}{2}+1$ to $2pn-\frac{n}{2}$.

(3) Vertex label of vertices $v_{2(n)}$, $v_{1(n+1)}$, $v_{2(n-2)}$, ..., $v_{22}$, $v_{13}$, $v_{11}$ are in strictly increasing manner from $2pn-\frac{n-2}{2}$ to $2pn+n+1$.

Therefore, we get edge labels as follow.

(1) $v_{(n+3-j)(i)}$ with vertex $v_{(2)(n+1-j)}$ and $v_{(1)(n+2-j)}$ where $i=1, 2, ..., p-1$, $j=1, 2, ..., n$ provides edge labeling 1 to $2np-2n$.

(2) Vertex $v_{2n}$ with vertex $v_{(2)(n+1)}$ provides edge labeling as $2np-2n+1$ and vertex $v_{(1)(n+1)}$ to $v_{(2)(n+1)}$ provides edge labeling as $2np-2n+2$.

(3) Vertex $v_{(1)(n+2-j)}$ with vertex $v_{1(n+1-j)}$ and vertex $v_{1(n+1-j)}$ with vertex $v_{2(n+1-j)}$ and vertex $v_{2(n+1-j)}$ with vertex $v_{2(n+-j)}$ where $j=1, 2, ..., n-1$ provides edges from $2np-2n+3$ to $2np+n-1$.

(4) $v_{12}$ with $v_{11}$ and $v_{11}$ with $v_{21}$ provides edge labeling as $2np+n$ to $2np+n+1$.

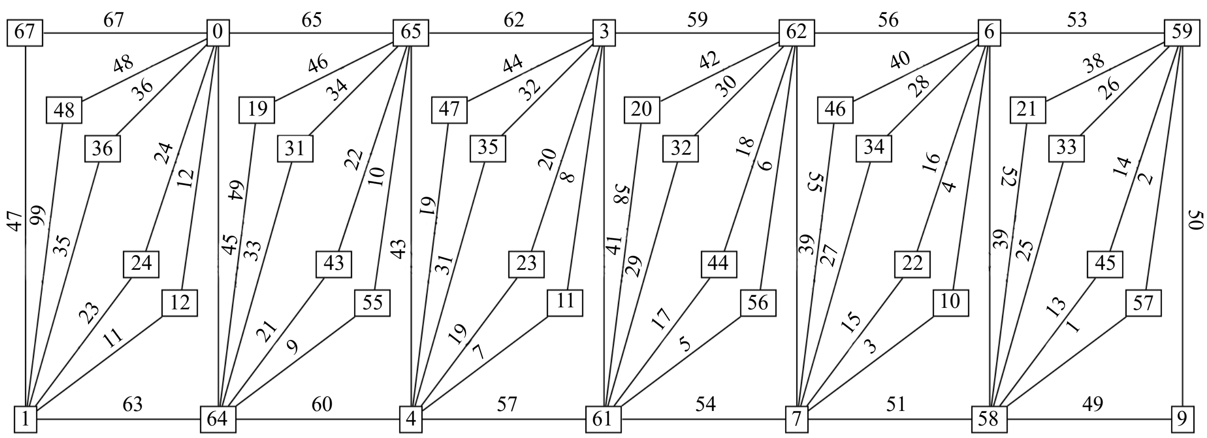


**Figure 5.** Graceful labeling of graph $L_{7,3}$

**Corollary 2.** Prime index graph of finite abelian $p$-group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ always has graceful labeling.

**Corollary 3.** Prime index graph of finite cyclic group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ always has graceful labeling.

*Proof.* If take $p = 1$, then graph $L_{n+1,1}$ is prime index graph of finite cyclic group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ which is always graceful.

## 6. Conclusion

In this article, we have discovered gracefulness of prime index graph of finite cyclic *p*-group $\mathbb{Z}_{p^n}$, a finite abelian group $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ and a finite abelian *p*-group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$.


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## Conflict of interest

The authors declare that they have no competing of interests regarding the publication of this paper.

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