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Research Article

A Note on Convexity Properties for Gaussian Hypergeometric Function

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Abstract: Gaussian hypergeometric function has been investigated in the context of geometric function theory regarding many aspects. Obtaining univalence conditions for this function is a line of research followed by many scholars. In the present study, methods specific to the differential superordination theory are used for obtaining properties of the Gaussian hypergeometric function regarding convexity of order $\left(-\frac{1}{2}\right)$. Also, a necessary and sufficient condition is proved such that Gaussian hypergeometric function is a close-to-convex function. The applicability of the theoretical findings is demonstrated by a numerical example.

Keywords: holomorphic function, analytic function, convexity of negative order, close-to-convex function, differential superordiantion, best subordinant

MSC: 30C80, 33C15, 30C45

1. Introduction and preliminaries

When it started to be studied in relation to univalence requirements, the famous Gaussian hypergeometric function attracted the attention of scholars in geometric function theory. Miller and Mocanu presented one of the earliest articles to demonstrate specific starlikeness and convexity features of this function in 1990 [1]. Miller and Mocanu regarded a, b, and c as real numbers in their research. Also, other authors who took into account the same restriction on parameters a, b, and c obtained additional univalence conditions [2–5].

In recent publications, interesting conditions for starlikeness and convexity of the Gaussian hypergeometric function were established considering a, b, c complex numbers. Two criteria for univalence are stated in [6], the relationship between the results provided in the research and the results Miller and Mocanu previously achieved in 1990 being highlighted. Other two univalence criteria for Gaussian hypergeometric function are stated in [7] as extensions of Miller and Mocanu's results seen in [1] using certain differential inequalities and the geometrical interpretation is given using particular sets inclusions in the complex plane. Carathéodory properties are shown for Gaussian hypergeometric function in [8] by employing methods specific to differential superordination theory.

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In this paper, criteria for the Gaussian hypergeometric function to be convex of order $\left(-\frac{1}{2}\right)$ are obtained by means of the theory of differential superordination developed by Miller and Mocanu [9]. Additionally, it is demonstrated that a necessary and sufficient condition exists for the Gaussian hypergeometric function to be a close-to-convex function.

The research's broad context is first established.

Denote by $U = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc of the complex plane and write $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\partial U = \{z \in \mathbb{C} : |z| = 1\}$.

Let $H(U)$ be the class of holomorphic functions in U and for $a \in \mathbb{C}$, $n \in \mathbb{N}^*$, the following subclasses of $H(U)$ are known:

$$H[a,\ n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots,\quad z \in U\},$$

and

$$A_n = \{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots,\ \ z \in U\},$$

with A_1 written simply A.

Let $0 \leq \alpha < 1$ and define the class of stralike functions of order α by:

$$S^*(\alpha) = \{f \in A : \mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha,\ \ z \in U\}.$$

For $\alpha = 0$, S^* denotes the class of starlike functions.

For $0 \leq \alpha < 1$,

$$K(\alpha) = \{f \in A : \mathrm{Re}\left[\frac{zf''(z)}{f(z)} + 1\right] > \alpha,\ \ z \in U\},$$

identifies all convex functions of order α. The class of convex functions, denoted by K, is found when $\alpha = 0$.

The class of functions $f \in A$ which are holomorphic and univalent in U and normed by $f(0) = 0$, $f'(0) = 1$ is denoted by S and defined as:

$$S = \{f \in A : f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,\ \ z \in U\}.$$

The theory of differential subordination is extensively presented in [10]. Miller and Mocanu proposed its dual, differential superordination theory, in [9]. The definitions and lemmas linked to the two dual theories that are necessary for the research examined in this paper are listed below.

Definition 1 Let f and F be members of $H(U)$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ and such that $f(z) = F(w(z))$. In such a case we write $f \prec F$ or $f(z) \prec F(z)$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Definition 2 [9] Let $\varphi(r,\ s,\ t;\ z) : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z),\ zp'(z),\ z^2 p''(z);\ z)$ are univalent in U and satisfy the (second-order) differential superordination.

$$h(z) \prec \varphi(p(z),\ zp'(z),\ z^2p''(z);\ z) \tag{1}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination or more simply a subordinant, if $q \prec p$ for all p satisfying (1). A subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1) is said to be the best subordinant of (1). Note that the best subordinant is unique up to a rotation of U.

Definition 3 [9]. We denote by Q the set of functions f that are analytic and injective on $\overline{U} \backslash E(q)$ where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$, for $\zeta \in \partial U \backslash E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Definition 4 [9] Let Ω be a set in $\mathbb{C}$ and $q \in H[a,\ n]$. The class of admissible functions $\phi_n[\Omega,\ q]$ consists of those functions $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\varphi(r,\ s,\ t;\ \zeta) \in \Omega \tag{2}$$

whenever

$$r = q(z),\ s = \frac{zq'(z)}{m},\ \mathrm{Re}\left(\frac{t}{s} + 1\right) \leq \frac{1}{m} \mathrm{Re}\left[\frac{zq''(z)}{q'(z)} + 1\right],\ \zeta \in \partial U,\ z \in U,\ m \geq n \geq 1.$$

When $n = 1$, $\phi_1[\Omega,\ q]$ is written as $\phi[\Omega,\ q]$.

In the special case when h is an analytic mapping of U onto $\Omega \neq \mathbb{C}$ we denote the class $\phi_n[h(U),\ q]$ by $\phi_n[h,\ q]$.

If $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$ then the admissibility condition (2) reduces to

$$\phi(r,\ s,\ t;\ \zeta) \in \Omega \tag{3}$$

where $\zeta \in \partial U,\ z \in U,\ m \geq n \geq 1$.

Lemma 1 [11] Let h be analytic in U, $q \in H[a,\ n]$, $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$ and suppose that $\varphi(q(z),\ tzq'(z);\ \zeta) \in h(U)$, for $z \in U$, $\zeta \in \partial U$ and $0 < t \leq \dfrac{1}{n} \leq 1$.

If $p \in Q(a)$ and $\varphi(p(z),\ zp'(z);\ z)$ is univalent in U, then

$$h(z) \prec \varphi(p(z),\ zp;\ (z);\ z)$$

implies

$$q(z) \prec p(z),\ z \in U.$$

Furthermore, if $\varphi(q(z),\ zq'(z);\ z) = h(z)$ has a univalent solution $q \in Q(a)$, then q is the best subordinant.

Lemma 2 [12] A necessary and sufficient condition for a function f, analytic in U satisfying the condition $f'(z) \neq 0$, to be close-to-convex is:

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) d\theta > -\pi,\ \ z = re^{i\theta},$$

for all $\theta_1,\ \ \theta_2$ satisfying $0 < \theta_1 < \theta_2 < 2\pi$ and any $r \in (0,\ 1)$.

Lemma 3 [12] Let $f \in H(U)$ satisfying $f'(0) \neq 0$. Then,

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0,$$

implies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)} + 1\right) > 0,\quad z \in U.$$

If $f \in A_n$, f is called starlike with respect to the origin (or simply starlike).

If $f \in H[a,\ 1]$, then

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0,$$

implies

$$\mathrm{Re}\frac{zf'(z)}{f(z) - 1} > 0,\ \ a,\ z \in U,$$

and f is called starlike with respect to a.

The domain $f(U)$ is called starlike with respect to $f(0) = a$ if for any $z \in U$, the segment which unites a to $f(z)$ is included into $f(U)$.

The definition of Gaussian hypergeometric function is given as found in [1]:

Definition 5 [1] Let a, b and c be complex numbers with $c \neq 0,\ -1,\ -2,\ \ldots$ The function

$$F(a,\ b,\ c;\ z) = {}_2F_1(a,\ b,\ c;\ z) = 1 + \frac{ab}{c} \cdot \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \ldots,\ z \tag{4}$$

is called Gaussian hypergeometric function, is analytic in U and satisfies the hypergeometric equation:

$$z(z-1) \cdot w''(z) + [c - (a+b+1)z] \cdot w'(z) - ab \cdot w(z) = 0.$$

If we let

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = d(d+1)(d+2)\dots(d+k-1) \text{ and } (d)_0 = 1$$

then (4) can be written in the form

$$F(a,\ b,\ c;\ z) = \sum_{k=0}^{\infty} \frac{(a)_k \cdot (b)_k}{(c)_k} \cdot \frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \cdot \frac{z^k}{k!},$$

with

$$\Gamma(z) = \int_0^{\infty} t^{z-1} \cdot e^{-t} dt,\ \mathrm{Re} z > 0,\ \ z \in U,$$

or,

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{z+n} + \int_0^{\infty} t^{z-1} \cdot e^{-t} dt,$$

where points $z = 0,\ -1,\ -2,\ \dots$ are pols of first order for function Γ.

Next, the definition of the subordination chain is recalled as found in [10]:

Definition 6 [10] A function $L(z,\ t)$, $z \in U$, $t \geq 0$ is a subordination chain if $L(\cdot,\ t)$ is analytic and univalent in U for all $t \geq 0$, $L(z,\ \cdot)$ is contiunously differentiable on $\mathbb{R}^+$ for all $z \in U$ and $L(z,\ s) \prec L(z,\ t)$ when $0 \leq s \leq t$.

A sufficient condition for a function $L(z,\ t)$, $z \in U$, $t \geq 0$, to be a subordination chain is found in the next lemma:

Lemma 4 [10] The function $L(z,\ t) = a_1(t)z + a_2(t)z^2 + \dots$ with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t \to \infty} |a_1(t)| = \infty$ is a subordination chain if

$$\mathrm{Re} \frac{z \dfrac{\partial L(z,\ t)}{\partial z}}{\dfrac{\partial L(z,\ t)}{\partial t}} > 0,\ \ z \in U,\ \ t \geq 0.$$

Definition 7 [12] A function $f \in H(U)$ is called close-to-convex if there exists a function g, convex in U, such that:

$$\mathrm{Re} \frac{f'(z)}{g'(z)} > 0, \quad z \in U.$$

In particular, if $\mathrm{Re} f'(z) > 0$, $z \in U$, we say that function f is close-to-convex with respect to the identical function $g(z) = z$, $z \in U$.

Recent studies have focused on the convexity of Gaussian hypergeometric function [14–16]. Our novel convexity results are shown in the following section utilizing certain distinct differential superordinations. A differential superordination is examined in the first theorem, and its best subordinant is provided. By using certain convex functions in the differential superordination examined in this theorem, its outcome is employed in the subsequent corollary to get the convexity of order $\left(-\frac{1}{2}\right)$ for Gaussian hypergeometric function. The second theorem establishes Gaussian hypergeometric function characteristic of being a close-to-convex function using the convexity of negative order already established. The study is ended by providing an illustration of how the findings from this paper can be put to use.

2. Main results

The best subordinant for a particular differential superordination studied is determined in the first theorem proved in this study.

Theorem 1 Let $h \in H(U)$ be given by:

$$h(z) = q(z) - a + zq'(z)(q(z) - a), \quad z \in U, \tag{5}$$

where $q \in H[a, 1]$ is convex in U satisfying:

$$\operatorname{Re}(q(z) - a) > 0, \; z \in U. \tag{6}$$

Let $p \in H[a, 1] \cap Q$ be convex in U and let $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$, $\varphi(p(z), zp'(z); z)$ be univalent in U.
If

$$h(z) = q(z) - a + zq'(z)(q(z) - a) \prec \varphi(p(z), zp'(z); z) \tag{7}$$

then

$$q(z) \prec p(z), \; z \in U,$$

and q is the best subordinant.

Proof. In the proof of this result, Lemma 1 will be applied. For that, consider the function $L : U \times [0, \infty) \to \mathbb{C}$ given by:

$$\begin{aligned} L(z, t) &= \varphi(q(z), tzq'(z); z) = a_1(t)z + a_2(t)z^2 + \dots \\ &= q(z) - a + tzq'(z)(q(z) - a), \; a_1(t) \neq 0, \; t \geq 0. \end{aligned} \tag{8}$$

Using Definition 1 and Lemma 1, we show that $L(z, t)$ given by (8) is a subordination chain. For that, relation (8) is derived with respect to z:

$$\begin{aligned} \frac{\partial L(z, t)}{\partial z} &= a_1(t) + 2a_2(t)z + \dots \\ &= q'(z) + tq'(z)(q(z) - a) + ztq''(z)(q(z) - a) + tz\left(q'(z)\right)^2. \end{aligned} \tag{9}$$

Differentiating (8) with respect to t, we have:

$$\frac{\partial L(z,\,t)}{\partial t}=zq'(z)\left(q(z)-a\right),\ z\in U. \tag{10}$$

We now evaluate:

$$\operatorname{Re}\frac{z\dfrac{\partial L(z,\,t)}{\partial z}}{\dfrac{\partial L(z,\,t)}{\partial t}}=\operatorname{Re}\left[\frac{1}{q(z)-a}+t\left(1+\frac{zq''(z)}{q'(z)}+\frac{zq'(z)}{q(z)-a}\right)\right],\ z\in U,\ t\geq 0. \tag{11}$$

From the hypothesis it is known that $q\in K$, hence we know that:

$$\operatorname{Re}\left(\frac{zq''(z)}{q'(z)}+1\right)>0,\ z\in U. \tag{12}$$

Since $q\in H[a,1]$, by employing results given in Lemma 3, we can write:

$$\operatorname{Re}\frac{zf'(z)}{f(z)-a}>0,\ a,\ z\in U. \tag{13}$$

and f is starlike with respect to a in U.

Using the relations (6), (12) and (13) in (11), we write:

$$\operatorname{Re}\frac{z\dfrac{\partial L(z,\,t)}{\partial z}}{\dfrac{\partial L(z,\,t)}{\partial t}}>0,\quad z\in U,\ \ t\geq 0. \tag{14}$$

By replacing $z=0$ in (9), we write:

$$\frac{\partial L(0,\,t)}{\partial z}=a_1(t)=q'(0)+tq'(0)\left(q(0)-a\right). \tag{15}$$

Since $\operatorname{Re}(q(z)-a)>0$ given by (6) in the hypothesis, for $z=0$ we obtain:

$$\operatorname{Re}(q(0)-a)>0,\ z\in U, \tag{16}$$

from which we conclude that $q(0)-a\neq 0$.

Since $q\in K$, it is univalent in U and we have that $q'(0)\neq 0$. Using this and relation (16) in (15), we obtain:

$$\frac{\partial L(0,\,t)}{\partial z}=a_1(t)=q'(0)\left[1+t\left(q(0)-a\right)\right].$$

We evaluate

$$\lim_{t\to\infty}|a_1(t)| = \lim_{t\to\infty}\left|q'(0)\left[1+t\left(q(0)-a\right)\right]\right| = \infty. \tag{17}$$

Using (14) and (17) in Lemma ,1we conclude that function $L(z,\ t)$ given by (8) is a subordination chain. By using Definition 6 we can write:

$$L(z,\ t) \prec L(z,\ 1),\ z\in U,\ \ 0\le t\le 1. \tag{18}$$

If we use $t=1$ in relation (8), we get:

$$L(z,1) = \varphi(q(z),\ zq'(z);\ z) = q(z)-a+1\cdot zq'(z)(q(z)-a) = h(z). \tag{19}$$

Using (19) in (18) we get:

$$L(z,\ t) \prec h(z),\ z\in U,\ \ t\ge 0. \tag{20}$$

By considering Definition 1, relation (20) is equivalent to:

$$\varphi(q(z),\ tzq'(z);\ z) \in h(U),\ z\in U. \tag{21}$$

Since $L(z,\ t) = \varphi(q(z),\ tzq'(z);\ z)$ is a subordination chain and differential superordination (7) holds, by applying Lemma 1, we assess that

$$q(z) \prec p(z),\ z\in U. \tag{22}$$

We acknowledge that the function q is the best subordinant since it is a univalent solution for the equation given by (2).

Remark In Corollary 2 from [7], it was proven that function $F(a,\ b,\ c;\ z)$ given by (4) with $ab\neq 0$ is convex, written equivalently as:

$$\mathrm{Re}\left\{1+\frac{z\left[F\left(a.\ b.\ c;\ z\right)\right]''}{\left[F\left(a,\ b,\ c;\ z\right)\right]'}\right\} > 0,\ \ ab\neq 0, z\in U.$$

If we take in Theorem 1 the convex functions

$$q(z) = 1+\frac{zF''\left(z,\ b,\ c;\ z\right)}{F'\left(a,\ b,\ c;\ z\right)},\ ab\neq 0,\ p(z) = \frac{1+2z}{1-z},\ z\in U,$$

the resulting corollary can be expressed as:

Corollary 1 Let h be an analytic function in U given by:

$$h(z)=1+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)}+z\left(1+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)}\right)'\cdot\left(1+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)}\right)$$

and

$$q(z)=1+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)},\ q(0)=1,$$

is a convex function in U satisfying:

$$\operatorname{Re}\left(1+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)}\right)>0,\ z\in U.$$

Let

$$p(z)=\frac{1+2z}{1-z},\ p\in H[1,\ 1]\cap Q,\ p(0)=1,$$

be a convex function in U and let $\varphi:\mathbb{C}^2\times\overline{U}\to\mathbb{C}$, given by

$$\varphi(p(z),\ zp'(z);\ z)=\varphi\left(\frac{1+2z}{1-z},\ \frac{3z}{(1-z)^2};\ z\right)$$

be univalent in U.

If

$$h(z)=q+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)}+z\left(1+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)}\right)'\cdot\left(1+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)}\right)$$

$$\prec\varphi\left(\frac{1+2z}{1-z},\ \frac{3z}{(1-z)^2};\ z\right)$$

then

$$1+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)}\prec\frac{1+2z}{1-z},\ z\in U,$$

which is equivalent to:

$$\mathrm{Re}\left(1+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)}\right)>-\frac{1}{2},$$

hence $F(a,\ b,\ c;\ z)$ is a convex function of order $-\frac{1}{2}$, written $F(a,\ b,\ c;\ z)\in K\left(-\frac{1}{2}\right)$.

Proof. We first demonstrate that $p(z)=\frac{1+2z}{1-z}$ is a convex function in U. Let $f(z)=\frac{1+z}{1-z}$. Since we know that this function is a conformal mapping of U into the half-plane $\{w\in\mathbb{C}:\mathrm{Re}\,w>0\}$ we obtain that

$$\mathrm{Re}\frac{1+z}{1-z}>0,\ \ z\in U. \tag{23}$$

We calculate:

$$p'(z)=\frac{3}{(1-z)^2},\ p''(z)=\frac{6}{(1-z)^3},\ \frac{zp''(z)}{p'(z)}+1=\frac{1+z}{1-z}.$$

Using (23) we obtain:

$$\mathrm{Re}\left(\frac{zp''(z)}{p'(z)}+1\right)=\mathrm{Re}\frac{1+z}{1-z}>0,\ \ z\in U,$$

which gives that function $p(z)=\frac{1+2z}{1-z}$ is convex in U.

Since $p\in K$, we determine that it is a conformal mapping of U into the half-plane $\left\{w\in\mathbb{C}:\mathrm{Re}w>-\frac{1}{2}\right\}$, hence,

$$\mathrm{Re}p(z)>-\frac{1}{2},\ \ z\in U.$$

According to Theorem 1's proof, relation (22) is used for

$$q(z)=1+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)},\ \ p(z)=\frac{1+2z}{1-z},$$

and we obtain:

$$1+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)}\prec\frac{1+2z}{1-z},\ \ z\in U. \tag{24}$$

Since we know that

$$1+\frac{zF''(z,\ b,\ c;\ 0)}{F'(a,\ b,\ c;\ 0)}=q(0)=1=p(0),$$

and that $p \in K$, the superordination given by (24) is interpreted as:

$$\mathrm{Re}\left(1+\frac{zF''(z,\ b,\ c;\ `z)}{F'(a,\ b,\ c;\ z)}\right) > \mathrm{Re}\left(\frac{1+2z}{1-z}\right) > -\frac{1}{2},\ z \in U,$$

and we establish that $F(a,\ b,\ c;\ z) \in K\left(-\frac{1}{2}\right)$ when $ab \neq 0$.

Remark 2 Since we have that $\mathrm{Re}\left(1+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)}\right) > -\frac{1}{2},\ z \in U,$ we next prove using Lemma 2 that function $F(a,\ b,\ c;\ z)$ is close-to-convex U which establishes that $F(a,\ b,\ c;\ z)$ is also univalent in U.

Theorem 2 Let $F(a,\ b,\ c;\ z)$ be given by (4) with

$$\mathrm{Re}\left(1+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)}\right) > -\frac{1}{2},\ z \in U.$$

Then function $F(a,\ b,\ c;\ z)$ is close-to-convex in U.

Proof. For applying Lemma 2, we evaluate:

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1+\frac{zF''(z,\ b,\ c;\ z)}{F'(a,\ b,\ c;\ z)}\right) d\theta > \int_{\theta_1}^{\theta_2}\left(-\frac{1}{2}\right) d\theta = -\frac{1}{2}(\theta_2-\theta_1) > -\pi,$$

from which, using Lemma 2, we establish that function $F(a,\ b,\ c;\ z)$ is close-to-convex in U.

Example 1 Let $a=-1$, $b=6+6i$, $c=2-2i$. Then we obtain that function $F(-1,\ 6+6i,\ 2-2i;\ z)=1-3iz$. We can prove that this is a close-to-convex function.

Let $g(z)=1-2iz$. This function is known to be convex in U since we have that $\mathrm{Re}\left(1+\frac{zg''(z)}{g'(z)}\right)=1\ >\ 0$. Next, we use Definition 7. For that, we calculate:

$$\mathrm{Re}\frac{F'(-1,\ 6+6i,\ 2-2i;\ z)}{g'(z)} = \mathrm{Re}\frac{-3i}{-2i} = \mathrm{Re}\frac{3}{2} > 0.$$

Now, from Definition 7, we conclude that function $F(-1,\ 6+6i,\ 2-2i;\ z)$ is close-to-convex with respect to function $g(z)=1-2iz,\ z \in U$.

3. Conclusion

The new convexity findings regarding Gaussian hypergeometric function are presented in Section 2 of this paper by applying differential superordinations results. A differential superordination is examined and its best subordinant is given in the Theorem 1. Corollary 1 uses the findings of Theorem 1 to obtain the convexity of order $-\frac{1}{2}$ for the Gaussian hypergeometric function while taking specific convex functions into account in the differential superordination examined in the first theorem. In Theorem 2, it is demonstrated that the Gaussian hypergeometric function has the attribute of being close-to-convex by using the property of negative order convexity proved before. An illustration of how the study's findings might be applied is provided as a conclusion for the study.

The results presented in this paper can be used in future studies connected to fractional calculus as it is seen in very recently published papers regarding Gaussian hypergeometric function [17], confluent hypergeometric function [18] and other hypergeometric functions [19]. Quantum calculus aspects can also be associated with Gaussian hypergeometric function inspired by results like those presented in [20, 21].

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Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings.

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