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Research Article

On an Efficient Iterative Method for Fixed Points

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Abstract: Real-world applications depend heavily on the fixed-point solution. In this paper, we have suggested an effective iterative method for fixed points. We have first given the approximate order of convergence for this method using Taylor's series. The radii of convergence balls for this method can then be calculated using a local convergence theorem that we then present. The semilocal convergence theorem, which determines the starting point's accuracy, is then presented. We have created some technical lemmas and theorems to serve this purpose. In contrast to an earlier study using the same type of method for nonlinear equations, we have not used the convergence conditions on higher-order Frechet derivatives in our study of convergence. Finally, some numerical examples are provided to support the theoretical findings we made. This highlights the uniqueness of this study.

Keywords: midpoint method, Striling method, order of convergence, local convergence, semilocal convergence, Lipschitz and center-Lipschitz condition

MSC: 65H05, 65H10

1. Introduction

Solving nonlinear equations is one of the most significant computational mathematics problems. In many fields of engineering and scientific discovery, systems that model different phenomena give rise to nonlinear equation systems that must be solved. In many circumstances, nonlinear equations or systems of nonlinear equations naturally appear as nonlinear problems in Banach spaces. In fact, solving nonlinear equations is enough to spur researchers to create new, computationally effective iterative techniques. Naturally, the fact that analytical solutions to the majority of nonlinear equation types are frequently unavailable is a major factor. As a result, numerical iterative methods are best suited for this task. With the development of iterative methods, it is also an important task to derive their convergence conditions, which make the method suitable. Semilocal [1-3] and local [4-6] convergence studies are generally performed for this purpose.

In literature, the Newton method and its variants [7, 8] are used for the computation of solutions to nonlinear equations in Banach spaces. Many researchers have studied these methods and used them for their purposes. They have developed different iterative methods and will continue their work to develop new efficient iterative methods. Recently, an efficient iterative method [9] was presented for the nonlinear equation, which is given by:

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$$s_k = r_k - G'(w_{k-1})G(r_k),$$
$$r_{k+1} = r_k - G'(w_k)G(r_k) \tag{1}$$

where $w_{-1} = r_0$ and $w_n = \dfrac{r_n + s_n}{2}$. It uses the evaluation of a function and a derivative at each step after the first step. As the evaluation of derivatives appears in the first step, it comes from the previous one. They have claimed its efficiency index is 1.553, which is better than Newton's method of 1.414. In order to enhance the applicability of (1), many researchers [10-13] have recently studied the convergence of this method and its variants under different convergence conditions.

However, we are concerned with the solution of fixed points [14, 15] of the nonlinear equation $G(r)$ in the present work, that can be presented in the form:

$$r = G(r) \tag{2}$$

Here, $G:\Omega_0 \subset \mathcal{P} \to \mathcal{P}$ is a nonlinear operator from an open convex domain $\Omega_0 \subset \mathcal{P}$ to itself. A quadratically convergent Striling's method is one of the iterative methods for (2), which is given by

$$r_{n+1} = r_n - \left(I - G'\left(G\left(r_n\right)\right)\right)^{-1}\left(r_n - G\left(r_n\right)\right), n \geq 0. \tag{3}$$

This method starts with r_0 and $(I - G'(G(r)))^{-1} \in \mathrm{B}(\mathcal{P})$ (the set of continuous linear operators from $\mathcal{P} \to \mathcal{P}$). Not much attention has been given to this method. However, the study of (3) can be found in these excellent research papers [16-22]. Sometimes, it has been observed that it performs better as compared to Newton's method. In order to enhance the applicability of (3), a cubically convergent iterative method was developed by Parhi and Gupta [23] that is given by

$$s_n = r_n - \left(I - G'\left(G\left(r_n\right)\right)\right)^{-1}\left(r_n - G\left(r_n\right)\right),$$
$$r_{n+1} = s_n - \left(I - G'\left(G\left(r_n\right)\right)\right)^{-1}\left(s_n - G\left(s_n\right)\right), n \geq 0. \tag{4}$$

In some cases, they have also shown that this method performs better as compared to Newton's method. In order to make (4) more suitable for different real-life applications, many researchers [24-26] have also studied this method and contributed their efforts towards this direction. To enhance the applicability of (1), we use it to (2) and the resultant method takes the form

$$s_n = r_n - \left(I - G'\left(w_{n-1}\right)\right)^{-1}\left(r_n - G\left(r_n\right)\right),$$
$$r_{n+1} = s_n - \left(I - G'\left(w_n\right)\right)^{-1}\left(r_n - G\left(r_n\right)\right), n \geq 0. \tag{5}$$

where $w_{-1} = r_0$ and $w_n = \dfrac{r_n + s_n}{2}$. It achieves the convergence order of $1+\sqrt{2}$ with one function evaluation and one derivative evaluation at each step, which makes the method more efficient in comparison to other methods found in the literature. This is because it uses the evaluation of $(I - G'(w_n))^{-1}$ that comes from the previous iteration. This presents the novelty of this method. For the convergence analysis, we have not used the condition on second-order Frechet derivatives that does not appear in the method but was used for the study of (1) in earlier studies mentioned above.

In this study, we present an efficient iterative method for the solution of (2). The order of convergence of this method is found to be $1+\sqrt{2}$. In order to find the suitability of the radii of convergence balls and starting points for the iteration, we have also presented the local and semilocal convergence theorems for the study of this method. Some lemmas and theorems are constructed for this purpose. In our study, we have not used the conditions on higher-order Frechet derivatives that do not appear in this method but were used before to achieve the convergence of this type of method. This shows the key features of this method.

The following is how the manuscript is set up: In Section 2, we derived the convergence theorems to obtain the order of convergence of (5) following Taylor's series expansion. In Section 3, we derived the local convergence theorems to find the radii of convergence balls for (5) and also obtained the a priori error estimate. In Section 4, we establish the semilocal convergence theorems, followed by some technical lemmas. Further, the numerical examples are constructed to justify the theoretical results obtained by us in Section 5. Finally, some conclusions and discussion are given after that.

It may be stated that the open and closed balls with the center at x and radius y will be denoted by $\mathfrak{V}(x, y)$ and $\overline{\mathfrak{V}}(x, y)$, respectively.

2. Order of convergence

In this section, we have developed the order of convergence for (5). We have employed Taylor's series approximation for this purpose.

Lemma 1. The following holds true:

(a) $\left(I-G'(w_n)\right)^{-1}=(I-P(r^*)(u_n+v_n))\left(I-G'(r^*)\right)^{-1}+O_2(u_n,v_n)$,

(b) $r_n-G(r_n)=(I-G'(r^*))(u_n+P(r^*)(u_n,u_n))+O_3(u_n)$.

Here, r^* satisfies $r^*=G(r^*), P(r^*)=\dfrac{(I-G'(r^*))^{-1}G''(r^*)}{2}$ and $u_n=r_n-r^*, v_n=s_n-r^*$.

Proof. First, we take $F(r)=r-G(r)$, which gives $F(r^*)=0$, and now we proceed with Taylor's series approximation of $F(r)$ along r^*, which provides

$$\begin{aligned} F(r) &= F(r^*)+F'(r^*)(r-r^*)+\frac{F''(r^*)}{2}(r-r^*,r-r^*)+O_3(r-r^*)\\ &= F'(r^*)(r-r^*)+\frac{F''(r^*)}{2}(r-r^*,r-r^*)+O_3(r-r^*). \end{aligned} \tag{6}$$

Now, we apply Taylor's series approximation of along r^*, which provides

$$\begin{aligned} F'(r) &= F'(r^*)+F''(r^*)(r-r^*)+O_2(r-r^*)n\\ &= F'(r^*)(I+\frac{F'(r^*)^{-1}F''(r^*)}{2}(r-r^*))+O_2(r-r^*)\\ &= F'(r^*)(I+2P(r^*)(r-r^*))+O_2(r-r^*). \end{aligned}$$

This provides

$$F'(r)^{-1}=(I-2P(r^*)(r-r^*))F'(r^*)^{-1}+O_2(r-r^*). \tag{7}$$

Here, we replace r by $\dfrac{r_n+s_n}{2}$ in (7), and this gives

$$\begin{aligned} \left(I-G'(\frac{r_n+s_n}{2})\right)^{-1} &= (I-P(r^*)(r_n-r^*+s_n-r^*))(I-G'(r^*))^{-1}+O_2(\frac{r_n+s_n}{2}-r^*)\\ &= (I-P(r^*)(u_n+v_n))(I-G'(r^*))^{-1}+O_2(u_n,v_n). \end{aligned}$$

Replace r by $\dfrac{r_n+s_n}{2}$ in (6) and using some algebraic manipulation, we get

$$r_n-G(r_n)=(I-G'(r^*))(u_n+P(r^*)(u_n,u_n))+O_3(u_n).$$

This proves the lemma.

Lemma 2. The following are true for (5):

(a) $u_{n+1}=P(r^*)(v_n,u_n)+O_3(u_n,v_n)$.

(b) $v_n=P(r^*)(u_{n-1},u_n)+O_3(u_{n-1},v_{n-1})$.

Proof. From (5),

$$u_{n+1} = u_n - \left(I - G'(\frac{r_n + s_n}{2})\right)^{-1} (r_n - G(r_n)).$$

Now, using the combining results of Lemma 1, we have

$$\begin{aligned} u_{n+1} &= u_n - \left((I - P(r^*)(u_n + v_n))(u_n + P(r^*)(u_n, u_n))\right) + O_3(u_n, v_n) \\ &= P(r^*)(u_n, u_n) + P(r^*)(v_n, u_n) - P(r^*)(u_n, u_n) + O_3(u_n, v_n) \\ &= P(r^*)(v_n, u_n) + O_3(u_n, v_n). \end{aligned} \tag{8}$$

From Lemma 1, it can be seen that

$$\left(I - G'(\frac{r_{n-1} + s_{n-1}}{2})\right)^{-1} = (I - P(r^*)(u_{n-1} + v_{n-1}))(I - G'(r^*))^{-1} + O_2(u_{n-1}, v_{n-1}).$$

Therefore,

$$\begin{aligned} v_n &= u_n - \left((I - P(r^*)(u_{n-1} + v_{n-1}))(u_n + P(r^*)(u_n, u_n))\right) + O_3(u_n, v_n) \\ &= P(r^*)(u_{n-1} + v_{n-1}, u_n) - P(r^*)(u_n, u_n) + O_3(u_n, v_n). \end{aligned} \tag{9}$$

Repeating the use of (8) in (9), this gives

$$\begin{aligned} v_n &= P(r^*)(u_{n-1}, u_n) + P(r^*)(u_{n-1}, P(r^*)(v_n, u_n)) - P(r^*)(P(r^*)(v_{n-1}, u_{n-1}), P(r^*)(v_{n-1}, u_{n-1})) + O_3(u_n, v_n) \\ &= P(r^*)(u_{n-1}, u_n) + O_3(u_{n-1}, v_{n-1}). \end{aligned}$$

This proves the lemma.

We are in a position to present the theorem, which represents the order of convergence of (5).

Theorem 1. Suppose Lemma 1 and Lemma 2 are true then the order of convergence of (5) is $1 + \sqrt{2}$.

Proof. We shall use the results of Lemma 1 and Lemma 2 to prove this theorem. First, we consider $\|P(r^*)\| = K, a_n = \|u_n\| = \|r_n - r^*\|$ and $b_n = \|v_n\| = \|s_n - r^*\|$. Now, we use (8) and (9), which gives

$$a_{n+1} = Ka_n b_n + O_3\left(a_{n-1}, b_{n-1}\right), \text{ and } b_n = Ka_{n-1}a_n + O_3\left(a_{n-1}, b_{n-1}\right).$$

Reiterate the use of b_n in a_{n+1}, this provides

$$a_{n+1} = Ka_n^2 a_{n-1} + O_3(a_{n-1}, b_{n-1}),$$

which is asymptotically equivalent to $a_{n+1} \approx a_n^2 a_{n-1}$. This leads to the order of convergence, which is given by the quadratic equation $\alpha^2 - 2\alpha - 1 = 0$. This produces the order of convergence $1 + \sqrt{2}$. This proves the theorem.

3. Local convergence

In this section, we present the local convergence theorem of (5), which estimates the radii of the convergence ball centered at the solution r^* satisfying (2). Let J, J_* be some nonnegative parameters, and then we construct a function $t(r)$ as follows:

$$t(r) = \frac{3Jr}{2(1 - J_* r)}.$$

It can be clearly seen that the function $t(r)$ strictly increases functions in its domain $\left(0, \frac{1}{J_*}\right)$. Now, we consider the function $f(r) = t(r) - 1$ as $f(0) = -1$ and $f\left(\frac{1}{J_*}\right)^- \to \infty$. This shows, using the intermediate value theorem, that $f(r)$ has at least one root in the interval $\left(0, \frac{1}{J_*}\right)$. It is a simple exercise to get this value by $x_2 = \frac{2}{3J + 2J_*}$ and $0 < t(r) < 1$ for $r \in (0, x_2)$. This helps us present the following lemmas. The following set of conditions '$\mathfrak{L}$' are being used to serve this purpose, which is defined as:

Definition 1. An element belongs to '$\mathfrak{L}$', if it satisfies certain conditions given by

$$\left.\begin{aligned} &A_*^{-1} = (I - G'(r^*))^{-1} \in \mathrm{B}(P), r^* = G(r^*), r, s \in \mathfrak{V}(r^*, x_2), \\ &\left\| A_*^{-1}(G'(r) - G'(r^*)) \right\| \le J_* \|r - r^*\|, \\ &\left\| A_*^{-1}(G'(r) - G'(s)) \right\| \le J\|r - s\| \text{ and } \mathfrak{V}(r^*, x_2) \in \Omega_0. \end{aligned}\right\} \tag{10}$$

Lemma 3. Suppose $w \in \mathfrak{V}(r^*, x_2)$ and conditions of $\mathfrak{L}$ hold true, then

$$\left\| (I - G'(w))^{-1} A_* \right\| \le \frac{1}{1 - J_* \|w - r^*\|}. \tag{11}$$

Proof. For $w \in \mathfrak{V}(r^*, x_2)$, we have

$$\begin{aligned} \left\| I - (A_*)^{-1}(I - G'(w)) \right\| &= \left\| A_*^{-1}(I - G'(w) - (I - G'(r^*))) \right\| \\ &\le \left\| A_*^{-1}(G'(w) - G'(r^*)) \right\| \le J_* \|w - r^*\| \le J_* x_2 < 1. \end{aligned}$$

Using Banach Lemma [27], this shows (11).

The output of this lemma allows us to immediately derive the lemma given below.

Lemma 4. Suppose condition $\mathfrak{L}$ along with all of its conditions (10) are true, then

(a) $\|s_0 - r^*\| \le \dfrac{3J\|r_0 - r^*\|}{2\left(1 - J_*\|r_0 - r^*\|\right)} \|r_0 - r^*\|,$

(b) $\|r_{n+1} - r^*\| \le \dfrac{J\left(2\|r_n - r^*\| + \|s_n - r^*\|\right)}{2\left(1 - J_*\|w_n - r^*\|\right)} \|r_n - r^*\|, n = 0, 1, 2, \ldots,$

(c) $\|s_n - r^*\| \le \dfrac{J\left(\|r_{n-1} - r^*\| + \|s_{n-1} - r^*\| + \|r_n - r^*\|\right)}{2\left(1 - J_*\|w_{n-1} - r^*\|\right)} \|r_n - r^*\|, n = 1, 2, \ldots.$

Proof. We begin the theorem using the mathematical induction on 'n'. From (5),

$$\begin{aligned} s_0 - r^* &= r_0 - r^* - (I - G'(r_0))^{-1}(r_0 - G(r_0) - r^* + G(r^*)) \\ &= (I - G'(r_0))^{-1}\left(G(r_0) - G(r^*) - G'(r_0)(r_0 - r^*)\right) \\ &= (I - G'(r_0))^{-1} A_* \int_0^1 A_*^{-1}\left(G'(r^* + \theta(r_0 - r^*)) - G'(r_0)\right) d\theta (r_0 - r^*). \end{aligned}$$

Taking norm both sides with assuming $r_0 \in \mathfrak{V}(r^*, x_2)$ and utilize Lemma (3), we get

$$\|s_0 - r^*\| \le \frac{J\left(\int_0^1 |1-\theta| \|r_0 - r^*\|\right) d\theta}{1 - J_* \|r_0 - r^*\|} \|r_0 - r^*\|$$
$$\le \frac{3J\|r_0 - r^*\|}{2\left(1 - J_*\|r_0 - r^*\|\right)} \|r_0 - r^*\|. \tag{12}$$

Now from (5),

$$r_1 - r^* = r_0 - r^* - (I - G'(w_0))^{-1}(r_0 - G(r_0) - r^* + G(r^*))$$
$$= (I - G'(w_0))^{-1}\left(G(r_0) - G(r^*) - G'(w_0)(r_0 - r^*)\right)$$
$$= (I - G'(w_0))^{-1} A_* \int_0^1 A_*^{-1}\left(G'(r^* + \theta(r_0 - r^*)) - G'(w_0)\right) d\theta (r_0 - r^*).$$

Taking the norm on both sides and using Lemma 3 as $w_0 \in \mathfrak{V}(r^*, x_2)$, we get

$$\|r_1 - r^*\| \le \frac{J\left(\int_0^1 \|r^* + \theta(r_0 - r^*) - w_0\|\right) d\theta}{1 - J_*\|w_0 - r^*\|} \|r_0 - r^*\|$$
$$= \frac{J\left(\int_0^1 \left\|r^* + \theta(r_0 - r^*) - \frac{r_0 + s_0}{2}\right\|\right) d\theta}{1 - J_*\|w_0 - r^*\|} \|r_0 - r^*\|$$
$$= \frac{J\left(\int_0^1 (1 - 2\theta)(r_0 - r^*) + (s_0 - r^*)\right) d\theta}{2(1 - J_*\|w_0 - r^*\|)} \|r_0 - r^*\|$$
$$\le \frac{J\left(2\|r_0 - r^*\| + \|s_0 - r^*\|\right)}{2(1 - J_*\|w_0 - r^*\|)} \|r_0 - r^*\|. \tag{13}$$

Now,

$$s_1 - r^* = r_1 - r^* - (I - G'(w_0))^{-1}(r_1 - G(r_1) - r^* + G(r^*))$$
$$= (I - G'(w_0))^{-1}\left(G(r_1) - G(r^*) - G'(w_0)(r_1 - r^*)\right)$$
$$= (I - G'(w_0))^{-1} A_* \int_0^1 A_*^{-1}\left(G'(r^* + \theta(r_1 - r^*)) - G'(w_0)\right) d\theta (r_1 - r^*).$$

We use Lemma 3 and taking norm on both sides, this provides

$$\left\|s_1 - r^*\right\| \le \frac{J\left(\int_0^1 \left\|r^* + \theta(r_1 - r^*) - w_0\right\|\right)}{1 - J_* \left\|w_0 - r^*\right\|} \left\|r_1 - r^*\right\|$$

$$= \frac{J\left(\int_0^1 \left\|r^* + \theta(r_1 - r^*) - \frac{r_0 + s_0}{2}\right\|\right) d\theta}{1 - J_* \left\|w_0 - r^*\right\|} \left\|r_1 - r^*\right\|$$

$$= \frac{J\left(\int_0^1 \left\|r_0 - r^* + s_0 - r^* - 2\theta(r_1 - r^*)\right\|\right) d\theta}{2\left(1 - J_* \left\|w_0 - r^*\right\|\right)} \left\|r_1 - r^*\right\|$$

$$\le \frac{J\left(\left\|r_0 - r^*\right\| + \left\|s_0 - r^*\right\| + \left\|r_1 - r^*\right\|\right)}{2\left(1 - J_* \left\|w_0 - r^*\right\|\right)} \left\|r_1 - r^*\right\|. \tag{14}$$

This proves the lemma for the first term. Now, we reiterate the use of above to $n + 1$ terms along the lines derived above which provide the proof of this lemma.

Now, we present the main local convergence theorem with the help of Lemma 3 and Lemma 4.

Theorem 2. Suppose $r_0 \in \mathfrak{V}(r^*, x_2)$ and $r^* = G(r^*)$. If the conditions of $\mathfrak{L}$ hold true, then the sequences start from $r_0 \in \mathfrak{V}(r^*, x_2)$ in (5), iterate $r_k (s_k, w_k) \in \mathfrak{V}\ r^*\ x_2$ and converge to r^*, which is the solution of (2). The error derived for this equation is the same as for Lemma 4. Moreover, this solution is unique in $\left(r^*, \frac{2}{J^*}\right)$.

Proof. At first, we show that $s_0 \in \mathfrak{V}(r^*, x_2)$. From (12), we have

$$\left\|s_0 - r^*\right\| \le \frac{3J\left\|r_0 - r^*\right\|}{2\left(1 - J_* \left\|r_0 - r^*\right\|\right)} \left\|r_0 - r^*\right\|$$

$$\le t\left(\left\|r_0 - r^*\right\|\right)\left\|r_0 - r^*\right\| < \left\|r_0 - r^*\right\|.$$

This shows $s_0 \in \mathfrak{V}(r^*, x_2)$, and now

$$\left\|w_0 - r^*\right\| = \left\|\frac{r_0 + s_0}{2} - r^*\right\| \le \frac{\left\|r_0 - r^*\right\| + \left\|s_0 - r^*\right\|}{2} < \left\|r_0 - r^*\right\|,$$

which means $w_0 \in \mathfrak{V}(r^*, x_2)$. It can be easily seen that

$$\left\|r_1 - r^*\right\| < t\left(\left\|r_0 - r^*\right\|\right)\left\|r_0 - r^*\right\| < \left\|r_0 - r^*\right\|,$$

which provides $r_1 \in \mathfrak{V}(r^*, x_2)$. Continue in the same way this time with Lemma 4, we get

$$\left\|s_n - r^*\right\| \le t\left(\left\|r_n - r^*\right\|\right)\left\|r_n - r^*\right\| < \left(t\left(\left\|r_0 - r^*\right\|\right)\right)^n \left\|r_0 - r^*\right\| \text{ and}$$
$$\left\|r_{n+1} - r^*\right\| \le t\left(\left\|r_n - r^*\right\|\right)\left\|r_n - r^*\right\| < \left(t\left(\left\|r_0 - r^*\right\|\right)\right)^n \left\|r_0 - r^*\right\|.$$

For $n \to \infty$, this gives $s_n \to r^*$ and $r_n \to r^*$. Suppose s^* is another solution of (2) satisfying $s^* = G(s^*)$. Then, combining with $r^* = G(r^*)$ provides

$$\left(I - \int_0^1 G'(s^* + \theta(r^* - s^*)) d\theta\right)(r^* - s^*) = 0. \tag{15}$$

Now, take $G_{s^*} = I - \int_0^1 G'(s^* + \theta(r^* - s^*))d\theta$ and

$$\begin{aligned}\left\|I - A_*^{-1}G_{s^*}\right\| &= \left\|\int_0^1 A_*^{-1}(G'(s^* + \theta(r^* - s^*)) - G'(r^*))d\theta\right\| \\ &\leq \int_0^1 J_* \left\|(s^* + \theta(r^* - s^*)) - r^*\right\| d\theta \\ &\leq \int_0^1 J_*(1-\theta)d\theta \left\|s^* - r^*\right\| \\ &\leq \frac{J_*}{2}\left\|s^* - r^*\right\| < 1.\end{aligned}$$

This shows that G_{s^*} is invertible and hence using (15), $r^* = s^*$. This proves the theorem.

4. Semilocal convergence

In this section, we provide the semilocal convergence analysis of (5) to estimate the starting points. We require the following set of conditions, which we define as condition $\mathfrak{L}_s$.

Definition 2. Suppose $\mathcal{C}_0, J_0, J_1$ are some nonnegative parameters, then a set of elements $\mathcal{C}_0, J_0, J_1 \in \mathfrak{L}_s$, satisfy some conditions as:

$$\left.\begin{array}{c}\mathcal{C}_0^{-1} \in \mathrm{B}(\mathcal{P}), \\ \left\|\mathcal{C}_0^{-1}(r_0 - G(r_0))\right\| \leq \mathfrak{E}, \\ \left\|\mathcal{C}_0^{-1}(G'(r) - G'(r_0))\right\| \leq J_0\left\|r - r_0\right\|, \\ \left\|\mathcal{C}_0^{-1}\left(G'(r) - G'(s)\right)\right\| \leq J_1\left\|r - s\right\| \text{ where } \mathcal{C}_0 = \left(I - G'\left(r_0\right)\right), \text{ and} \left(r, s, r_0\right) \in \Omega_0\end{array}\right\} \tag{16}$$

Now, we begin the work of our semilocal convergence theorem using the following lemma:

Lemma 5. Let α_k and β_k be two sequences defined by $\alpha_0 = 0, \beta_0 = \mathfrak{E}, \alpha_1 = \dfrac{\mathfrak{E}}{1 - \dfrac{J_0\mathfrak{E}}{2}}$, and after that

$$\beta_k = \alpha_k + \frac{J(\beta_{k-1} - \alpha_{k-1} + \alpha_k - \alpha_{k-1})}{2(1 - \dfrac{J_0(\alpha_{k-1} + \beta_{k-1})}{2})}(\alpha_k - \alpha_{k-1}), \tag{17}$$

$$\alpha_{k+1} = \alpha_k + \frac{J(\beta_{k-1} - \alpha_{k-1} + \alpha_k - \alpha_{k-1})}{2(1 - \dfrac{J_0(\alpha_k + \beta_k)}{2})}(\alpha_k - \alpha_{k-1}). \tag{18}$$

Suppose η is the smallest positive root of the equation

$$q(a) = \frac{J_0}{2}a^3 + \frac{J_0}{2}a^2 + J(a-1), \tag{19}$$

satisfying

$$0 < \frac{J(\beta_0 + \alpha_1)}{2(1 - \frac{J_0(\alpha_1 + \beta_1)}{2})} \leq \eta \leq 1 - J_0\alpha_1. \tag{20}$$

Then, (17) and (18) follow

$$\beta_k - \alpha_k \leq \eta(\alpha_k - \alpha_{k-1}) \tag{21}$$

$$\alpha_{k+1} - \alpha_k \leq \eta(\alpha_k - \alpha_{k-1}). \tag{22}$$

Further, the sequences β_k and α_k are bounded above by $\alpha_* = \frac{\alpha_1}{1-\eta}$ and $\beta_k \leq \alpha_{k+1}$.

Proof. First, we show the existence of η. So, from (19), $q(0) = -J$ and $q(1) = J_0$, it follows that $q(a)$ has at least positive root in (0,1). We denote the same by η. From the construction of sequences, we have to show

$$\frac{J(\beta_{k-1} - \alpha_{k-1} + \alpha_k - \alpha_{k-1})}{2(1 - \frac{J_0(\alpha_{k-1} + \beta_{k-1})}{2})} \leq \eta, \tag{23}$$

and

$$\frac{J(\beta_{k-1} - \alpha_{k-1} + \alpha_k - \alpha_{k-1})}{2(1 - \frac{J_0(\alpha_k + \beta_k)}{2})} \leq \eta. \tag{24}$$

In order to show both (23) and (24), it is sufficient to show (24) as

$$\frac{J(\beta_{k-1} - \alpha_{k-1} + \alpha_k - \alpha_{k-1})}{2(1 - \frac{J_0(\alpha_k + \beta_k)}{2})} > \frac{J(\beta_{k-1} - \alpha_{k-1} + \alpha_k - \alpha_{k-1})}{2(1 - \frac{J_0(\alpha_{k-1} + \beta_{k-1})}{2})}.$$

Now, reiterate the term of (21) and (22), this gives

$$\beta_k < \alpha_k + \eta^k\alpha_1 \text{ and } \alpha_{k+1} < \alpha_k + \eta^k\alpha_1,$$

which together gives

$$\alpha_{k+1} \leq \alpha_k + \eta^k\alpha_1 \leq \alpha_{k-1} + \eta^{k-1}\alpha_1 + \eta^k\alpha_1 \leq \left(1 + \eta + \eta^2 + \ldots + \eta^k\right)\alpha_1 = \frac{1-\eta^{k+1}}{1-\eta}\alpha_1. \tag{25}$$

Clearly, $\beta_k < \alpha_{k+1}$. Therefore, (24) it is true if

$$\frac{J}{2}(\eta^{k-1}\alpha_1 + \eta^{k-1}\alpha_1) \leq \eta\left(1 - \frac{J_0}{2}(\beta_k + \alpha_k)\right),$$

which holds true if

$$J\eta^{k-1}\alpha_1 + \frac{J_0}{2}\eta\left(\frac{1-\eta^{k+1}}{1-\eta} + \frac{1-\eta^k}{1-\eta}\right)\alpha_1 - \eta \leq 0.$$

This motivates us to introduce the function p_k depending upon 'k' by

$$p_{k-1}(\alpha) = J\alpha^{k-2}\alpha_1 + \frac{J_0}{2}\left(1+\alpha+\ldots+\alpha^k+1+\alpha+\ldots+\alpha^{k-1}\right)\alpha_1 - 1.$$

We write k in place of $k-1$ and using some algebraic arrangement, this provides

$$p_k(\alpha) = p_{k-1}(\alpha) + q(\alpha)\alpha^{k-2}\alpha_1,$$

letting $p_k(\eta) = p_{k-1}(\eta)$. Further, we define p_∞ on (0, 1) by

$$p_\infty(\eta) = \lim_{k\to\infty} p_k(\eta).$$

So, the above is true if $p_\infty(\eta) \le 0$. This implies

$$J_0\frac{\eta}{1-\eta}\alpha_1 - \eta \le 0,$$

which provides $\eta \le 1 - J_0\alpha_1$. This gives the right-hand inequality of (20) and from the use of (24), we obtain the left-side inequality of (20). By (25), we get the upper bound α_* for the sequences β_k and α_k by approaching $k \to \infty$. This proves the lemma.

Now, we construct another lemma to establish a relationship between the sequences $\{\alpha_k\}$, $\{\beta_k\}$, and terms of (5).

Lemma 6. The following holds true:

(a) $\|s_n - r_n\| \le \beta_n - \alpha_n$,

(b) $\|(I - G'(w_n))^{-1}\mathcal{C}_0\| \le \dfrac{1}{1-\dfrac{J_0(\alpha_n+\beta_n)}{2}}$,

(c) $\|r_{n+1} - r_n\| \le \alpha_{n+1} - \alpha_n$.

Proof. We use the condition $\mathfrak{L}_s$ and Lemma 5 to show this lemma. For $n = 0$, we have $\|s_0 - r_0\| \le \mathfrak{E} = \beta_0 - \alpha_0$. This shows $s_0 \in \mathfrak{V}(r_0, \alpha_*)$, and we can write $\|s_0 - r_0\| \le \alpha_* - \alpha_0$. Now, $w_0 - r_0 = \dfrac{s_0 - r_0}{2}$ and so, $w_0 \in \mathfrak{V}(r_0, \alpha_*)$. This gives

$$\begin{aligned}\|I - \mathcal{C}_0^{-1}(I - G'(w_0))\| &= \|\mathcal{C}_0^{-1}(G'(w_0) - G'(r_0))\| \\ &\le J_0\|w_0 - r_0\| \le \frac{J_0\mathfrak{E}}{2} = \frac{J_0(\beta_0+\alpha_0)}{2}.\end{aligned}$$

From Banach lemma, it gives

$$\|(I - G'(w_0))^{-1}\mathcal{C}_0\| \le \frac{1}{1-\dfrac{J_0(\alpha_0+\beta_0)}{2}}. \tag{26}$$

Now from (5),

$$r_1 - r_0 = -(I - G'(w_0))^{-1}\mathcal{C}_0\mathcal{C}_0^{-1}(r_0 - G(r_0)).$$

Taking the norm on both sides and use of $\mathfrak{L}_s$ and (26), we get

$$\|r_1-r_0\| \le \|(I-G'(w_0))^{-1}\mathcal{C}_0\| \|\mathcal{C}_0^{-1}(r_0-G(r_0))\| \le \frac{\mathfrak{E}}{1-\frac{J_0(\alpha_0+\beta_0)}{2}} = \alpha_1-\alpha_0.$$

This proves the lemma for $n = 0$. Suppose this validates for $n \le k-1$. For $n = k$, we have

$$\begin{aligned} r_k-G(r_k) &= r_k-G(r_k)-(r_{k-1}-G(r_{k-1}))+(r_{k-1}-G(r_{k-1})) \\ &= r_k-G(r_k)-(r_{k-1}-G(r_{k-1}))-(I-G'(w_{k-1}))(r_k-r_{k-1}) \\ &= \int_0^1 \left(G'(w_{k-1})-G'(r_{k-1}+\theta(r_k-r_{k-1}))\right)d\theta(r_k-r_{k-1}). \end{aligned}$$

Multiplying both sides by $\mathcal{C}_0^{-1}$ and taking the norm provides

$$\begin{aligned} \|\mathcal{C}_0^{-1}(r_k-G(r_k))\| &\le J\int_0^1 \left\|\frac{r_{k-1}+s_{k-1}}{2}-r_{k-1}-\theta(r_k-r_{k-1})\right\| d\theta \|r_k-r_{k-1}\| \\ &\le \frac{J}{2}\int_0^1 \left(\|s_{k-1}-r_{k-1}\|+2\theta\|r_k-r_{k-1}\|\right)d\theta\|r_k-r_{k-1}\| \\ &\le \frac{J}{2}(\beta_{k-1}-\alpha_{k-1}+\alpha_k-\alpha_{k-1})(\alpha_k-\alpha_{k-1}). \end{aligned} \tag{27}$$

Further,

$$s_k-r_k = -(I-G'(w_{k-1}))^{-1}\mathcal{C}_0\mathcal{C}_0^{-1}(r_k-G(r_k)).$$

We use (27) and taking the norm on both sides, we get

$$\begin{aligned} \|s_k-r_k\| &\le \|(I-G'(w_{k-1}))^{-1}\mathcal{C}_0\| \, \|\mathcal{C}_0^{-1}(r_k-G(r_k))\| \\ &\le \frac{J(\beta_{k-1}-\alpha_{k-1}+\alpha_k-\alpha_{k-1})}{2(1-\frac{J_0(\alpha_{k-1}+\beta_{k-1})}{2})}(\alpha_k-\alpha_{k-1}) = \beta_k-\alpha_k \end{aligned}$$

and

$$\begin{aligned} \|s_k-r_0\| &\le \|s_k-r_k\|+\|r_k-r_{k-1}\|+\ldots+\|r_1-r_0\| \\ &\le \beta_k-\alpha_k+\alpha_k-\alpha_{k-1}+\ldots+\alpha_1-\alpha_0 = \beta_k. \end{aligned}$$

This shows that $s_k \in \mathfrak{V}(r_0,\alpha_*)$, and similarly for $r_k \in \mathfrak{V}(r_0,\alpha_*)$ and $\|r_k-r_0\| \le \alpha_k-\alpha_0$. Now, we show the existence of $\|(I-G'(w_k))^{-1}\|$. For

$$\begin{aligned} \|I-\mathcal{C}_0^{-1}(I-G'(w_k))\| &= \|\mathcal{C}_0^{-1}(G'(w_k)-G'(r_0))\| \\ &\le J_0\|w_k-r_0\| \\ &\le \frac{J_0}{2}\left(\|r_k-r_0\|+\|s_k-r_0\|\right) \\ &= \frac{J_0(\alpha_k+\beta_k)}{2}. \end{aligned}$$

By the use of Banach lemma, it gives

$$\left\|(I-G'(w_k))^{-1}\mathcal{C}_0\right\| \le \frac{1}{1-\frac{J_0(\alpha_k+\beta_k)}{2}}. \tag{28}$$

From the above, we have

$$r_{k+1}-r_k = -(I-G'(w_k))^{-1}\mathcal{C}_0\mathcal{C}_0^{-1}(r_k - G(r_k)).$$

Taking the norm on both sides and use of $\mathfrak{L}_s$ and (28), we get

$$\begin{aligned}\|r_{k+1}-r_k\| &\le \left\|(I-G'(w_k))^{-1}\mathcal{C}_0\right\|\left\|\mathcal{C}_0^{-1}(r_k-G(r_k))\right\| \\ &\le \frac{J(\beta_{k-1}-\alpha_{k-1}+\alpha_k-\alpha_{k-1})}{2(1-\frac{J_0(\alpha_k+\beta_k)}{2})}(\alpha_k-\alpha_{k-1}) = \alpha_{k+1}-\alpha_k,\end{aligned}$$

and

$$\begin{aligned}\|r_{k+1}-r_0\| &\le \|r_{k+1}-r_k\| + \|r_k - r_{k-1}\| + \ldots + \|r_1 - r_0\| \\ &\le \alpha_{k+1}-\alpha_k+\alpha_k-\alpha_{k-1}+\ldots+\alpha_1-\alpha_0 = \alpha_{k+1}.\end{aligned}$$

This shows that $r_{k+1} \in \mathfrak{V}(r_0, \alpha_*)$. This proves the lemma.

Next, we present the main semilocal convergence theorem for (5) to derive the fixed points of operator G satisfying $r^* = G(r^*)$.

Theorem 3. Suppose $G: \Omega_0 \subset \mathcal{P} \to \mathcal{P}$ is a fixed point operator and the conditions of Lemma 5 and Lemma 6 hold true. Then, starting with $r_0 \in \mathfrak{V}(r_0, \alpha_*) \subset \Omega_0$, the algorithm (5) converges to the fixed point of G satisfying $r^* = G(r^*)$. Again, r^*, r_k, and s_k remain in $\mathfrak{V}(r_0, \alpha_*)$ and hold the error equation

$$\|r_n - r^*\| \le \alpha_* - \alpha_n. \tag{29}$$

Furthermore, r^* remains the unique solution of (2) in $\Omega_0 \cap \mathfrak{V}(r_0, \frac{2}{J_0})$.

Proof. From Lemma 6, it is easy to see that $\{r^*, r_k, s_k\} \in \mathfrak{V}(r_0, \alpha_*)$. First, we show that for each 'n', $\mathfrak{V}(r_n, \alpha_* - \alpha_n) \subset \mathfrak{V}(r_{n-1}, \alpha_* - \alpha_{n-1})$. Let $z \in \mathfrak{V}(r_1, \alpha_* - \alpha_1)$, then

$$\|z - r_0\| \le \|z - r_1\| + \|r_1 - r_0\| \le \alpha_* - \alpha_1 + \alpha_1 - \alpha_0 = \alpha_* - \alpha_0.$$

Therefore, $\mathfrak{V}(r_1, \alpha_* - \alpha_1) \subset \mathfrak{V}(r_0, \alpha_* - \alpha_0)$ and $\|r_1 - r^*\| \le \alpha_* - \alpha_1$. Following in the same way, we can show that $\mathfrak{V}(r_n, \alpha_* - \alpha_n) \subset \mathfrak{V}(r_{n-1}, \alpha_* - \alpha_{n-1}) \ldots \subset \mathfrak{V}(r_1, \alpha_* - \alpha_1) \subset \mathfrak{V}(r_0, \alpha_* - \alpha_0)$. The convergence of the scalar sequences α_n and β_n indicates the convergence of r_n from the last lemmas. Now, we show that the uniqueness of r^*. Let s^* be another solution of (2) satisfying $s^* = G(s^*)$. Then, combined with $r^* = G(r^*)$, which provides

$$\left(I - \int_0^1 G'(s^* + \theta(r^* - s^*))d\theta\right)(r^* - s^*) = 0.$$

Now, taking $G_{s_c} = I - \int_0^1 G'(s^* + \theta(r^* - s^*))d\theta$ and

$$\left\| I - \mathcal{C}_0^{-1} G_{s_c} \right\| = \left\| \int_0^1 \mathcal{C}_0^{-1} (G'(s^* + \theta(r_0 - s^*)) - G'(r_0)) d\theta \right\|$$

$$\leq \int_0^1 J_0 \left\| s^* + \theta(r_0 - s^*) - r_0 \right\| d\theta$$

$$\leq \int_0^1 J_0 (1 - \theta) d\theta \left\| s^* - r_0 \right\|$$

$$\leq \frac{J_0}{2} \left\| s^* - r_0 \right\| < 1.$$

This shows that G_{sc} is invertible and hence, $r^* = s^*$. This proves the theorem.

5. Numerical examples

In this section, we provide some numerical examples to validate the results obtained by us in the previous theorems. We begin with some numerical examples to validate the local convergence theorems derived in Section 3. We shall also compare the radii of convergence balls with the earlier method, which holds the same number of function evaluations [21].

Example 1. Consider the function on $\Omega_0 = \mathcal{B}(0,1)$, by

$$G(r) = e^r - r - 1.$$

Example 2. Consider the function on $\Omega_0 = \mathbb{R}^3$ in the form $G(r) = r + F(r)$, by

$$F(a,b,c) = \left(e^a - 1 - \frac{e-1}{b}^2 + b, c \right)^T.$$

Example 3. [28] Consider Planck's radiation law problem, which calculates the energy density within an isothermal body and is given by

$$A(\aleph) = \frac{8\pi c P \aleph^{-5}}{e^{\frac{cP}{\aleph BT}} - 1}$$

where the constants have their respective meanings. In order to optimize $A(\aleph)$, this reduces the problem of finding the solution to the equation

$$G(r) = e^{-r} + \frac{r}{5} - 1.$$

We consider here $\Omega_0 = [-1, 1]$.

The list of the parameter appears in all these examples and the radii of balls derived from Theorem 2 are tabulated in Table 1. A comparison of radii of convergence balls with similar methods is also given in this table. We achieved an improved radii of convergence balls in all of the examples. Now, we provide two examples to validate the results of examining the semilocal convergence results derived in Section 4.

Table 1. Radii of convergence balls and a comparison

Example	J	J_*	r^*	Method 5	Method 3
1	e	$e-1$	0	0.1725	0.0774
2	$e-1$	$e\frac{1}{(e^2-1)}$	(0, 0, 0)	0.2269	0.0883
3	$\frac{5}{4}e$	$\frac{5}{4}(e-1)$	0	0.16668	0.0658

Example 4. Suppose C[0, 1], the space of continuous function defined on [0,1] equipped with the sup-norm. Let $\Omega_0 = \{r \in C[0,1]; \|r\| \leq \lambda\}$, such that $\lambda > 0$ and consider the problem

$$r(a) = f(a) + \xi \int_x^y \mathcal{K}(a,b) r(a)^3 \, db$$

that leads to solving the problem

$$G(r)a = f(a) + \xi \int_x^y \mathcal{K}(a,b) r(a)^3 \, db.$$

Here, $\mathcal{K}(a,b)$ denotes the Green's function. This gives

$$[G'(r)(s)](a) = 3\xi \int_x^y \mathcal{K}(a,b) r(a)^2 s(b) db,$$

choose $r_0(a) = f(a) = 1$, it follows that

$$\|G'(r)\| \leq \frac{3\xi}{8} \text{ as } \left\| \int_0^1 \mathcal{K}(a,b) db \right\| = \frac{1}{8}$$

and using Banach lemma,

$$\|\mathcal{C}_0^{-1}\| = \|(I - G'(r_0))^{-1}\| \leq \frac{8}{8-3|\xi|}.$$

Now,

$$r_0 - G(r_0) = r_0(a) - f(s) - \xi \int_0^1 \mathcal{K}(a,b) r_0(a)^3 \, db,$$

which provides

$$\|r_0 - G(r_0)\| \leq \frac{|\xi|}{8}.$$

Thus, $\mathfrak{E} = \frac{|\xi|}{8-3|\xi|}$. On the other hand,

$$\begin{aligned}\|\mathcal{C}_0^{-1}(G'(r) - G'(s))\| &\leq \|\mathcal{C}_0^{-1}\| \frac{3|\xi|}{8} \|r(a) + s(a)\| \|r(a) - s(a)\| \\ &\leq \frac{6\lambda|\xi|}{8-3|\xi|} \|r - s\|,\end{aligned}$$

and

$$\left\| \mathcal{C}_0^{-1}(G'(r) - G'(r_0)) \right\| \le \left\| \mathcal{C}_0^{-1} \right\| \frac{3|\xi|}{8} \left\| r(a) + r_0(a) \right\| \left\| r(a) - r_0(a) \right\|$$
$$\le \frac{3(1+\lambda)|\xi|}{8 - 3|\xi|} \left\| r - r_0 \right\|.$$

We choose $\lambda = 2$ and $\xi = 0.5$, this provides $J = 0.92308$, $J_0 = 0.69231$, and $\mathfrak{C} = 0.076923$. Now, the condition of Theorem 3 is true as

$$\frac{J(\beta_0 + \alpha_1)}{2(1 - \frac{J_0(\alpha_1 + \beta_1)}{2})} = 0.07826220153 \le \eta = 0.69401975 \le 0.94528875 = 1 - J_0 \alpha_1.$$

Thus, the solution exists in $\mathcal{B}(r_0, 0.25827600)$ and is unique in $\overline{\mathcal{B}}(r_0, 2.8889) \cap \Omega_0$. Now, we compute the errors of the scalar sequences $\{\alpha_{k+1} - \alpha_k\}$ and $\{\beta_k - \alpha_k\}$ appear in Lemma 5, which is given in Table 2.

Table 2. Error of scalar sequences for Example 4

k	$\{\alpha_{k+1} - \alpha_k\}$	$\{\beta_k - \alpha_k\}$
0	0.079027	0.076923
1	0.006030	0.0058438
2	$3.5116e-05$	$3.5036e-05$
3	$1.2082e-09$	$1.2081e-09$

Example 5. Consider Chandrasekhar's type integral equations that arise in the theory of radiative transfer, neutron transport, the kinetic theory of gases, etc.

$$r(a) = f(a) + \xi \int_0^1 \frac{a}{a+b} r(a)^3 \, db,$$

that leads to solving the problem

$$G(r)a = f(a) + \xi \int_0^1 \frac{a}{a+b} r(a)^3 \, db.$$

Now,

$$[G'(r)(s)](a) = 3\xi \int_0^1 \frac{a}{a+b} r(a)^2 s(b) db,$$

choose $r_0(a) = f(a) = 1$, it follows that

$$\left\| G'(r_0) \right\| \le \frac{3\xi}{8} \text{ as } \left\| \int_0^1 \frac{a}{a+b} db \right\| = \log(2)$$

and using Banach lemma,

$$\left\| \mathcal{C}_0^{-1} \right\| = \left\| (I - G'(r_0))^{-1} \right\| \leq \frac{1}{1 - 3\log(2) \mid \xi \mid}.$$

Now,

$$r_0 - G(r_0) = r_0(a) - f(s) - \xi \int_0^1 \frac{a}{a+b} r_0(a)^3 \, db,$$

which provides

$$\left\| r_0 - G(r_0) \right\| \leq \mid \xi \mid \log(2).$$

Thus, $\mathfrak{E} = \frac{\mid \xi \mid \log(2)}{1 - 3 \mid \xi \mid \log(2)}$. On the other hand,

$$\begin{aligned} \left\| \mathcal{C}_0^{-1}(G'(r) - G'(s)) \right\| &\leq \left\| \mathcal{C}_0^{-1} \right\| 3 \mid \xi \mid \log(2) \left\| r(a) + s(a) \right\| \left\| r(a) - s(a) \right\| \\ &\leq \frac{6\lambda \mid \xi \mid \log(2)}{1 - 3 \mid \xi \mid \log(2)} \left\| r - s \right\|, \end{aligned}$$

and

$$\begin{aligned} \left\| \mathcal{C}_0^{-1}(G'(r) - G'(r_0)) \right\| &\leq \left\| \mathcal{C}_0^{-1} \right\| 3 \mid \xi \mid \log(2) \left\| r(a) + r_0(a) \right\| \left\| r(a) - r_0(a) \right\| \\ &\leq \frac{3(1+\lambda) \mid \xi \mid \log(2)}{1 - 3 \mid \xi \mid \log(2)} \left\| r - r_0 \right\|. \end{aligned}$$

We choose $\lambda = 1.45$ and $\xi = 0.15$, this provides $J = 1.3146$, $J_0 = 0.1106$, and $\mathfrak{E} = 0.1511$. Now, the condition of Theorem 3 is true as

$$\frac{J(\beta_0 + \alpha_1)}{2\left(1 - \frac{J_0(\alpha_1 + \beta_1)}{2}\right)} = 0.3006846 \leq \eta = 0.6762217 \leq 0.8168109 = 1 - J_0 \alpha_1.$$

Thus, the solution exists in $\mathcal{B}(r_0, 0.5094352)$ and is unique in $\overline{\mathcal{B}}(r_0, 1.8008) \cap \Omega_0$. Now, we compute the errors of the scalar sequences $\{\alpha_{k+1} - \alpha_k\}$ and $\{\beta_k - \alpha_k\}$ appear in Lemma 5, which is given in Table 3.

Table 3. Error of scalar sequences for Example 5

k	$\{\alpha_{k+1} - \alpha_k\}$	$\{\beta_k - \alpha_k\}$
0	0.16494	0.15112
1	0.04304	0.03740
2	0.002966	0.0028593
3	$1.4832e - 05$	$1.4799e - 05$
4	$3.7726e - 10$	$3.7726\ e - 10$

Remark 1. It is worth mentioning that the method presented here is one of the most efficient iterative methods for

fixed points available in the literature. The efficiency index of Method 1 is the same as the efficiency index of Method 5, which is one of the best efficient iterative methods. So, we do not compare here for the efficiency index. However, a detailed study of the computational order of convergence and its efficiency for the nonlinear system of equations is to be done in the future.

Remark 2. In order to compare the performance of any iterative method, we use the term efficiency index [29] to compare the efficiencies of different iterative methods. In the classical sense, it is given by $EI = \rho^{1/g}$, where ρ is the order of convergence and g is the number of function evaluations. For this purpose, we take the total number of function evaluations that appear in Method 3, Method 4, and our suggested approach Method 5 in Table 4.

Table 4. A comparison of the efficiency index

Methods	g	ρ	EI
Method 3	2	2	$2^{1/2} = 1.414$
Method 4	3	3	$3^{1/3} = 1.442$
Method 5	2	$1+\sqrt{2}$	$(1+\sqrt{2})^{1/2} = 1.553$

6. Conclusion

Approximating the fixed points of an operator is a very popular area of research. In this paper, we have provided an efficient iterative method for approximating the fixed points of a nonlinear operator. Moreover, we provide a finer convergence analysis of this method by using the condition on the first Frechet derivative instead of the condition on the higher-order Frechet derivative. Our contribution to this can be listed as follows:

- We convert an iterative method for nonlinear operator equations to an efficient iterative method for approximating the fixed point of an operator.
- We have provided the order of convergence of this method using Taylor's series approximations.
- We have provided a finer convergence analysis and have not used the condition on higher-order derivatives recently used by others [10-13] for this type of method.
- We have provided the local as well as the semilocal convergence analysis and compared the radii of convergence balls to a recent study of the well-known quadratically convergent Striling's method.
- We also provide the error terms at each step for both local and semilocal convergence analyses.
- In Remark 2, we compare the efficiency index of the suggested iterative methods with their competitors in the classical sense given by Ostrowski [29].

In the future, we will try to develop new iterative methods for fixed-point problems and utilize them to solve integral equations, differential equations, boundary value problems, and initial value problems that arise in the mathematical modeling of different real-life applications. Moreover, the dynamical study is also to be performed in the future. In this way, we have presented an effective study.

Conflict of interest

The authors declare no conflicts of interest in this article.

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