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Research Article

# Efficient Finite Difference Approaches for Solving Initial Boundary Value Problems in Helmholtz Partial Differential Equations

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**Abstract:** This study presents numerical solutions for initial boundary value problems of homogeneous and non-homogeneous Helmholtz equations using first- and second-order difference schemes. The stability of these methods is rigorously analyzed, ensuring their reliability and convergence for a wide range of problem instances. The proposed schemes' robustness and applicability are demonstrated through several examples, accompanied by an error analysis table and illustrative graphs that visually represent the accuracy of the solutions obtained. The results confirm the effectiveness and efficiency of the proposed schemes, making them valuable tools for solving Helmholtz equations in practical applications.

***Keywords*:** Helmholtz equation, initial boundary value problems (IBVP), finite difference scheme, computational techniques, stability analysis

**MSC:** 65M06, 65M12, 35G16, 35J05

## 1. Introduction

Partial differential equations (PDEs) are widely used in various fields of science and engineering to model complex physical phenomena. Solving PDEs analytically is often impossible or impractical; therefore, numerical methods are commonly used [1, 2]. The Helmholtz equation is a second-order partial differential equation that arises in many physical systems involving wave phenomena, such as acoustics, electromagnetics, and quantum mechanics [3-9]. It describes the behavior of waves propagating in a medium with a given propagation constant, or wave number [6]. The Helmholtz equation has been extensively studied and has numerous applications in seismology, medical imaging, and telecommunications. Some notable references for the Helmholtz equation include the works of Hermann von Helmholtz, who originally derived the equation in the 19th century, and the textbooks of Mathews and Walker [10] and Morse and Feshbach [11], which provide comprehensive treatments of the equation and its applications.


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The numerical solution of partial differential equations is a fundamental problem in many areas of science and engineering [12]. Recently, many works have been done to find numerical solutions for different kinds of partial differential equations and fractional differential equation. Novel numerical methods proposed for solving distributed order fractional differential equations in the time domain using different types of wavelet and hybrid functions approaches [13-15]. Ahmad et al. [16] investigate the resonance, fusion, and fission dynamics of bifurcation solitons and hybrid rogue wave structures of the Sawada-Kotera equation. Saifullah et al. [17] analyze the interaction of lump solutions with kink-soliton solutions of the generalized perturbed Korteweg–de Vries (KdV) equation using the Hirota-bilinear approach. Khaliq et al. [18] propose a novel expansion method to obtain new wave solutions of the (2+1)-dimensional generalized Hirota-Satsuma-Ito equation. These studies contribute to understanding PDEs and their solutions and can be useful in various fields such as physics, engineering, and applied mathematics.

Numerical methods for solving the Helmholtz equation are widely used due to their efficiency and accuracy [19]. Several numerical methods have been proposed for solving the Helmholtz equation in recent years. Canino et al. [7] proposed a high-order Nyström discretization method for numerically solving the Helmholtz equation in two and three dimensions. The method was efficient and accurate for high-frequency waves. Kress and Sloan [8] presented a numerical method for solving a logarithmic integral equation of the first kind, which arises in the context of the Helmholtz equation. The method was efficient and accurate, especially for small wave numbers. Wang et al. [4] developed an efficient and accurate numerical method for solving the Helmholtz equation in polar and spherical coordinates. The method was based on a high-order finite difference scheme. It was more efficient than other existing methods.

Bayliss et al. [9] proposed a numerical method for solving the Helmholtz equation for wave propagation problems in underwater acoustics. The method was based on a finite difference scheme and was accurate and efficient. Hamzah et al. [3] developed a numerical method for solving anisotropic boundary value problems governed by the Helmholtz equation. The method was based on a finite element method and was efficient and accurate. Erlangga [5] proposed a robust and efficient iterative method for solving the Helmholtz equation. The method was based on a preconditioned conjugate gradient method. It was found to be more efficient than other existing methods.

Bao et al. [6] present a numerical method for solving the Helmholtz equation with high wavenumbers, which is used to model wave propagation in various fields of science and engineering. The method proposed in the paper is able to accurately and efficiently solve the Helmholtz equation, even for high-frequency waves. Kabanikhin et al. [20] studied initial boundary value problems (IBVP) for Helmholtz equations and they have made a comparative analysis of methods for regularizing Helmholtz equations under initial boundary conditions.

Finite difference schemes are one such method that has been extensively used for the numerical solution of the Helmholtz equation [19, 21]. Finite difference schemes have been extensively used for the numerical solution of the elliptic equation due to their simplicity, low computational cost, and ease of implementation. Several authors have proposed different finite difference schemes with varying degrees of accuracy and computational complexity [22-25]. Conversely, there has been considerable research on the numerical solution of inhomogeneous Helmholtz equations. Karachik and Antropova [26] presented a solution to the inhomogeneous polyharmonic equation and the inhomogeneous Helmholtz equation. At the same time, Yang et al. [27] proposed a truncation method for the Cauchy problem of the inhomogeneous Helmholtz equation. Zhang et al. [28] developed a sixth-order finite difference scheme for the Helmholtz equation with inhomogeneous Robin boundary conditions. Muleshkov et al. [29] found particular solutions for axisymmetric Helmholtz-type operators. These studies demonstrate the importance of finding efficient and accurate numerical solutions for inhomogeneous Helmholtz equations and provide a foundation for future research in this area.

In this paper, we present an efficient finite difference approach for solving initial boundary value problems in Helmholtz equations. Our proposed approach reduces the computational cost and memory requirements while maintaining high accuracy. We demonstrate the effectiveness of our method through several numerical examples.

## 2. The problem

The Helmholtz equation, attributed to Hermann von Helmholtz, has applications in Physics and Mathematics. This PDE is expressed mathematically as:

$$\nabla^2 u + A^2 u = 0$$

where, $\nabla^2$ is Laplacian operator, wave number, is amplitude. When $A^2 = 0$, the Helmholtz equation reduces to Laplace's equation. When $A^2 < 0$, the Helmholtz equation becomes the space part of the diffusion equation. In this study, we will consider the following IBVP Helmholtz equation:

$$\begin{cases} u_{tt}(x,t) + u_{xx}(x,t) + A^2 u(x,t) = f(x,t) \qquad 0 \le x \le L, \\ u(0,t) = u(\pi,t) = 0, \qquad 0 \le t \le T, \\ u(x,0) = \psi(x), u_t(x,0) = \varphi(x). \end{cases} \tag{1}$$

Where $\psi(x)$, $\varphi(x)$ ($x \in [0,\pi]$), ($(x,t) \in [0,\pi] \times [0,1]$) are smooth functions; problem (1) presents an IBVP Helmholtz equation [30].

## 3. First- and second-order difference schemes for Helmholtz equation

In this section, we introduce the explicit and implicit methods for the approximate solution of the IBVP Helmholtz equation in (1). By using central differences with respect to x and t, we get the following difference equation:

$$\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + A^2 u_n^k = 0. \tag{2}$$

By rewriting in terms of $u_n$ in (2), we get

$$\left(\frac{1}{h^2}\right) u_{n+1}^k + \left(\frac{1}{\tau^2}\right) u_n^{k+1} + \left(\frac{-2}{\tau^2} + \frac{-2}{h^2} + A^2\right) u_n^k + \left(\frac{1}{\tau^2}\right) u_n^{k-1} + \left(\frac{1}{h^2}\right) u_{n-1}^k = 0. \tag{3}$$

By supposing that $a = \frac{1}{h^2}, b = \frac{1}{\tau^2}$, and $c = -\frac{2}{\tau^2} - \frac{2}{h^2} + A^2$. Putting in (3) yields

$$a u_{n+1}^k + b u_n^{k+1} + c u_n^k + b u_n^{k-1} + a u_{n-1}^k = 0. \tag{4}$$

Equation (4) can be transformed into matrices according to [31].

Similarly, we will introduce an implicit difference scheme for the problem (1) and the first-order difference scheme in (2), By substituting all terms on the right side of equation (2) with an average derived from the values at time steps $k - 1$ and $k + 1$, we obtain:

$$\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + \frac{1}{2h^2}\left(u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1} + u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}\right) + \frac{A^2}{2}\left(u_n^{k-1} + u_n^{k+1}\right) = 0. \tag{5}$$

The difference equation in (5) is called the second-order difference scheme for the problem (1). The stability of the difference scheme (5) is guaranteed in [23, 31-34]. By rewriting (5) in terms of $u_n$, we obtain:

$$\left(\frac{1}{2h^2}\right)u_{n+1}^{k+1}+\left(\frac{1}{2h^2}\right)u_{n+1}^{k-1}+\left(\frac{1}{\tau^2}-\frac{1}{h^2}+\frac{A^2}{2}\right)u_n^{k-1}+\left(\frac{-2}{\tau^2}\right)u_n^{k}+\left(\frac{1}{\tau^2}-\frac{1}{h^2}+\frac{A^2}{2}\right)u_n^{k-1}+$$
$$\left(\frac{1}{2h^2}\right)u_{n-1}^{k+1}+\left(\frac{1}{2h^2}\right)u_{n-1}^{k-1}=0. \tag{6}$$

By putting $a=\frac{1}{2h^2}$, $b=\frac{1}{\tau^2}-\frac{1}{h^2}+\frac{A^2}{2}$, $c=\frac{-2}{\tau^2}$, and in (6), we obtain

$$au_{n+1}^{k+1}+au_{n+1}^{k-1}+bu_n^{k+1}+cu_n^{k}+bu_n^{k-1}+au_{n-1}^{k+1}+au_{n-1}^{k-1}=0. \tag{7}$$

The difference equations found in both (3) and (7) can be reformulated into the corresponding matrix expressions, as demonstrated in references [23, 31]:

$$AU_{n+1}+BU_n+CU_{n-1}=D\varphi_n, 1\le n\le M-1, u_0=u_M=0, \tag{8}$$

where, $A$, $B$, and $C$ are $(N+1)\times(N+1)$ matrix, $U_{n+1}$, $U_n$, $U_{n-1}$ and $\varphi_n$ is $(N+1)\times 1$ vectors [30, 31]. We have applied a modified Gauss elimination method for solving (8). Next, we seek solutions in the form of a matrix equation, as described in reference [35]:

$$u_j=\alpha_{j+1}u_{j+1}+\beta_{j+1}; u_M=0; j=M-1,\ldots,2,1.$$

Where $\alpha_j$ are $(N+1)\times(N+1)$ square matrices and are column vectors presented by

$$\alpha_{j+1}=-(B+C\alpha_j)^{-1}A,$$

and

$$\beta_{j+1}=(B+C\alpha_j)^{-1}(D\varphi-C\beta_j), j=1,2,\ldots,M-1,$$

where $j=1, 2,\ldots, M-1$, $\beta_1$ is the $(N+1)\times 1$ zero column vector, and $\alpha_1$ is the $(N+1)\times(N+1)$ zero matrix. Matlab software was implemented for computing results for different values of $N$ and $M$. Comparison was made between numerical and exact solutions. The maximum error is indicated where $h=\pi/M$, and $\tau=1/N$. Maximum absolute error calculated by

$$E_M^N=\max_{\substack{1\le k\le N-1\\1\le n\le M-1}}\left|u\left(t_k,x_n\right)-u_n^k\right|, \tag{9}$$

Here, the approximation solution is represented by $u_n^k$ while the analytical solution represented by $u(t_k, x_n)$ at points $(t_k, x_n)$.

**Remark 1:** For proving the stability of (3), we examine the amplification factor (it must have a magnitude less than 1); we'll assume that the solution has the form:

$$u_n^k=G^nH^k$$

where $G$ is the amplification factor in the spatial variable ($n$), and $H$ is the amplification factor in the time variable ($k$), putting it in (3), and dividing both sides by $G^n H^k$, we obtain:

$$\left(\frac{1}{h^2}\right)G+\left(\frac{1}{\tau^2}\right)H+\left(\frac{-2}{\tau^2}+\frac{-2}{h^2}+A^2\right)+\frac{\left(\frac{1}{\tau^2}\right)}{H}+\frac{\left(\frac{1}{h^2}\right)}{G}=0.$$

This equation depends on $G$ and $H$. The difference equation (3) is stable when $|G| \leq 1$ and $|H| \leq 1$.

**Remark 2:** To prove the stability of (5), we'll use the Von Neumann stability analysis. Substitute the Fourier mode $u_n^k = e^{iwn} g^k$ in (5), where $w$ is the angular frequency and $g$ is the growth factor, divide both sides of the equation by $e^{iwn}$:

$$\frac{g^{k+1}-2g^k+g^{k-1}}{\tau^2}+\frac{1}{2h^2}(e^{ihw}g^{k+1}-2g^{k+1}+e^{-ihw}g^{k+1}+e^{ihw}g^{k-1}-2g^{k-1}$$
$$+e^{-ihw}g^{k-1})+\frac{A^2}{2}(g^{k-1}+g^{k+1})=0,$$

the scheme is stable if the absolute value of the growth factor is less than or equal to 1 for all values of $w$: $|g| \leq 1$.

## 4. Numerical experiments

In this section, we apply the first-order difference scheme from (2) and the second-order scheme from (5) to several distinct numerical examples of Helmholtz PDEs with initial-boundary conditions. All examples have known analytical solutions, which will be compared to the numerical results to test the method's efficacy. We will consider the general form of the IBVP Helmholtz partial differential equation in our applications.

$$\begin{cases} u_{tt}(t,x)+u_{xx}(t,x)+A^2u(t,x)=f(x,t) & 0\leq x\leq L, \\ u(t,0)=u(t,\pi)=0, & 0\leq t\leq T, \\ u(0,x)=\psi(x), \quad u_t(0,x)=\varphi(x). \end{cases} \tag{10}$$

**Example 1.** Investigate the following IBVP for Helmholtz PDE, where $A^2 = 2$

$$\begin{cases} \nabla^2 u+2u=0 & 0\leq x\leq \pi, 0\leq t \\ u(0,t)=u(\pi,t)=0, \\ u(x,0)=\sin(x), u_t(x,0)=0. \end{cases} \tag{11}$$

The exact solution of (11) can be found as $u(x,t) = \sin(x)\cos(t)$. First-order difference scheme for problem (11) is given by:

$$\begin{cases} \dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+2u_n^k=0 \\ u_0^k=u_\pi^k=0 \quad 0\leq n\leq N, 0\leq m\leq M \\ u_n^0=\sin(x_n), \dfrac{u_n^1-u_n^0}{\tau}=0. \end{cases} \tag{12}$$

The second-order difference scheme for (11) can be derived as follows:

$$
\begin{cases}
\dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}+u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{2h^2}+2\dfrac{(u_n^{k+1}+u_n^{k-1})}{2}=0 \\
u_0^k=u_\pi^k=0, \qquad 0\le n\le N, 0\le m\le M \\
u_n^0=\sin(x_n), \quad \dfrac{u_n^1-u_n^0}{\tau}=\dfrac{\tau}{2}\dfrac{u_n^2-2u_n^1+u_n^0}{\tau^2}.
\end{cases}
\tag{13}
$$

The error analysis for the problem (11) can be found in Table 1. In contrast, the visualizations for the exact and approximate solutions are displayed in Figure 1.

**Table 1**. Absolute error and relative error difference scheme in (11)

| $h = \pi/M$, $\tau = 1/N$ | $N = M = 10$ | $N = M = 30$ | $N = M = 45$ |
|---|---|---|---|
| First-order difference in (12) | $3.8618 \times 10^{-2}$ | $1.6242 \times 10^{-2}$ | $9.6724 \times 10^{-3}$ |
| Second-order difference in (13) | $1.4633 \times 10^{-3}$ | $1.8094 \times 10^{-4}$ | $2.0007 \times 10^{-2}$ |

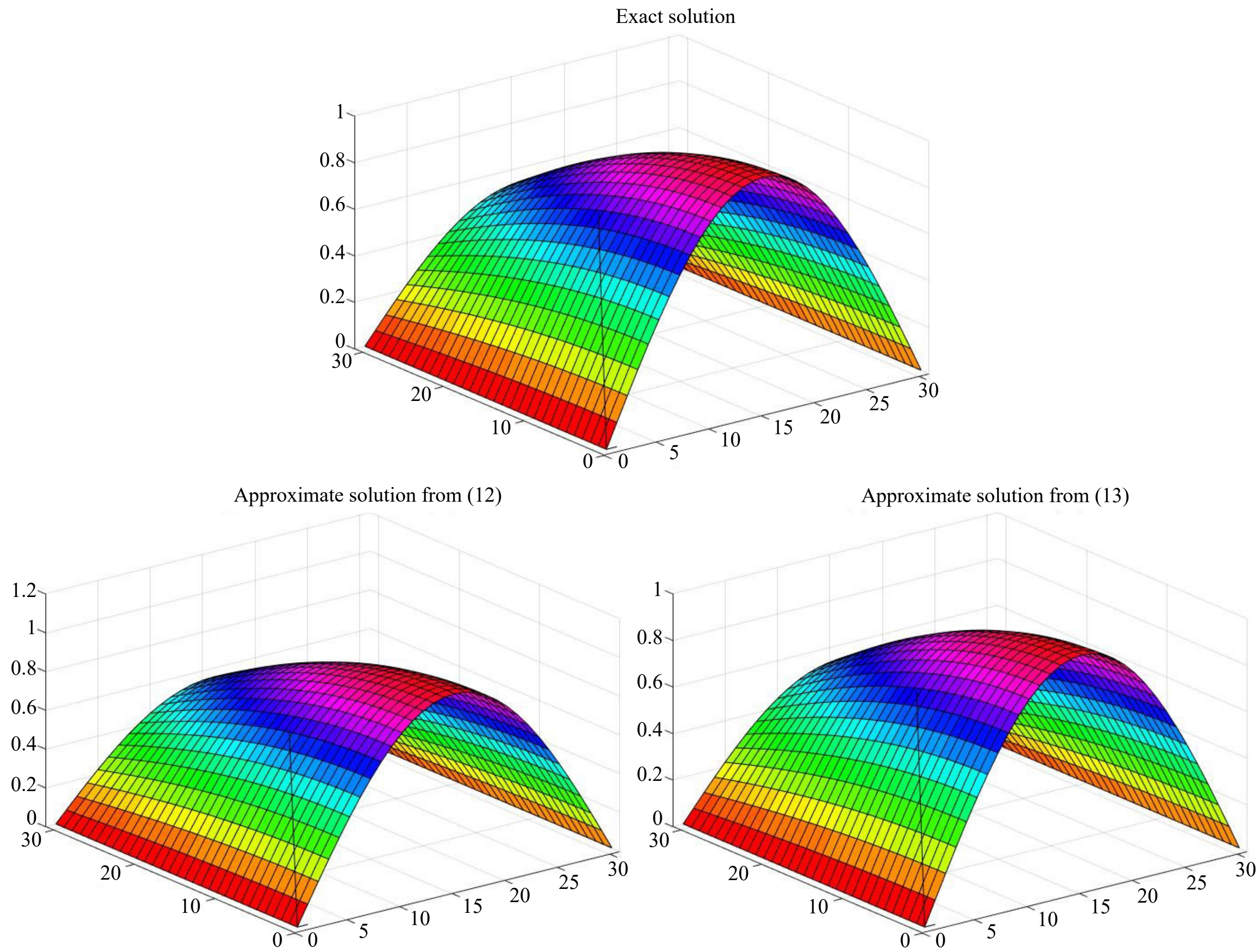


**Figure 1.** Numerical solution and exact solution of (11), where $M = N = 30$

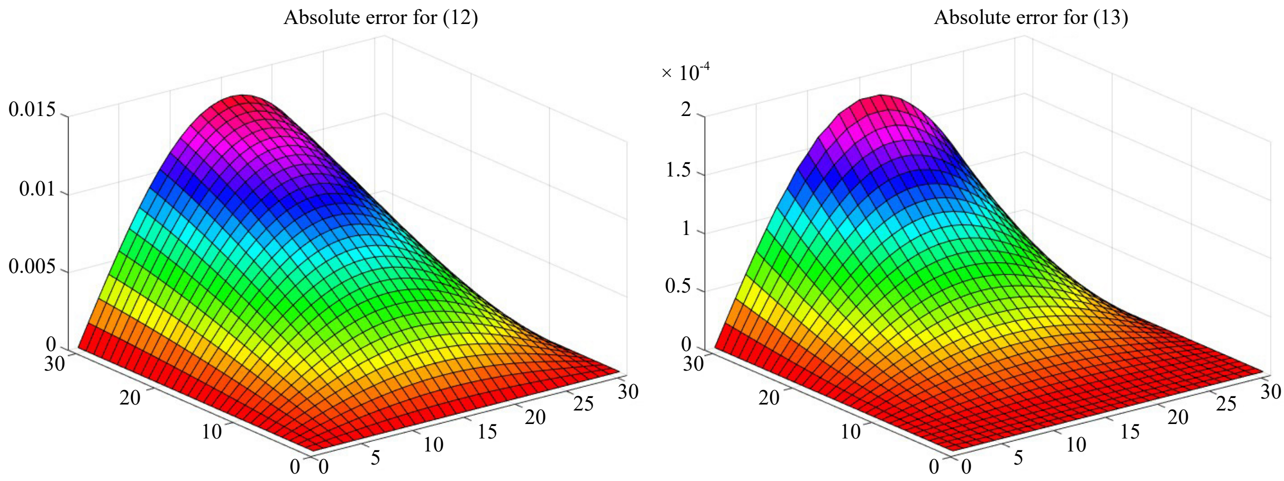


**Figure 2.** Absolute error of first-order and second-order difference scheme for (11), where $M = N = 30$

Figure 2 shows the absolute error for presented difference schemes in (12) and (13), it shows that the second-order difference scheme is more accurate than the first-order difference scheme.

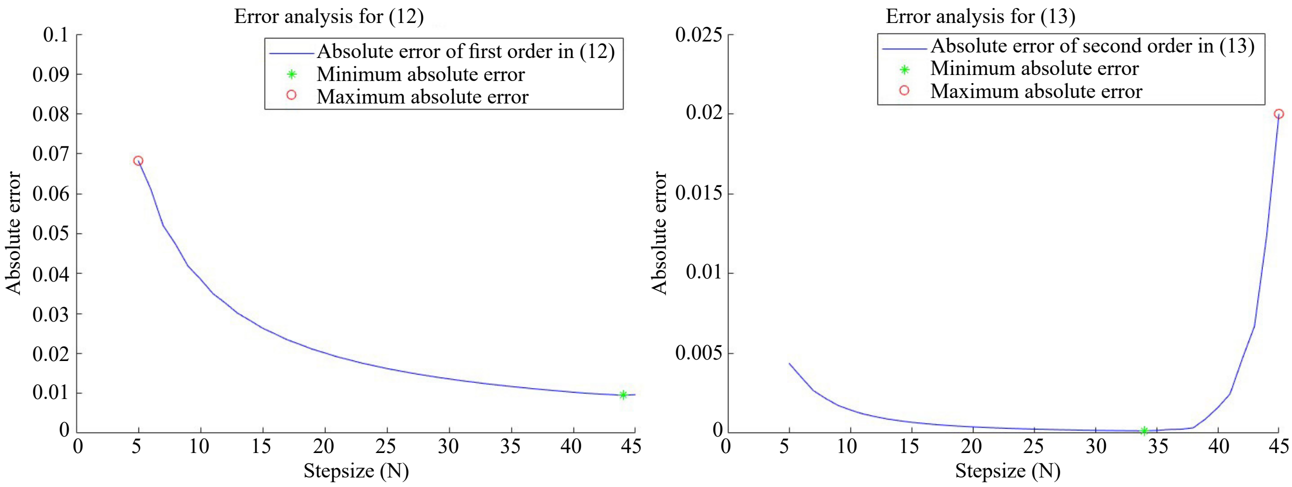


**Figure 3.** Error analysis of (12) and (13) in different points of $N$ and $M$

Figure 3 shows the maximum absolute error of the first- and second-order difference schemes for example 1 that presented in equation (12) and equation (13), respectively, as a function of the $N$-stepsize on the X-axis. The Y-axis represents the maximum absolute error for each $N$-stepsize. From the graph, it can be observed that the maximum absolute error decreases as the $N$-stepsize increases for both schemes. However, the second-order difference scheme has a significantly lower maximum absolute error compared to the first-order difference scheme. This indicates that the second-order difference scheme is more accurate in approximating the derivative of the function than the first-order difference scheme.

It can also be seen that the rate of decrease in maximum absolute error decreases as the $N$-stepsize increases. However, the graph of error analysis in (13) shows that the second-order difference scheme is appropriate for $N$ values less than 40; larger values will lose accuracy, as shown in **Remark 3.**

**Example 2.** Consider the following problem for a homogenous Helmholtz equation

$$\begin{cases} u_{xx}(x,t) + u_{tt}(x,t) + A^2 u(x,t) = 0 \quad 0 \le x \le \pi, 0 \le t \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = \psi(x),, u_t(x,0) = \varphi(x). \end{cases} \tag{14}$$

The exact solution to the problem depends on the value of the first-order difference scheme for the problem (14) can be present as follows:

$$\begin{cases} \dfrac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + \dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + A^2 u_n^k = 0 \\ u_0^k = u_\pi^k = 0, \quad 0 \le n \le N, 0 \le m \le M \\ u_n^0 = \psi(x_n), \dfrac{u_n^1 - u_n^0}{\tau} = \varphi(x_n). \end{cases} \tag{15}$$

Second-order difference scheme for the problem (14) is defined as follows:

$$\begin{cases} \dfrac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + \dfrac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1} + u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{2h^2} + A^2 \dfrac{(u_n^{k+1} + u_n^{k-1})}{2} = 0 \\ u_0^k = u_\pi^k = 0, \quad 0 \le n \le N, 0 \le m \le M \\ u_n^0 = \psi(x_n), \dfrac{u_n^1 - u_n^0}{\tau} = \varphi(x_n) + \dfrac{\tau}{2} \dfrac{u_n^2 - 2u_n^1 + u_n^0}{\tau^2}. \end{cases} \tag{16}$$

In the following table, three different examples have been presented for homogenous IBVP Helmholtz equation; exact solutions have been presented according to the value of $A^2$.

**Table 2**. Approximate solution of (14) in different values of $A^2$, where $h = \pi/M$, $\tau = 1/N$

| | | First-order difference in (15) | | Second-order difference in (15) | |
|---|---|---|---|---|---|
| $A^2$ | **Exact solution** | $N = M = 30$ | $N = M = 45$ | $N = M = 30$ | $N = M = 45$ |
| 0 | $\sin(x)\exp(-t)$ | 1.9751e-02 | 1.4115e-02 | 6.0666e-04 | 2.1278e-02 |
| 0.5 | $\sin(x)\exp(\sqrt{0.5}t)$ | 9.7218e-03 | 7.1699e-03 | 3.8760e-04 | 3.4590e-02 |
| 1 | $\sin(2x)\exp(\sqrt{3}t)$ | 9.7516e-02 | 6.2312e-02 | 9.0827e-03 | 4.3965e-02 |
| 2 | $\sin(x)\cos(t)$ | 1.3614e-02 | 9.6665e-03 | 1.8094e-04 | 2.0007e-02 |
| 5 | $\sin(2x)\cos(t)$ | 8.0118e-03 | 6.9133e-03 | 5.8944e-03 | 2.1376e-02 |

**Example 3.** Consider the following problem for the inhomogenous Helmholtz equation:

$$\begin{cases} u_{xx}(x,t) + +u_{tt}(x,t) + A^2 u(x,t) = f(x,t) \quad 0 \le x \le \pi, 0 \le t \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = \sin(x), u_t(x,0) = -\sin(x). \end{cases} \tag{17}$$

Where the exact solution is $u(x,t) = \sin(x)\exp(-t)$.
First-order difference scheme for the problem (17) is

$$
\begin{cases}
\dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\dfrac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+A^2u_n^k=f_n^k \\
u_0^k=u_\pi^k=0, \quad 0\le n\le N, 0\le m\le M \\
u_n^0=\sin(x_n), \dfrac{u_n^1-u_n^0}{\tau}=-\sin(x_n).
\end{cases} \tag{18}
$$

Similarly, the second-order difference scheme is obtained in the following manner:

$$
\begin{cases}
\dfrac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+\dfrac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}+u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{2h^2}+A^2\dfrac{(u_n^{k+1}+u_n^{k-1})}{2}=f_n^k \\
u_0^k=u_\pi^k=0, \qquad 0\le n\le N, 0\le m\le M \\
u_n^0=\sin(x_n), \dfrac{u_n^1-u_n^0}{\tau}=-\sin(x_n)+\dfrac{\tau}{2}\dfrac{u_n^2-2u_n^1+u_n^0}{\tau^2}.
\end{cases} \tag{19}
$$

**Table 3**. Approximate solution of (19) in different values of $A^2$, where $h=\pi/M$, $\tau=1/N$

| | | First-order difference in (15) | | Second-order difference in (16) | |
|---|---|---|---|---|---|
| $A^2$ | $f(x,t)$ | $N=M=30$ | $N=M=45$ | $N=M=30$ | $N=M=45$ |
| 0.1 | $\exp(-t)\sin(x)/10$ | 1.9446e-02 | 1.3313e-02 | 6.2092e-04 | 1.4849e-02 |
| 0.5 | $\exp(-t)\sin(x)/2$ | 1.8257e-02 | 1.2753e-02 | 6.7284e-04 | 2.4926e-02 |
| 1 | $\exp(-t)\sin(x)$ | 1.6834e-02 | 1.2130e-02 | 7.3088e-04 | 1.5081e-02 |
| 2 | $2\exp(-t)\sin(x)$ | 1.4193e-02 | 9.6916e-03 | 8.2487e-04 | 1.8062e-02 |
| 5 | $5\exp(-t)\sin(x)$ | 8.4301e-03 | 5.6192e-03 | 9.6208e-04 | 2.2145e-02 |
| 10 | $10\exp(-t)\sin(x)$ | 5.5885e-03 | 3.7153e-03 | 9.3278e-04 | 1.4466e-02 |

From Table 2, we conclude that the method is accurate and applies to homogenous types of IBVP Helmholtz equations. It is observed that the second-order difference scheme exhibits greater accuracy compared to the first-order approach in small values of $N$ and $M$. Regarding the first-order difference scheme, the correct result is accurate when $N=M=45$. By giving examples in the table, we demostrate that the method is applicable for different values of $A^2$ Helmholtz. Table 3 presents different examples of inhomogeneous Helmholtz equations, where first- and second-order difference schemes are applied for different values of $A^2$. Results show that the second-order difference scheme is more accurate than the first-order difference scheme in small stepsize. By increasing the stepsize, the first-order difference scheme is more accurate. Moreover, the value of $A^2$ affects the results of the inhomogenous IBVP Helmholtz equation; the result's accuracy increases by increasing the value of $A^2$.

**Remark 3.** The present method is applicable to small values of $N$ and $M$. By increasing the values of $N$ and $M$, the singular matrix will appear and result in loss of accuracy.

## 5. Conclusion

The present study presents numerical solutions for the IBVP of the Helmholtz equation. Both first- and second-order difference schemes were proposed, and the stability of the problem was guaranteed. Several examples were provided to demonstrate the accuracy and efficiency of the method. Comparisons have been made between numerical and exact solutions, and the error analysis table and illustration graphs are presented. The results showed that the proposed method was accurate and efficient for both homogenous and non-homogenous types of Helmholtz equations. Overall, this study provides valuable insights into the numerical solution of the Helmholtz equation and demonstrates the effectiveness of the proposed method in solving IBVP.

### 5.1 *Future works*

Future work could focus on extending the proposed method to higher dimensions and expanding its applicability to other partial differential equations. Additionally, further refinement of the error analysis could enhance the understanding of the method's performance under different conditions, paving the way for more advanced and efficient numerical techniques in numerical analysis.

## Conflict of interest

The authors declare that they have no known competing of interest.

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