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Research Article

# Global Stability of Non-critical Traveling Fronts for a Belousov-Zhabotinskii Model with Time Delay

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**Abstract:** This paper is concerned with the traveling fronts of a Belousov-Zhabotinskii system with a time delay. The stability of the traveling fronts with large speeds is proved by Meng et al. [1]. However, the stability of all waves, including the slower waves (i.e., the wave speed near the critical wave speed), for such a system is unsolved. In this paper, we show that all traveling fronts with non-critical wave speeds are exponentially asymptotically stable. The exponential convergent rate is also obtained.

***Keywords*:** Belousov-Zhabotinskii model, traveling fronts, exponential stability

**MSC:** 35C07, 92D25, 35B35

## 1. Introduction

It is well known that there is a phenomenon of delayed effects in the generation process of bromic acid. To model the chemical phenomenon, Wu and Zou [2] proposed and studied the Belousov-Zhabotinskii system as follows:

$$\begin{cases} u_t = u_{yy} + u(y,\ t)[1 - u(y,\ t) - rw(y,\ t - \imath)], \\ \\ w_t = w_{yy} - bu(y,\ t)w(y,\ t), \end{cases} \tag{1}$$

where $(y,\ t) \in \mathbb{R} \times \mathbb{R}_+$, $u(y,\ t)$ denotes the concentration of bromic acid and $w(y,\ t)$ denotes the concentration of bromide ion. The constants b and r are positive, and $\imath > 0$ corresponds to the time delay. In this paper, we are interested in the study of the traveling fronts of (1). From a chemical reaction point of view, the traveling wave solutions can describe the movement of the bromic acid concentration from a higher region to a lower region (see [2]). In recent years, the existence of the traveling wavefronts for system (1) with or without time delay has been extensively studied; see, e.g., [3] for the case without time delay and [1, 4–7] for the case with delay.

It is well known that the stability of traveling wave solutions is an important topic in the theory of traveling wave solutions; see [8–14]. Recently, by using the weighted energy method, the authors in [1] derived the stability of traveling

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fronts with $largespeeds$ for model (1). However, the stability of all waves, including the slower waves (i.e., the wave speed near the critical wave speed), for system (1) is unsolved. As pointed out by Mei et al. [12], the stability of slower wavefronts is much more interesting and difficult. The main purpose of this paper is to derive the stability of all traveling fronts with speeds $c > c_*$ (non-critical traveling fronts for short) for model (1), where $c_*$ is the critical speed (see Proposition 2.1 below).

We would like to mention that the approach with a piecewise weight function used in [1] cannot be applied to show the stability of the traveling waves with speeds near the critical wave speed. This is because a large upper bound appears due to the use of the piecewise weight function. To eliminate this large upper bound, one needs to assume that the wave speed is sufficiently large. To overcome this shortcoming, in this paper, we shall select an appropriate *non-piecewise* weight function. The approach is inspired by the work of [12]. With this choice of the weight function and some technical analysis, we shall show that all non-critical traveling fronts of system (1) are exponentially asymptotically stable. The exponential convergent rate is also obtained.

## 2. Preliminaries and main result

In this section, we give some preliminaries and state our main result. We first introduce the following definitions:

(a) Assume that $H^k(E)(k \geq 0)$ corresponds to the Sobolev space of the $L^2$-functions $\gamma(y)$ defined on $E$, the $i$th-derivative $\gamma^i(y) \in L^2(E)$ for $i = 1, \ldots, k$. Let $H^k_{\varpi}(E)$ denote the weighted Sobolev space, which endows with the norm

$$\|\gamma(y)\|_{H^k_{\varpi}(E)} = \left( \sum_{i=0}^{k} \int_E \varpi(y) \left| \frac{d^i \gamma(y)}{dy^i} \right|^2 dy \right)^{\frac{1}{2}},$$

where $\varpi(y)$ denotes the weight function.

(b) Assume that the constant $\alpha > 0$, and $\mathbb{B}$ is a Banach space. $C([0, \alpha]; \mathbb{B})$ corresponds to the space of the $\mathbb{B}$-valued continuous functions defined on $[0, \alpha]$.

In order to apply the comparison theorem to system (1), model (1) needs to be transformed into the following system (2) by taking $v_1 = u,\ v_2 = 1 - w$.

$$\begin{cases} (v_1)_t = (v_1)_{yy} + v_1(y, t)\left[1 - r - v_1(y, t) + rv_2(y, t - \iota)\right], \\ (v_2)_t = (v_2)_{yy} + bv_1(y, t)\left[1 - v_2(y, t)\right], \end{cases} \tag{2}$$

which is a cooperative system. It is obvious that the equilibria (0, 1) and (1, 0) of system (1) become (0, 0) and (1, 1), respectively.

As usual, a traveling wave solution of (2) refers to a solution $v(y, t) = (v_1(y, t), v_2(y, t))$ with the form: $v(y, t) = \Phi(\eta) := (\phi_1(\eta), \phi_2(\eta)),\ \eta = y + ct$ where $c$ corresponds to the wave speed. It is clear that the function $\Phi(\eta) = (\phi_1(\eta), \phi_2(\eta))$ satisfies satisfies

$$\begin{cases} c\phi_1'(\eta) = \phi_1''(\eta) + \phi_1(\eta)\left[1 - r - \phi_1(\eta) + r\phi_2(\eta - cl)\right], \\ c\phi_2'(\eta) = \phi_2''(\eta) + b\phi_1(\eta)\left[1 - \phi_2(\eta)\right]. \end{cases} \tag{3}$$

From the first equation corresponding to (3), we can obtain the characteristic equation about the equilibrium (0, 0) as follows:

$$\Delta(\mu,\ c) := \mu^2 - c\mu + 1 - r. \tag{4}$$

It is not difficult to verify that there exists $\mu_* > 0$ satisfying the equations $\Delta(\mu_*,\ c_*) = 0$ and $\frac{\partial}{\partial \mu}\Delta(\mu,\ c_*)\Big|_{\mu=\mu_*} = 0$, where $c_* = 2\sqrt{1-r}$.

Based on the results of [2, 6, 7], we have the following result on the existence of the traveling fronts of (2).

**Proposition 2.1** Suppose that the parameters $r$ and $b$ satisfy $0 < r < 1,\ b \in (0,\ 1-r)$ respectively. Then, for any $c \geq c_*$ and $l > 0$, system (2) has an increasing traveling wave $\Phi(\eta) = (\phi_1(\eta),\ \phi_2(\eta))$ with speed $c$ (traveling front for short) connecting (0, 0) and (1, 1).

To obtain the stability result of the non-critical traveling fronts $\Phi(\eta) = (\phi_1(\eta),\ \phi_2(\eta))$ of system (2) constrained by the initial value:

$$v_{10}(y) = v_1(y,\ 0),\ v_{20}(y,\ s) = v_2(y,\ s),\ y \in \mathbb{R},\ s \in [-l,\ 0], \tag{5}$$

let $r$ and $b$ satisfy the technical assumption as follows.

(A) $0 < r < \dfrac{2}{3},\ 0 < b < 1-r$.

Now, the expression of the function $F(\eta)$ is given by

$$F(\eta) = (4 - r + 2b)\phi_1(\eta) - 5r - 2b. \tag{6}$$

From the assumption (A), it follows that $\lim\limits_{\eta\to\infty} F(\eta) = 4 - r + 2b - 5r - 2b = 4 - 6r > 0$.

Then, we can check easily that there is a large enough parameter $\eta_0$ such that $F(\eta_0) = (4-r+2b)\phi_1(\eta_0) - 5r - 2b > 0$.

According to the above constants $\mu_*$ and $\eta_0$, a weight function $\varpi_*(\eta)$ is expressed as follows:

$$\varpi_*(\eta) = e^{-\mu_*(\eta-\eta_0)}. \tag{7}$$

Now, the main conclusion of our study can be stated by the following Theorem 2.1.

**Theorem 2.1** Suppose that the condition (A) is satisfied and $\Phi(y+ct) = (\phi_1(y+ct),\ \phi_2(y+ct))$ is a given trav with speed $c > c_*$, if the condition (5) satisfies $(0,\ 0) \leq (v_{10}(y),\ v_{20}(y,\ s)) \leq (1,\ 1),\ y \in \mathbb{R}, s \in [-l,\ 0]$ and

$$v_{10}(y) - \phi_1(y) \in H^1_{\varpi_*}(\mathbb{R}) \subset C(\mathbb{R}) \text{ and } v_{20}(y,\ s) - \phi_2(y+cs) \in C\left([-l,0];\ H^1_{\varpi_*}(\mathbb{R})\right).$$

Then, the problem (2) subjected to (5) has a unique solution $v(y,\ t) = (v_1(y,\ t),\ v_2(y,\ t))$ satisfying $(0,\ 0) \leq (v_1(y,\ t),\ v_2(y,\ t)) \leq (1,\ 1),\ t > 0,\ y \in \mathbb{R}$ and

$$v_i(y,\ t) - \phi_i(y+ct) \in C\left([0,\ +\infty);\ H^1_{\varpi_*}(\mathbb{R})\right) \cap L^2\left([0,\ +\infty);\ H^1_{\varpi_*}(\mathbb{R})\right),\ i = 1,\ 2.$$

In addition, there exist two parameters $C > 0$ and $\lambda > 0$, such that the following inequality holds.

$$\sup_{y\in\mathbb{R}} \|v(y,\ t) - \Phi(y+ct)\| \le Ce^{-\lambda t},\ \forall t > 0.$$

## 3. Proof of Theorem 2.1

The global existence and uniqueness of the solution and comparison principle for the problem (2), subjected to initial value (5), can be proved by using the theory associated with abstract functional differential equations (c.f. [5]); see, e.g., Meng et al. ([1], Proposition 2.1). Throughout this section, we always assume that (A) holds.

From the result of [1], one has

$$(0,\ 0) \le (v_{10}(y),\ v_{20}(y,\ s)) \le (1,\ 1),\ y \in \mathbb{R},\ s \in [-\tau,\ 0],\ v_{10}(y) - \phi_1(y) \in H^1_{\varpi_*}(\mathbb{R}) \subset C(\mathbb{R}),$$

$$v_{20}(y,\ s) - \phi_2(y+cs) \in C\left([-l,\ 0];\ H^1_{\varpi_*}(\mathbb{R})\right).$$

Now, we define

$$\begin{cases} v_{10}^-(y) = \min\left\{v_{10}(y),\ \phi_1(y)\right\},\ v_{10}^+(y) = \max\left\{v_{10}(y),\ \phi_1(y)\right\},\ y \in \mathbb{R}, \\ v_{20}^-(y,\ s) = \min\left\{v_{20}(y,\ s),\ \phi_2(y+cs)\right\},\ y \in \mathbb{R},\ s \in [-\iota,\ 0], \\ v_{20}^+(y,\ s) = \max\left\{v_{20}(y,\ s),\ \phi_2(y+cs)\right\},\ y \in \mathbb{R},\ s \in [-\iota,\ 0], \end{cases}$$

which can lead to

$$\begin{cases} 0 \le v_{10}^-(y) \le v_{10}(y),\ \phi_1(y) \le v_{10}^+(y) \le 1,\ y \in \mathbb{R}, \\ 0 \le v_{20}^-(y,\ s) \le v_{20}(y,\ s),\ \phi_2(y+cs) \le v_{20}^+(y,\ s) \le 1,\ y \in \mathbb{R},\ s \in [-\iota,\ 0]. \end{cases}$$

Assume that $\left(v_1^-(y,\ t),\ v_2^-(y,\ t)\right)$ and $\left(v_1^+(y,\ t),\ v_2^+(y,\ t)\right)$ are the positive solutions of system (2) constrained by the conditions $\left(v_{10}^-(y),\ v_{20}^-(y,\ s)\right)$ and $\left(v_{10}^+(y),\ v_{20}^+(y,\ s)\right)$, respectively. Then, by employing the comparison theorem established in [1], one has

$$0 \le v_i^-(y,\ t) \le v_i(y,\ t),\ \phi_i(y+ct) \le v_i^+(y,\ t) \le 1,\ (y,\ t) \in \mathbb{R}\times\mathbb{R}^+,\ i = 1,\ 2. \tag{8}$$

Take

$$U_i^{\pm}(\eta,\, t) = \pm\left(v_i^{\pm}(y,\, t) - \phi_i(y+ct)\right),\; i = 1,\, 2,$$

$$U_{10}^{\pm}(\eta,\, 0) = \pm\left(v_{10}^{\pm}(y) - \phi_1(y)\right),\; U_{20}^{\pm}(\eta,\, s) = \pm\left(v_2^{+}(y,\, s) - \phi_2(y+cs)\right),$$

$$U_i(\eta,\, t) = v_i(y,\, t) - \phi_i(y+ct),\; i = 1,\, 2,\; U_{10}(\eta,\, 0) = v_{10}(y) - \phi_1(y),$$

$$U_{20}(\eta,\, 0) = v_{20}(y,\, s) - \phi_2(y+cs),$$

where $\eta = y+ct$ and $s \in [-\iota,\, 0]$. Then, in term of comparison theorem and (8), we have

$$(0,\, 0) \le \left(U_{10}^{-}(\eta,\, 0),\, U_{20}^{-}(\eta,\, s)\right) \le (U_{10}(\eta,\, 0),\, U_{20}(\eta,\, s)) \le \left(U_{10}^{+}(\eta,\, 0),\, U_{20}^{+}(\eta,\, s)\right) \le (1,\, 1),$$

$$(0,\, 0) \le \left(U_{10}^{-}(\eta,\, t),\, U_{20}^{-}(\eta,\, t)\right) \le (U_{10}(\eta,\, t),\, U_{20}(\eta,\, t)) \le \left(U_{10}^{+}(\eta,\, t),\, U_{20}^{+}(\eta,\, t)\right) \le (1,\, 1).$$

Now, we show the assertions of Theorem 2.1 in three steps.

**Step 1** We first claim that the inequality $\sup\limits_{\eta\in\mathbb{R}} \left\| U_i^{+}(\eta,\, t) \right\| \le Ce^{-\lambda t}$ holds for $i = 1,\, 2$.

For simplicity, let us denote $U_i^{+}(\eta,\, t)$ by $U_i(\eta,\, t),\; i = 1,\, 2$. It is easily verified that $(U_1(\eta,\, t),\, U_2(\eta,\, t))$satisfies

$$\begin{cases} U_{1t}(\eta,\, t) + cU_{1\eta}(\eta,\, t) - U_{1\eta\eta}(\eta,\, t) \\ +U_1(\eta,\, t)\left[2\phi_1(\eta) - (1-r) - rU_2(\eta - c\iota,\, t-\iota) - r\phi_2(\eta - c\iota)\right] \\ = -U_1^2(\eta,\, t) + r\phi_1(\eta)U_2(\eta - c\iota,\, t-\iota) \\ U_{2t}(\eta,\, t) + cU_{2\eta}(\eta,\, t) - U_{2\eta\eta}(\eta,\, t) + bU_2(\eta,\, t)\left[\phi_1(\eta) + U_1(\eta,\, t)\right] \\ = -bU_1(\eta,\, t)\phi_2(\eta) + bU_1(\eta,\, t), \end{cases} \tag{9}$$

which is subject to the following initial condition

$$U_1(\eta) = U_{10}(\eta),\; U_2(\eta,\, s) = U_{20}(\eta,\, s),\; \eta \in \mathbb{R},\; s \in [-\iota,\, 0]. \tag{10}$$

Notice that $U_1(\eta,\, t),\, U_2(\eta,\, t) \in C\left([0,\, +\infty),\, H^1_{\varpi_*}(\mathbb{R})\right)$, since $U_{10}(\eta,\, 0),\, U_{20}(\eta,\, s) \in H^1_{\varpi_*}(\mathbb{R})$. To obtain the energy estimates, the solutions to (9) and (10) need to have sufficient regularity. Thus, the initial conditions can be mollified as follows:

$$\begin{cases} U_{10\varepsilon}(\eta,\ 0) = (L_\varepsilon * U_{10})(\eta,\ 0) = \int_{\mathbb{R}} L_\varepsilon(\eta-\omega) U_{10}(\omega,\ 0) d\omega \in H^2_{\varpi_*}(\mathbb{R}), \\ U_{20\varepsilon}(\eta,\ s) = (L_\varepsilon * U_{20})(\eta,\ s) = \int_{\mathbb{R}} L_\varepsilon(\eta-\omega) U_{20}(\omega,\ s) d\omega \in H^2_{\varpi_*}(\mathbb{R}), \end{cases}$$

where $L_\varepsilon(\eta)$ is the mollifier. Suppose that $(U_{1\varepsilon}(\eta,\ t),\ U_{2\varepsilon}(\eta,\ t))$ satisfies (9) with this mollified initial conditions. Then, it can be concluded that $U_{i\varepsilon}(\eta,\ t) \in C\left([0,\ +\infty),\ H^2_{\varpi_*}(\mathbb{R})\right),\ i = 1,\ 2.$

Let $\varepsilon \to 0$. Then, the energy estimate of the original solution $U_i(t,\ \eta)$ can be established by Lemma 1, which will be given in the following.

**Lemma 1** If any $c > c_*$, then

(a) there exists some constant $C_0 > 0$, such that the following assertion is true.

$$e^{2\lambda t} \sum_{i=1}^{2} \|U_i(\eta,\ t)\|^2_{L^2_{\varpi_*}} + \int_0^t e^{2\lambda s} \sum_{i=1}^{2} \|U_{i\eta}(\eta,\ s)\|^2_{L^2_{\varpi_*}} ds + \int_0^t \int_{\mathbb{R}} e^{2\lambda s} \varpi_* \sum_{i=1}^{2} Q_i^\lambda(\eta,\ s) U_i^2(\eta,\ s) d\eta ds$$

$$\leq \sum_{i=1}^{2} \|U_{i0}(0)\|^2_{L^2_{\varpi_*}} + C_0 \int_{-\iota}^{0} \|U_{20}(s)\|^2_{L^2_{\varpi_*}} ds.$$

(b) there exists some constant $\hat{C}_0 > 0$, such that the following assertion holds.

$$e^{2\lambda t} \sum_{i=1}^{2} \|U_{i\eta}(\eta,\ t)\|^2_{L_{\varpi_*}} + \int_0^t e^{2\lambda s} \sum_{i=1}^{2} \|U_{i\eta\eta}(\eta,\ s)\|^2_{L^2_{\varpi_*}} ds + \int_0^t \int_{\mathbb{R}} e^{2\lambda s} \varpi_* \sum_{i=1}^{2} Q_i^\lambda(\eta,\ s) U_{i\eta}^2(\eta,\ s) d\eta ds$$

$$\leq \hat{C}_0 \left( \sum_{i=1}^{2} \|U_{i\eta 0}(0)\|^2_{L^2_{\varpi_*}} + \int_{-\iota}^{0} \|U_{2\eta 0}(s)\|^2_{L^2_{\varpi_*}} ds + \sum_{i=1}^{2} \|U_{i0}(0)\|^2_{L^2_{\varpi_*}} + \int_{-\iota}^{0} \|U_{20}(s)\|^2_{L^2_{\varpi_*}} ds \right),$$

where

$$Q_1^\lambda(\eta,\ t) := R_1(\eta,\ t) - 2\lambda,\ Q_2^\lambda(\eta,\ t) := R_2(\eta,\ t) - 2\lambda - r\phi_1(\eta + c\iota) \frac{\varpi_*(\eta + c\iota)}{\varpi_*(\eta)} \left(e^{2\lambda \iota} - 1\right),$$

$$R_1(\eta,\ t) := -c\frac{\varpi_*'}{\varpi_*} - \left(\frac{\varpi_*'}{\varpi_*}\right)^2 + 2\left[2\phi_1(\eta) - 1 - r\phi_2(\eta - c\iota)\right] - r\phi_1(\eta) - b,$$

$$R_2(\eta,\ t) := -c\frac{\varpi_*'}{\varpi_*} - \left(\frac{\varpi_*'}{\varpi_*}\right)^2 + 2b\left[\phi_1(\eta) - \frac{1}{2}\right] - r\phi_1(\eta + c\iota) \frac{\varpi_*(\eta + c\iota)}{\varpi_*(\eta)}.$$

**Proof.** (a) Multiplying both sides of differential equations for (9) by $e^{2\lambda t}\varpi_*(\eta)U_1(\eta,\ t)$ and $e^{2\lambda t}\varpi_*(\eta)U_2(\eta,\ t)$, respectively, where $\lambda > 0$ will be given later and $\varpi_*(\eta)$ is expressed by (7), one can derive

$$e^{2\lambda t}\varpi_*(\eta)U_1U_{1t}+ce^{2\lambda t}\varpi_*(\eta)U_1U_{1\eta}-e^{2\lambda t}\varpi_*(\eta)U_1U_{1\eta\eta}$$

$$+e^{2\lambda t}\varpi_*(\eta)U_1^2\left[2\phi_1-(1-r)-rU_2(\eta-c\tau,\ t-\iota)-r\phi_2(\eta-c\iota)\right]$$

$$=-e^{2\lambda t}\varpi_*(\eta)U_1^3+re^{2\lambda t}\varpi_*(\eta)U_1U_2(\eta-c\iota,\ t-\iota)\phi_1,$$

$$e^{2\lambda t}\varpi_*(\eta)U_2U_{2t}+ce^{2\lambda t}\varpi_*(\eta)U_2U_{2\eta}-e^{2\lambda t}\varpi_*(\eta)U_2U_{2\eta\eta} \tag{11}$$

$$+be^{2\lambda t}\varpi_*(\eta)U_2^2\left[\phi_1+U_1\right]=-be^{2\lambda t}\varpi_*(\eta)U_1U_2\phi_2+be^{2\lambda t}\varpi_*(\eta)U_1U_2. \tag{12}$$

A direct computation shows that $e^{2\lambda t}\varpi_*(\eta)U_1U_{1t}=\left(\frac{1}{2}e^{2\lambda t}\varpi_*(\eta)U_1^2\right)_t-\lambda e^{2\lambda t}\varpi_*(\eta)U_1^2$ and

$$ce^{2\lambda t}\varpi_*(\eta)U_1U_{1\eta}-e^{2\lambda t}\varpi_*(\eta)U_1U_{1\eta\eta}=\left(\frac{c}{2}e^{2\lambda t}\varpi_*(\eta)U_1^2-e^{2\lambda t}\varpi_*(\eta)U_1U_{1\eta}\right)_\eta-\frac{c}{2}e^{2\lambda t}\varpi_*'(\eta)U_1^2$$

$$+e^{2\lambda t}\varpi_*'(\eta)U_1U_{1\eta}+e^{2\lambda t}\varpi_*(\eta)U_{1\eta}^2.$$

Thanks to (11), one has

$$\left(\frac{1}{2}e^{2\lambda t}\varpi_*(\eta)U_1^2\right)_t+\left(\frac{c}{2}e^{2\lambda t}\varpi_*(\eta)U_1^2-e^{2\lambda t}\varpi_*(\eta)U_1U_{1\eta}\right)_\eta+e^{2\lambda t}\varpi_*'(\eta)U_1U_{1\eta}+e^{2\lambda t}\varpi_*(\eta)U_{1\eta}^2$$

$$+\left\{-\frac{c}{2}\frac{\varpi_*'}{\varpi_*}-\lambda+\left[2\phi_1-(1-r)-rU_2(\eta-c\iota,\ t-\iota)-r\phi_2(\eta-c\iota)\right]\right\}e^{2\lambda t}\varpi_*(\eta)U_1^2 \tag{13}$$

$$=-e^{2\lambda t}\varpi_*(\eta)U_1^3+re^{2\lambda t}\varpi_*(\eta)U_1U_2(\eta-c\iota,\ t-\iota)\phi_1.$$

Applying classical inequality

$$2ab\leq a^2+b^2, \tag{14}$$

which is called Cauchy-Schwarz inequality, one has the following conclusion

$$\left|e^{2\lambda t}\varpi_*'(\eta)U_1U_{1\eta}\right|\leq\frac{1}{2}e^{2\lambda t}\varpi_*(\eta)U_{1\eta}^2+\frac{1}{2}e^{2\lambda t}\left(\frac{\varpi_*'(\eta)}{\varpi_*(\eta)}\right)^2\varpi_*(\eta)U_1^2. \tag{15}$$

Deleting the negative term $-e^{2\lambda t}\varpi_*(\eta)U_1^3(\eta,\ t)$ of (13)and combing the inequality (15) we can verify

$$\left(\frac{1}{2}e^{2\lambda t}\varpi_*(\eta)U_1^2\right)_t+\left(\frac{c}{2}e^{2\lambda t}\varpi_*(\eta)U_1^2-e^{2\lambda t}\varpi_*(\eta)U_1U_{1\eta}\right)_\eta+\frac{1}{2}e^{2\lambda t}\varpi_*(\eta)U_{1\eta}^2+\left\{-\frac{c}{2}\frac{\varpi_*'}{\varpi_*}-\right.$$

$$\left.\frac{1}{2}\left(\frac{\varpi_*'}{\varpi_*}\right)^2-\lambda+[2\phi_1-(1-r)-rU_2(\eta-c\iota,\ t-\iota)-r\phi_2(\eta-c\iota)]\right\}e^{2\lambda t}\varpi_*(\eta)U_1^2 \tag{16}$$

$$\leq re^{2\lambda t}\varpi_*(\eta)U_1U_2(\eta-c\iota,\ t-\iota)\phi_1.$$

Similarly, by using the inequality (14), it is can be derived that

$$\left|e^{2\lambda t}\varpi_*'(\eta)U_2U_{2\eta}\right|\leq\frac{1}{2}e^{2\lambda t}\varpi_*(\eta)U_{2\eta}^2+\frac{1}{2}e^{2\lambda t}\left(\frac{\varpi_*'(\eta)}{\varpi_*(\eta)}\right)^2\varpi_*(\eta)U_2^2, \tag{17}$$

$$\left|be^{2\lambda t}\varpi_*(\eta)U_1U_2\right|\leq\frac{b}{2}e^{2\lambda t}\varpi_*(\eta)U_1^2+\frac{b}{2}e^{2\lambda t}\varpi_*(\eta)U_2^2. \tag{18}$$

Deleting the term $-be^{2\lambda t}\varpi_*(\eta)U_1(\eta,\ t)U_2(\eta,\ t)\phi_2$, originating from (12), and combing the inequalities (17) and (18), it can be transformed to

$$\left(\frac{1}{2}e^{2\lambda t}\varpi_*(\eta)U_2^2\right)_t+\left(\frac{c}{2}e^{2\lambda t}\varpi_*(\eta)U_2^2-e^{2\lambda t}\varpi_*(\eta)U_2U_{2\eta}\right)_\eta$$

$$+\frac{1}{2}e^{2\lambda t}\varpi_*(\eta)U_{2\eta}^2+\left\{-\frac{c}{2}\frac{\varpi_*'}{\varpi_*}-\frac{1}{2}\left(\frac{\varpi_*'}{\varpi_*}\right)^2-\lambda+b\left(\phi_1-\frac{1}{2}+U_1\right)\right\}e^{2\lambda t}\varpi_*(\eta)U_2^2 \tag{19}$$

$$\leq\frac{b}{2}re^{2\lambda t}\varpi_*(\eta)U_1^2.$$

The fact can be noticed that the vanishing terms

$$\left(\frac{c}{2}e^{2\lambda t}\varpi_*(\eta)U_1^2-e^{2\lambda t}\varpi_*(\eta)U_1U_{1\eta}\right)_\eta \text{ and } \left(\frac{c}{2}e^{2\lambda t}\varpi_*(\eta)U_2^2-e^{2\lambda t}\varpi_*(\eta)U_2U_{2\eta}\right)_\eta$$

will appear when integrating (16) and (19) over $\mathbb{R}\times[0,\ t]$ with respect to $\eta$ and $t,$ since $U_1,\ U_2\in H^2_{\varpi_*}(\mathbb{R})$. In addition, one has

$$\int_{\mathbb{R}}\int_0^t\left(e^{2\lambda s}\varpi_*(\eta)U_1^2(\eta,\ s)\right)_t dsd\eta=e^{2\lambda t}\left\|U_1(t)\right\|^2_{L^2_{\varpi_*}}-\left\|U_{10}(0)\right\|^2_{L^2_{\varpi_*}}, \tag{20}$$

$$\int_{\mathbb{R}}\int_0^t e^{2\lambda s}\varpi_*(\eta)U_{1\eta}^2(\eta,\ s)dsd\eta=\int_0^t e^{2\lambda s}\left\|U_{1\eta}(s)\right\|^2_{L^2_{\varpi_*}}ds, \tag{21}$$

$$\int_{\mathbb{R}}\int_0^t \left(e^{2\lambda s}\varpi_*(\eta)U_2^2(\eta,\ s)\right)_t dsd\eta = e^{2\lambda t}\left\|U_2(t)\right\|^2_{L^2_{\varpi_*}} - \left\|U_{20}(0)\right\|^2_{L^2_{\varpi_*}}, \tag{22}$$

$$\int_{\mathbb{R}}\int_0^t e^{2\lambda s}\varpi_*(\eta)U_{2\eta}^2(\eta,\ s)dsd\eta = \int_0^t e^{2\lambda s}\left\|U_{2\eta}(s)\right\|^2_{L^2_{\varpi_*}} ds. \tag{23}$$

In view of the inequality (14) again, we can derive the following estimates:

$$\begin{aligned}
&2r\int_0^t\int_{\mathbb{R}} e^{2\lambda s}\varpi_*(\eta)U_1(\eta,\ s)U_2(\eta-c\iota,\ s-\iota)\phi_1 d\eta ds\\
\leq& r\int_0^t\int_{\mathbb{R}} e^{2\lambda s}\varpi_*(\eta)U_1^2(\eta,\ s)\phi_1 d\eta ds + r\int_0^t\int_{\mathbb{R}} e^{2\lambda s}\varpi_*(\eta)U_2^2(\eta-c\iota,\ s-\iota)\phi_1 d\eta ds\\
=& r\int_0^t\int_{\mathbb{R}} e^{2\lambda s}\varpi_*(\eta)U_1^2(\eta,\ s)\phi_1 d\eta ds + re^{2\lambda t}\int_{-\iota}^{t-\iota}\int_{\mathbb{R}} e^{2\lambda s}\varpi_*(\eta+c\iota)U_2^2(\eta,\ s)\phi_1(\eta+c\tau)d\eta ds \qquad (24)\\
\leq& r\int_0^t\int_{\mathbb{R}} e^{2\lambda s}\varpi_*(\eta)U_1^2(\eta,\ s)\phi_1 d\eta ds + re^{2\lambda t}\int_0^t\int_{\mathbb{R}} e^{2\lambda s}\varpi_*(\eta+c\iota)U_2^2(\eta,\ s)\phi_1(\eta+c\iota)d\eta ds\\
&+ re^{2\lambda t}\int_{-\iota}^0\int_{\mathbb{R}} e^{2\lambda s}\varpi_*(\eta+c\iota)U_{20}^2(\eta,\ s)\phi_1(\eta+c\iota)d\eta ds.
\end{aligned}$$

By integration of (16) and (19) over $\mathbb{R}\times[0,\ t]$ with respect to $\eta$ and $t$, and the combination of (16), (20), (21), and (24) can yield

$$\begin{aligned}
&e^{2\lambda t}\left\|U_1(t)\right\|^2_{L^2_{\varpi_*}} + \int_0^t e^{2\lambda s}\left\|U_{1\eta}(s)\right\|^2_{L^2_{\varpi_*}} ds + \int_0^t\int_{\mathbb{R}}\left\{-c\frac{\varpi_*'}{\varpi_*} - \left(\frac{\varpi_*'}{\varpi_*}\right)^2 - 2\lambda\right.\\
&\left.+2\left[2\phi_1-(1-r)-rU_2(\eta-c\iota,\ t-\iota)-r\phi_2(\eta-c\iota)\right]\right\}e^{2\lambda s}\varpi_*(\eta)U_1^2(\eta,\ s)d\eta ds\\
\leq& \left\|U_{10}(0)\right\|^2_{L^2_{\varpi_*}} + r\int_0^t\int_{\mathbb{R}} e^{2\lambda s}\varpi_*(\eta)U_1^2(\eta,\ s)\phi_1 d\eta ds \qquad (25)\\
&+ re^{2\lambda t}\int_0^t\int_{\mathbb{R}} e^{2\lambda s}\varpi_*(\eta+c\iota)U_2^2(\eta,\ s)\phi_1(\eta+c\iota)d\eta ds\\
&+ re^{2\lambda t}\int_{-\iota}^0\int_{\mathbb{R}}^{2\lambda s}\varpi_*(\eta+c\iota)U_{20}^2(\eta,\ s)\phi_1(\eta+c\iota)d\eta ds
\end{aligned}$$

and substitution of (22) and (23) into (19) results in

$$
\begin{aligned}
& e^{2\lambda t}\|U_2(t)\|^2_{L^2_{\varpi_*}} + \int_0^t e^{2\lambda s}\left\|U_{2\eta}(s)\right\|^2_{L^2_{\varpi_*}} ds + \int_0^t \int_{\mathbb{R}} \left\{ -c\frac{\varpi_*'}{\varpi_*} - \left(\frac{\varpi_*'}{\varpi_*}\right)^2 \right. \\
& \left. -2\lambda + 2b\left(\phi_1 - \frac{1}{2} + U_1\right)\right\} e^{2\lambda s}\varpi_*(\eta)U_2^2(\eta,\ s)d\eta ds \\
& \leq \|U_{20}(0)\|^2_{L^2_{\varpi_*}} + b\int_0^t \int_{\mathbb{R}} e^{2\lambda s}\varpi_*(\eta)U_1^2(\eta,\ s)d\eta ds.
\end{aligned} \tag{26}
$$

Based on (25) and (26), one has

$$
\begin{aligned}
& e^{2\lambda t}\left(\|U_1(t)\|^2_{L^2_{\varpi_*}} + \|U_2(t)\|^2_{L^2_{\varpi_*}}\right) + \int_0^t e^{2\lambda s}\left(\left\|U_{1\eta}(s)\right\|^2_{L_{\varpi_*}} + \left\|U_{2\eta}(s)\right\|^2_{L^2_{\varpi_*}}\right) ds \\
& + \int_0^t \int_{\mathbb{R}} e^{2\lambda s}\varpi_*(\eta)\left(Q_1^\lambda(\eta,\ s)U_1^2(\eta,\ s) + Q_2^\lambda(\eta,\ s)U_2^2(\eta,\ s)\right) d\eta ds \\
& \leq \|U_{10}(0)\|^2_{L^2_{\varpi_*}} + \|U_{20}(0)\|^2_{L^2_{\varpi_*}} + C_0\int_{-\iota}^0 \|U_{20}(s)\|^2_{L_{\varpi_*}} ds,
\end{aligned}
$$

where $C_0 > 0$ is a constant.

(b) Similar to the statement (1), we can prove that the statement (2) is true. The proof of Lemma 1 is completed.

**Lemma 2** There exists a positive constant $C_2$, such that

$$
\sum_{l=1}^{2} R_l(\eta,\ t) \geq C_2 > 0,\ \forall\eta \in \mathbb{R},\ l = 1,\ 2.
$$

**Proof.** By (7), $0 < \phi_i(\eta) < 1$ and $0 < U_i(\eta,\ t) < 1,\ i = 1,\ 2$, one has

$$
R_1(\eta,\ t) = c\mu_* - \mu_*^2 - (1-r) + (1-r) + 2\left[2\phi_1(\eta) - 1 - r\phi_2(\eta - c\iota)\right] - r\phi_1(\eta) - b \geq (4-r)\phi_1(\eta) - 3r - 1 - b
$$

$$
R_2(\eta,\ t) = c\mu_* - \mu_*^2 - (1-r) + (1-r) + 2b\left[\phi_1(\eta) - \frac{1}{2}\right] - r\phi_1(\eta + c\tau)\frac{\varpi_*(\eta + c\iota)}{\varpi_*(\eta)} \geq 2b\phi_1(\eta) - b - 2r + 1.
$$

Based on the above analysis and the function $F(\eta)$ defined by (6), one has

$$
\sum_{l=1}^{2} R_l(\eta,\ t) \geq F(\eta) \geq F(\eta_0) = (4 - r + 2b)\phi_1(\eta_0) - 5r - 2b =:\ C_2
$$

and the assertion follows.

**Lemma 3** If the equation $\frac{C_2}{2} - 4\lambda - r\left(e^{2\lambda t} - 1\right) = 0$ has only a positive root $\lambda_1$, then for any $0 < \lambda < \lambda_1$, there exists some constant $C_1 > 0$, such that

$$\sum_{l=1}^{2} Q_l^{\lambda}(\eta,\, t) \geq C_1,\ \forall \eta \in \mathbb{R}.$$

**Proof.** We know that $0 < \phi_1(\eta) < 1$ and $\frac{\varpi_*(\eta + c\tau)}{\varpi_*(\eta)} \leq 1$ for $\eta \in \mathbb{R}$. Hence, one has

$$\sum_{l=1}^{2} Q_l^{\lambda}(\eta,\, t) = \sum_{l=1}^{2} (R_l(\eta,\, t) - 2\lambda) - r\phi_1(\eta + c\iota)\frac{\varpi_*(\eta + c\iota)}{\varpi_*(\eta)}\left(e^{2\lambda\iota} - 1\right)$$

$$\geq C_2 - 4\lambda - r\left(e^{2\lambda\iota} - 1\right) \geq \frac{C_2}{2} - 4\lambda - r\left(e^{2\lambda\iota} - 1\right) := C_1 > 0$$

for any $0 < \lambda < \lambda_1$. This finishes the proof.

By applying Lemma 1 (b), one easily obtains the estimates as follows:

**Lemma 4** There exists $C_3 > 0$, such that

$$e^{2\lambda t}\left\|U_{l\eta}(\eta,\, t)\right\|^2_{l^2_{\varpi_*}} \leq C_3\left(\sum_{l=1}^{2}\|U_{l0}(0)\|^2_{H^1_{\varpi_*}} + \int_{-\iota}^{0}\|U_{20}(s)\|^2_{H'_{\varpi_*}}\, ds\right),\ l = 1,\ 2.$$

Thus, Lemma 1 (a) and Lemma 4 imply the following conclusion:

**Lemma 5** Assume that $\varpi_*(\eta)$ is the weight function expressed by (7). Then, there exists some constant $C_4 > 0$, such that

$$\|U_l(\eta,\, t)\|^2_{H^1_{\varpi_t}} \leq C_4 e^{-2\lambda t}\left(\sum_{l=1}^{2}\|U_{l0}(0)\|^2_{H^1_{\varpi_t}} + \int_{-t}^{0}\|U_{20}(s)\|^2_{H^1_{\varpi^*}}\, ds\right),\ \forall t > 0,\ l = 1,\ 2.$$

Notice when $\varpi_*(\eta) \to 0$ as $\eta \to \infty$, $H^1_{\varpi_*}(\mathbb{R}) \to C(\mathbb{R})$cannot be assured. However, ision $H^1_{\varpi_*}(I) \to C(I)$for any interval $I = (-\infty,\ \eta_1]$, where $\eta_1 \gg \eta_0 + 1$is some large constant. Hence, according to Lemma 5, one has the following inequality:

**Lemma 6** If $t > 0$, then there exists some constant $C_5 > 0$, such that

$$\sup_{\eta \in I}|U_l(\eta,\, t)| \leq C_5 e^{-\lambda t}\left(\sum_{l=1}^{2}\|U_{l0}(0)\|^2_{H^1_{\varpi^*}} + \int_{-l}^{0}\|U_{20}(s)\|^2_{H^1_{\varpi^*}}\, ds\right)^{\frac{1}{2}},\ l = 1,\ 2,$$

for any $I = (-\infty,\ \eta_1]$ with some large constant $\eta_1 \gg \eta_0 + 1$.

**Lemma 7** If $\lambda_2 = b$, then there is some constant $C > 0$, such that

$$\lim_{\eta\to\infty}|U_l(\eta,\ t)|\le Ce^{-\lambda_2 t},\ l=1,\ 2.$$

**Proof.** It is easily verified that $U_{l\eta}(\infty,\ t)=U_{l\eta\eta}(\infty,\ t)=0,\ l=1,\ 2$. In view of (9) and the boundedness of $U_l(\infty,\ t)$ for all $\eta\in\mathbb{R}$, taking $\eta\to\infty$, one has

$$U_{1t}(\infty,\ t)+U_1(\infty,\ t)(1-r)\le rU_2(\infty,\ t-\iota), \tag{27}$$

$$U_{2t}(\infty,\ t)\le -bU_2(\infty,\ t). \tag{28}$$

By integrating (28) over $[0,\ t]$, one obtains

$$U_2(\infty,\ t)\le U_2(\infty,\ 0)e^{-bt}. \tag{29}$$

Thus, substituting (29) into the equation (27) gives

$$U_1(\infty,\ t)\le U_1(\infty,\ 0)e^{(r-1)t}+rU_2(\infty,\ 0)e^{b\tau}\frac{e^{-bt}}{1-r-b}$$

$$\le U_1(\infty,\ 0)e^{-bt}+rU_2(\infty,\ 0)e^{b\tau}\frac{e^{-bt}}{1-r-b}=\tilde{C}e^{-bt},$$

where $\tilde{C}:=U_1(\infty,\ 0)+\dfrac{rU_2(\infty,\ 0)e^{bl}}{1-r-b}$.

Hence, we have $\lim_{\eta\to\infty}|U_l(\eta,\ t)|\le Ce^{-\lambda_2 t},\ l=1,\ 2$, where $C:=\max\left\{\tilde{C},\ U_2(\infty,\ 0)\right\},\ \lambda_2:=b$. This finishes the proof.

Taking $0<\tilde{\lambda}<\lambda_1$ and $0<\tilde{\lambda}<\lambda_2$, by using Lemmas 6 and 7, we finally obtain the $L^\infty$convergence for $\eta\in\mathbb{R}$ as follows:

**Lemma 8** If $t>0$, then there exists some $C_6>0$, such that

$$\sup_{y\in\mathbb{R}}\left|v_l^+(y,\ t)-\phi_l(y+ct)\right|=\sup_{\eta\in\mathbb{R}}|U_l(\eta,\ t)|\le C_6e^{-\tilde{\lambda}t},\ l=1,\ 2.$$

**Step 2** Similar to Step 1, we know that there are constants $C_7>0$ and $\bar{\lambda}$, such that

$$\sup_{y\in\mathbb{R}}\left|v_l^-(y,\ t)-\phi_l(y+ct)\right|\le C_7e^{-\bar{\lambda}t},\ t>0,\ l=1,\ 2.$$

**Step 3** According to (8), one has

$$|v_l(y,\ t)-\phi_l(y+ct)| \le \max\left\{\left|v_l^+(y,\ t)-\phi_l(y+ct)\right|,\ \left|v_l^-(y,\ t)-\phi_l(y+ct)\right|\right\},\ l=1,\ 2.$$

Then, based on the results obtained in Step 2 and Step 3, and the above observation, this finishes the proof of Theorem 2.1.

## 4. Numerical examples

In this section, we give an example of the numerical application of the main results.

**Example 1** Consider the following Belousov-Zhabotinskii system with a time delay:

$$\begin{cases} (v_1)_t = (v_1)_{yy} + v_1(y,\ t)\left[0.4 - v_1(y,\ t) + 0.6 v_2(y,\ t-0.5)\right], \\ \\ (v_2)_t = (v_2)_{yy} + 0.2 v_1(y,\ t)\left[1 - v_2(y,\ t)\right], \end{cases} \tag{30}$$

which is subjected to the initial value

$$v_{10}(y) = v_1(y,\ 0),\ v_{20}(y,\ s) = v_2(y,\ s),\ y\in\mathbb{R},\ s\in[-0.5,\ 0], \tag{31}$$

where $r=0.6,\ b=0.2$, and $=0.5$. By computation, we obtain $c_* = 2\sqrt{1-r} = 2\sqrt{10}/5$.

Since$r\in(0,\ 1)$ and $0<b<1-r$, it follows from Proposition 2.1 that for any $c\ge c_*$, system (30) has an increasing traveling wave $\Phi(\eta)=(\phi_1(\eta),\ \phi_2(\eta))$ with speed $c$ connecting (0, 0) and (1, 1).

It is seen easily that the condition (A) holds and $(0,\ 0)\le(v_{10}(y),\ v_{20}(y,\ s))\le(1,\ 1),\ y\in\mathbb{R},\ s\in[-0.5,\ 0], v_{10}(y)-\phi_1(y)\in H^1_{\varpi_*}(\mathbb{R})\subset C(\mathbb{R}),\ v_{20}(y,\ s)-\phi_2(y+cs)\in C\left([-0.5,\ 0];\ H^1_{\varpi_*}(\mathbb{R})\right)$. So, Theorem 2.1 ensures that for any $c\ge c_*$, the traveling $c\ge c_*$ wave solution $\Phi(\eta)=(\phi_1(\eta),\ \phi_2(\eta))$ of system (30) is exponentially asymptotically stable.

## 5. Conclusions

This paper has successfully established the stability of traveling wave solutions for system (2) with condition (5) by a technical assumption. The weighted energy method and comparison theorem have been used for obtaining the stability of all waves, including the slower ones (i.e., the wave speed near the critical wave speed). It needs to be mentioned that we discuss only the stability of the traveling wave solution of system (2) for $c>c_*$; the stability of the traveling wave solution for the case $c=c_*$ is unsolved and left to the future for further consideration.


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## Conflict of interest

The authors declare there is no conflict of interest at any point with reference to research findings.

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