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Research Article

Analyzing Observability of Fractional Dynamical Systems Employing ψ-Caputo Fractional Derivative

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Abstract: In this present article, we inquire into the observability of linear and nonlinear fractional dynamical systems in terms of ψ-Caputo fractional derivative. The observability Grammian matrix, which is positive definite and expressed by the Mittag-Leffler functions, is used to obtain the necessary and sufficient conditions of observability of linear fractional dynamical systems, and Banach's fixed point theorem is used to get the sufficient conditions for the observability of nonlinear fractional systems. Three numerical examples are given to demonstrate the applicability of theoretical results for linear and nonlinear cases.

***Keywords*:** fractional dynamical systems, observability Grammian, ψ-Caputo fractional derivative, fixed point theorem

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1. Introduction

Dynamical systems with fractional derivatives and integrals are a growing topic in applied mathematics, with extensive applications in material science and engineering disciplines to describe the dynamics in different ways in anomalous mediums. The fundamental anomalous or heavy tail decay processes are better explained by the fractional derivative and integral models than by the more conventional integer order derivative approaches. This enables them to record the influence of system memory. In mathematical modelling, there are some research papers that are highly contributed by the application of fractional calculus in electrical network theory with fractal derivatives [1], chemistry with electronics [2], materials having viscoelasticity [3], thermal power systems and heat diffusion conduction [4] and moreover, differential equations with fractional order have recently proved to be valuable techniques to the modelling of many physical phenomena [5–8].

In the literature on fractional calculus, the most prevalent fractional derivatives are Caputo fractional derivative, the Riemann-Liouville derivative, Caputo-Fabrizio fractional derivative and Atangana-Baleanu fractional derivative. Researchers working in this field have recently shown a keen interest in the generalization of fractional derivatives and conformal fractional derivatives. Almeida introduced ψ-Caputo fractional derivative and [9] looked into the ψ-Caputo fractional derivative in relation to another function, as well as the links between Fermat's theorem, fractional derivative, Taylor's theorem, fractional integral and semigroup theory. Mfadel et al. [10] used the Krasnoselskii fixed point theorem to investigate the existence results for functional hybrid nonlinear differential equations with ψ-Caputo derivatives of

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order 0 to 1. In [11] examined blow-up solution equivalence results of fractional differential inequalities with a singular potential section, distinct orders, and polynomial nonlinearity in the view of ψ-Caputo derivatives. For reference, [12–16] these are certain articles on differential equations that heavily rely on the ψ-Caputo fractional derivative.

In control theory, qualitative properties like observability, controllability, stability, stabilizability and reachability are crucial concepts of dynamical systems. In this article, we examine one of the most fundamental structural aspects of a fractional dynamical system, known as observability. In short, the ability of a system to discover its initial state by observing its input-output performance is the ultimate notion of observability of a fractional dynamical system. In [17] presented new findings on the observability of fractional dynamical systems. In [18] presented the observability of fractional linear systems with multiple distinct orders in 2014. In [19] proposed the observability characteristics of a non-homogeneous fractional conformable system that is suitable for various electrical applications. Some publications [20–24] discuss the conclusions of the observability of fractional dynamical systems. Motivated by the fact there is no work reported on the observability of linear and nonlinear fractional dynamical systems in terms of ψ-Caputo fractional derivative. The importance of the current article is ψ-Caputo fractional derivative because ψ-Caputo fractional derivative has a non-singular kernel and a generalized one. For example, if $\psi(\varsigma) = \varsigma$, then we get the Caputo fractional derivative, if $\psi(\varsigma) = \log \varsigma$, then we get the Hadamard fractional derivative, and if $\psi(\varsigma) = \varsigma^{\rho}$, then we get the generalized Caputo fractional derivative. So we are interested in investigating the observability of linear and nonlinear fractional dynamical systems in terms of ψ-Caputo fractional derivative.

The remainder of this article is organized as follows. Section 2 comprises definitions of numerous fractional integrals and derivatives, as well as preliminary findings for the development of the article. The observability of linear and nonlinear fractional dynamical systems in the case of ψ-Caputo fractional derivative are examined in Sections 3 and 4, respectively. Finally, examples are provided at the end of the paper to further understand the findings in Section 5.

2. Preliminaries and basic results

In this section, we make use of a few definitions and some preliminary results that are essential to the development of our main findings.

Let $\mathscr{F} \subset \mathbb{R}$ be any finite interval such that $\mathscr{F} = [\tau_0, \tau_1]$. Let $X = \{g | g \colon \mathscr{F} \to \mathbb{R}^k \times \mathbb{R}^m$ is continuous$\}$ be a Banach space with respect to the norm $\|(g_1, g_2)\| = \|g_1\| + \|g_2\|$, where $\|g_1\| = sup\{|g_1(\varsigma)| : \varsigma \in \mathscr{F}\}$ and $\|g_2\| = sup\{|g_2(\varsigma)| : \varsigma \in \mathscr{F}\}$. For $1 \leq q < \infty$, denotes $L^q(\mathscr{F}, \mathbb{R})$ be the Banach space that contains set of all measurable functions $g \colon \mathscr{F} \to \mathbb{R}$ in relation to the norm $\|g\|^q_{L_q} = \int_{\tau_0}^{\tau_1} |g(\varsigma)|^q d\varsigma$.

Definition 1 [25] Let f be a real valued measurable function on $[\tau_0, \tau_1]$ and ψ be a differentiable increasing real valued function on $[\tau_0, \tau_1]$ such that $\psi'(t) \neq 0$ for all $t \in [\tau_0, \tau_1]$. Then, for any $\vartheta > 0$, the left-sided ψ-Riemann-Liouville fractional integral of order ϑ for a measurable function f with respect to ψ is defined by

$$I^{\vartheta;\psi}_{\tau_0+} f(t) = \frac{1}{\Gamma(\vartheta)} \int_{\tau_0}^{t} \psi'(\varsigma) \left(\psi(t) - \psi(\varsigma)\right)^{\vartheta-1} f(\varsigma) d\varsigma. \tag{1}$$

Definition 2 [25] Let $\vartheta > 0$ and $k \in \mathbb{N}$ such that $k = [\vartheta] + 1$, where $[\vartheta]$ be the integer part of ϑ. Let $k \in \mathbb{N}$ and $\psi, f \colon [\tau_0, \tau_1] \to \mathbb{R}$ be the k-times continuously differentiable functions such that for all $t \in [\tau_0, \tau_1]$, $\psi'(t) \neq 0$ and ψ is increasing on $[\tau_0, \tau_1]$. Then, for any $\vartheta > 0$, the left-sided ψ-Riemann-Liouville fractional derivative of a function f of order ϑ is expressed as

$$D_{\tau_0+}^{\vartheta;\,\psi} f(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k I_{\tau_0+}^{k-\vartheta;\,\psi} f(t)$$

$$= \frac{1}{\Gamma(n-\vartheta)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k \int_{\tau_0}^{t} \psi'(\varsigma)\left(\psi(t)-\psi(\varsigma)\right)^{k-\vartheta-1} f(\varsigma)d\varsigma. \tag{2}$$

Definition 3 [25] Let $k \in \mathbb{N}$ and ψ, f: $[\tau_0,\, \tau_1] \to \mathbb{R}$ be the k-times continuously differentiable functions such that for all $t \in [\tau_0,\, \tau_1]$, $\psi'(t) \neq 0$ and ψ is increasing on $[\tau_0,\, \tau_1]$. Then, for any $\vartheta > 0$, the left-sided ψ-Caputo fractional derivative of a function f of order ϑ is expressed as

$$^{C}D_{\tau_0+}^{\vartheta;\,\psi} f(t) = I_{\tau_0+}^{k-\vartheta;\,\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k f(t)$$

$$= \begin{cases} \frac{1}{\Gamma(k-\vartheta)}\int_{\tau_0}^{t} \psi'(\varsigma)\left(\psi(t)-\psi(\varsigma)\right)^{k-\vartheta-1} f_{\psi}^{[k]}(\varsigma)d\varsigma, & k = [\vartheta]+1 \text{ for } \vartheta \notin \mathbb{N} \\ f_{\psi}^{[k]}(t), & k = \vartheta \text{ for } \vartheta \in \mathbb{N} \end{cases} \tag{3}$$

where $f_{\psi}^{[k]}(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k f(t)$ and if f: $[\tau_0,\, \tau_1] \to \mathbb{R}$ be the k-times continuously differentiable functions, the left-sided ψ-Caputo fractional derivative of a function f of order ϑ is expressed as

$$^{C}D_{\tau_0+}^{\vartheta;\,\psi} f(t) = D_{\tau_0+}^{\vartheta;\,\psi}\left[f(t) - \sum_{m=0}^{k-1}\frac{f_{\psi}^{[m]}(\tau_0)}{m!}\left(\psi(t)-\psi(\tau_0)\right)^m\right], \tag{4}$$

Remark 1 The ψ-Caputo fractional derivative generalizes the other fractional derivatives. For instance, we obtain, Caputo fractional derivative for $\psi(t) = t$, Hadamard fractional derivative for $\psi(t) = \log_e t$ and generalized Caputo fractional derivative for $\psi(t) = t^{\rho}$, $\rho > 0$.

Proposition 1 A complex function $\mathbb{E}_{\vartheta,\,\rho}(z)$ is called the Mittag-Leffler function with parameters ϑ, $\rho > 0$ and which has the following expression:

$$\mathbb{E}_{\vartheta,\,\rho}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\vartheta k+\rho)}. \tag{5}$$

For all values of z, the function $\mathbb{E}_{\vartheta,\,\rho}(z)$ is convergent. The Mittag-Leffler function for a matrix $A_{n\times n}$ is given by

$$\mathbb{E}_{\vartheta,\,\rho}(A) = \sum_{k=0}^{\infty}\frac{A^k}{\Gamma(\vartheta k+\rho)}. \tag{6}$$

Lemma 1 [27] [Banach's Fixed Point Theorem] If $X$is a complete metric space and the mapping T: $X \to X$ is a contraction, then T has a unique fixed point in X.

3. Linear systems

Consider the fractional linear dynamical system with ψ-Caputo derivative of the form

$$ {}^{C}\mathbb{D}^{\vartheta;\psi}_{\tau_0+} z(\varsigma) = Az(\varsigma), \quad \varsigma \in [\tau_0, \tau_1] \tag{7}$$

with linear observation

$$ w(\varsigma) = Kz(\varsigma), \tag{8}$$

where the expression ${}^{C}\mathbb{D}^{\vartheta;\psi}_{\tau_0+}(\cdot)$ denotes ψ-Caputo fractional derivative of order $0 < \vartheta < 1$. The vector $z \in \mathbb{R}^k$ denotes state vector and $w \in \mathbb{R}^m$ with $m < k$. The entries of matrices $A_{k\times k}$ and $K_{m\times k}$ are constant over $\mathbb{R}$.

Definition 4 (Observable). If the linear observaition

$$ w(\varsigma) = Kz(\varsigma) = 0, \quad \varsigma \in [\tau_0, \tau_1]$$

implies

$$ z(\varsigma) = 0, \quad \varsigma \in [\tau_0, \tau_1]$$

then the linear system (7) and (8) is called observable on $[\tau_0, \tau_1]$.

Theorem 2 (Observability Grammian). The linear systems (7) and (8) is said to be observable on $[\tau_0, \tau_1]$ if and only if observability Grammian

$$ W[\tau_0, \tau_1] = \int_{\tau_0}^{\tau_1} E_\vartheta(A^*(\psi(\varsigma) - \psi(\tau_0)^\vartheta)K^*KE_\vartheta(A(\psi(\varsigma) - \psi(\tau_0))^\vartheta)d\varsigma \tag{9}$$

is positive definite.

Proof. Let the initial condition $z(\tau_0) = z_0$, then solution representation $z(\varsigma)$ of (7) is [26]

$$ z(\varsigma) = E_\vartheta(A(\psi(\varsigma) - \psi(\tau_0))^\vartheta)z_0 \tag{10}$$

with $w(\varsigma) = Kz(\varsigma) = KE_\vartheta(A(\psi(\varsigma) - \psi(\tau_0))^\vartheta)z_0$. Consider the quadratic form in z_0,

$$||w||^2 = \int_{\tau_0}^{\tau_1} w^*(\varsigma)w(\varsigma)d\varsigma$$

$$= z_0{}^* \int_{\tau_0}^{\tau_1} E_\vartheta(A^*(\psi(\varsigma)-\psi(\tau_0)^{\vartheta})K^*KE_\vartheta(A(\psi(\varsigma)-\psi(\tau_0))^{\vartheta})d\varsigma\, z_0$$

$$= z_0{}^* W[\tau_0,\, \tau_1]z_0.$$

□

Which says that the $k \times k$ matrix $W[\tau_0, \tau_1]$ is symmetric. If $W[\tau_0, \tau_1]$ is a positive definite, then $w = 0$ means that $z_0{}^*W[\tau_0, \tau_1]z_0 = 0$. Then we clearly obtain $z = 0$. We get that systems (7) and (8) observable on $[\tau_0, \tau_1]$.

Conversely, assume that $W[0, 1]$ is not positive definite. Then there exists a $z_0 = 0$ that fulfils $z_0{}^*W[\tau_0, \tau_1]z_0 = 0$. we get $w = 0$, since $||w||^2 = 0$. But which is contradiction to $w = 0$, because $z(\varsigma) = E_\vartheta(A(\psi(\varsigma)-\psi(\tau_0))^{\vartheta})z_0 \neq 0$, for $\varsigma \in [\tau_0, \tau_1]$. From this, the systems (7) and (8) are not observable on $[\tau_0, \tau_1]$. As a result, we conclude that $W[\tau_0, \tau_1]$ is positive definite.

Assume that linear dynamical systems (7) and (8) observable on $[\tau_0, \tau_1]$, then the initial state $z(\tau_0) = z_0$ for the solution of the system on the same interval $[\tau_0, \tau_1]$ is easily reconstructed by utilizing the linear observation $w(\varsigma) = KE_\vartheta(A(\psi(\varsigma)-\psi(\tau_0))^{\vartheta})z_0$.

Definition 5 (Reconstruction kernel). On $[\tau_0, \tau_1]$, the $Q(\varsigma)_{k\times k}$ satisfies

$$\int_{\tau_0}^{\tau_1} Q(\varsigma)KE_\vartheta(A(\psi(\varsigma)-\psi(\tau_0))^{\vartheta})d\varsigma = I \tag{11}$$

if and only if the matrix $Q(\varsigma)_{k\times k}$ is an reconstruction kernel of the system (7) and (8) on $[\tau_0, \tau_1]$.

Theorem 3 (Observability Grammian). There exists a reconstruction kernel $Q(\varsigma)$ on $[\tau_0, \tau_1]$ if and only if the system (7) and (8) observable on $[\tau_0, \tau_1]$.

Proof. Suppose the existence of a reconstruction kernel $Q(\varsigma)$ means that

$$\int_{\tau_0}^{\tau_1} Q(\varsigma)w(\varsigma)d\varsigma = \int_{\tau_0}^{\tau_1} Q(\varsigma)KE_\vartheta(A(\psi(\varsigma)-\psi(\tau_0))^{\vartheta})d\varsigma = z_0 \tag{12}$$

and $w(\varsigma) = 0$, then $z_0 = 0$. So $z(\varsigma) = 0$, and which is direct to the observability of the system (7) and (8) on $[\tau_0, \tau_1]$. □

For the converse case, assume that the observability of the system (7) and (8) on $[\tau_0, \tau_1]$. By Theorem (2), we have the observability grammian is takes the form

$$W[\tau_0,\, \tau_1] = \int_{\tau_0}^{\tau_1} E_\vartheta(A^*(\psi(\varsigma)-\psi(\tau_0)^{\vartheta})K^*KE_\vartheta(A(\psi(\varsigma)-\psi(\tau_0))^{\vartheta})d\varsigma > 0. \tag{13}$$

Assume

$$Q_0(\varsigma) = W^{-1}[\tau_0,\, \tau_1]E_\vartheta(A^*(\psi(\varsigma)-\psi(\tau_0)^{\vartheta})K^*, \quad \varsigma \in [\tau_0,\, \tau_1]. \tag{14}$$

Then we have

$$I = W^{-1}[\tau_0, \tau_1]\int_{\tau_0}^{\tau_1} E_{\vartheta}(A^*(\psi(\varsigma)-\psi(\tau_0)^{\vartheta})K^*KE_{\vartheta}(A^*(\psi(\varsigma)-\psi(\tau_0)^{\vartheta})d\varsigma$$

$$= \int_{\tau_0}^{\tau_1} Q_0(\varsigma)KE_{\vartheta}(A^*(\psi(\varsigma)-\psi(\tau_0)^{\vartheta})d\varsigma.$$

This indicates that (14) is a reconstruction kernel on $[\tau_0, \tau_1]$.

4. Non-linear systems

Consider the nonlinear fractional dynamical system in the sense of ψ-Caputo fractional derivative of the form

$${}^{C}\mathbb{D}_{\tau_0+}^{\vartheta;\psi} z(\varsigma) = Az(\varsigma) + f(\varsigma, z(\varsigma)), \quad \varsigma \in [\tau_0, \tau_1] \tag{15}$$

with linear observation

$$w(\varsigma) = Kz(\varsigma), \quad \varsigma \in [\tau_0, \tau_1] \tag{16}$$

where the expression ${}^{C}\mathbb{D}_{\tau_0+}^{\vartheta;\psi}(\cdot)$ denotes ψ-Caputo fractional derivative of order $0 < \vartheta < 1$. The vector $z \in \mathbb{R}^k$ denotes state vector and $w \in \mathbb{R}^m$ with $m < k$. The continuous function f is the $\mathbb{R}^k$ valued function from $[\tau_0, \tau_1] \times \mathbb{R}^k \times \mathbb{R}^m$ and the entries of matrices $A_{k\times k}$, $K_{m\times k}$ are constant over $\mathbb{R}$. We believe that the parameter w observes the system (15). Because $m < k$, the expression (15) does not enable immediate determination of z and w, the observability problem of (15) is handled as follows: it is essential to identify the unknown state z at the current time ς, given the value w across the interval $[\theta, \varsigma]$, where θ is some prior time.

Definition 6 If there exists $\theta < \varsigma$ in such a manner that the system state z at time ς can be determined given knowledge of output data throughout the interval $[\theta, \varsigma]$ then the system (15) and (16) is called to be observable at time ς. In general, a system is called to be completely observable indicates that it is observable at each point in the interval $\varsigma \in [\tau_0, \tau_1]$.

For an arbitrary initial state, we consider that the nonlinear system (15) has a unique solution. For every $\tau \in (\theta, \varsigma)$, assume $z = z(\tau)$ be an initial condition, then the solution of the nonlinear system (15) is given by

$$z(\varsigma) = E_{\vartheta}(A(\psi(\varsigma)-\psi(\tau))^{\vartheta})z(\tau) + \int_{\tau}^{\varsigma} \psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}E_{\vartheta,\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})f(r, z(r))dr. \tag{17}$$

After rearrangement, we get

$$z(\tau) = \left[E_{\vartheta}(A(\psi(\varsigma)-\psi(\tau))^{\vartheta})\right]^{-1}\left[z(\varsigma) - \int_{\tau}^{\varsigma} \psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}E_{\vartheta,\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})f(r, z(r))dr\right] \tag{18}$$

and the corresponding observation $w(\tau)$ is

$$w(\tau)=\Big[E_{\vartheta}(A(\psi(\varsigma)-\psi(\tau))^{\vartheta})\Big]^{-1}\Big[Kz(\varsigma)-K\int_{\tau}^{\varsigma}\psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}$$

$$E_{\vartheta,\,\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})f(r,\,z(r))dr\Big]$$

$$=\frac{1}{\Big[E_{\vartheta}(A(\psi(\varsigma)-\psi(\tau))^{\vartheta})\Big]^{2}}\Big[Kz(\varsigma)-K\int_{\tau}^{\varsigma}\psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}$$

$$E_{\vartheta,\,\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})f(r,\,z(r))dr\Big]E_{\vartheta}(A(\psi(\varsigma)-\psi(\tau))^{\vartheta}).$$

Multiply by $E_{\vartheta}(A^{*}(\psi(\varsigma)-\psi(\tau))^{\vartheta})K^{*}$ with $w(\tau)$ and take integrate from θ to ς, we get

$$\int_{\theta}^{\varsigma}[E_{\vartheta}(A(\psi(\varsigma)-\psi(\tau))^{\vartheta})]^{2}E_{\vartheta}(A^{*}(\psi(\varsigma)-\psi(\tau))^{\vartheta})K^{*}w(\tau)d\tau$$

$$=\int_{\theta}^{\varsigma}E_{\vartheta}(A^{*}(\psi(\varsigma)-\psi(\tau))^{\vartheta})K^{*}KE_{\vartheta}(A(\psi(\varsigma)-\psi(\tau))^{\vartheta})d\tau z(\varsigma)$$

$$-\int_{\theta}^{\varsigma}\psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}E_{\vartheta,\,\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})f(r,\,z(r))$$

$$\times\left(\int_{\theta}^{r}E_{\vartheta}(A^{*}(\psi(\varsigma)-\psi(\tau))^{\vartheta})K^{*}KE_{\vartheta}(A(\psi(\varsigma)-\psi(\tau))^{\vartheta})d\tau\right)dr \tag{19}$$

$$=W(\theta,\,\varsigma)z(\varsigma)-\int_{\theta}^{\varsigma}\psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}E_{\vartheta,\,\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})$$

$$\times W(\theta,\,r)f(r,\,z(r))dr.$$

If the truncated linear system (15) and (16) is observable, then we obtain $z(\varsigma)$ by rearranging above equation (19)

$$z(\varsigma)=W^{-1}(\theta,\,\varsigma)\int\limits_{\theta}^{\varsigma}\left[E_{\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})\right]^{2}E_{\vartheta}(A^{*}(\psi(\varsigma)-\psi(r))^{\vartheta})K^{*}y(r)dr$$

$$+W^{-1}(\theta,\,\varsigma)\int\limits_{\theta}^{\varsigma}\psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}E_{\vartheta,\,\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})W(\theta,\,r)f(r,\,z(r))dr.$$

Now let

$$M_{1}(\varsigma,\,\theta,\,r)=W^{-1}(\theta,\,\varsigma)\left[E_{\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})\right]^{2}E_{\vartheta}(A^{*}(\psi(\varsigma)-\psi(r))^{\vartheta})K^{*}$$

$$M_{2}(\varsigma,\,\theta,\,r)=W^{-1}(\theta,\,\varsigma)E_{\vartheta,\,\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})W(\theta,\,r),$$

we get the following equation

$$z(\varsigma)=\int\limits_{\theta}^{\varsigma}M_{1}(\varsigma,\,\theta,\,r)y(r)dr+\int\limits_{\theta}^{\varsigma}\psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}M_{2}(\varsigma,\,\theta,\,r)f(r,\,z(r))dr. \tag{20}$$

On the interval $[\theta,\,\varsigma]$, the equation (20) illustrates the connection between the unknown state z and the observed output w. As a consequence, we get the following result.

Theorem 4 If the system (15) and (16) satisfies the following conditions:

1. $det(W(\theta,\,\varsigma))\geq c$, where c is a constant with $c>0$, 2. at any w, the system (20) has a continuous unique solution on $[\theta,\,\varsigma]$ (a) for a completely observable system, for all ς and for some $\theta<\varsigma$ and (b) for an observable system at time ς for some $\theta<\varsigma$.

Then the system (15) and (16) is called (a) completely observable on $[\tau_0,\,\tau_1]$ and globally (a) observable at ς.

In (20), replace θ by τ, because time variable θ not necessarily to be fixed. After this modification, (20) is placed into (18). We obtain that

$$z(\tau)=\left[E_{\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})\right]^{-1}\Bigg(\int\limits_{\tau}^{\varsigma}M_{1}(\varsigma,\,\tau,\,r)y(r)dr+\int\limits_{\tau}^{\varsigma}\psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}$$

$$\times M_{2}(\varsigma,\,\tau,\,r)f(r,\,z(r))dr-\int\limits_{\tau}^{\varsigma}\psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}E_{\vartheta,\,\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})f(r,\,z(r))dr\Bigg).$$

For $\tau<t$,

$$z(\tau)=\int_{\tau}^{\varsigma} M_3(\varsigma,\,\tau,\,r)y(r)dr+\int_{\tau}^{\varsigma} \psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}M_4(\varsigma,\,\tau,\,r)f(r,\,z(r))dr, \tag{21}$$

where

$$M_3(\varsigma,\,\tau,\,r)=\left[E_{\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})\right]^{-1}M_1(\varsigma,\,\tau,\,r)$$

$$M_4(\varsigma,\,\tau,\,r)=\left[E_{\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})\right]^{-1}\left[M_2(\varsigma,\,\tau,\,r)-E_{\vartheta,\,\vartheta}(A(\psi(\varsigma)-\psi(r))^{\vartheta})\right].$$

Now replace (20) by (21) in the theorem (4), and we get the same results with the change of variables. Then, for the nonlinear system, we examine the use of Banach's fixed point theorem 1. For all $\varsigma\in[\tau_0,\,\tau_1]$, assume a special system

$${}^{C}\mathbb{D}^{\vartheta}_{\tau_0+}z(\varsigma)=Az(\varsigma)+\eta f(\varsigma,\,z(\varsigma)), \tag{22}$$

with linear observation

$$w(\varsigma)=Kz(\varsigma), \tag{23}$$

where $\eta>0$ and choose a constant $k\geq 0$ such that

$$\|f(\varsigma,\,z_1)-f(\varsigma,\,z_2)\|\leq k\,\|z_1-z_2\|\,. \tag{24}$$

Theorem 5 If the system (22) and (23) satisfies the following conditions:
1. $det(W(\theta,\,\varsigma))\geq c$, where c is a constant with $c>0$,
2. choose a constant η with $\eta>0$,

$$\eta<\frac{1}{k(\varsigma,\,\theta)}\ , \tag{25}$$

(a) for a completely observable system, for all ς and for some $\theta<\varsigma$ and (b) for an observable system at time ς, for some $\theta<\varsigma$.

Then the nonlinear system (22) and (23) is called (a) completely observable on $[\tau_0,\,\tau_1]$ and globally (b) observable at ς.

Proof. The solution representation $z(\varsigma)$ for (22) with initial condition $z=z(\tau)$ is

$$z(\tau) = \left[E_\vartheta(A(\psi(\varsigma) - \psi(\tau))^\vartheta)\right]^{-1} \left[z(\varsigma) - \eta \int_\tau^\varsigma \psi'(r)(\psi(\varsigma) - \psi(r))^{\vartheta-1} \right.$$

$$\left. E_{\vartheta,\,\vartheta}(A(\psi(\varsigma) - \psi(r))^\vartheta) f(r, z(r)) dr\right]. \tag{26}$$

□

Similarly to how (20) is obtained from (18), the following equation is obtained from (26),

$$z(\varsigma) = W^{-1}(\theta,\, \varsigma) \int_\theta^\varsigma \left[E_\vartheta(A(\psi(\varsigma) - \psi(r))^\vartheta)\right]^2 E_\vartheta(A^*(\psi(\varsigma) - \psi(r))^\vartheta) K^* y(r) dr$$

$$+ \eta\, W^{-1}(\theta,\, \varsigma) \int_\theta^\varsigma \psi'(r)(\psi(\varsigma) - \psi(r))^{\vartheta-1} E_{\vartheta,\,\vartheta}(A(\psi(\varsigma) - \psi(r))^\vartheta) W(\theta,\, r) f(r, z(r)) \Big) dr. \tag{27}$$

Applying (27) into (26), we get

$$z(\tau) = \left[E_\vartheta(A(\psi(\varsigma) - \psi(r))^\vartheta)\right]^{-1} \left[W^{-1}(\theta,\, \varsigma) \int_\theta^\varsigma K^* \left[E_\vartheta(A(\psi(\varsigma) - \psi(r))^\vartheta)\right]^2 \right.$$

$$\times E_\vartheta(A^*(\psi(\varsigma) - \psi(r))^\vartheta) y(r) dr + \eta W^{-1}(\theta,\, \varsigma) \int_\theta^\varsigma \psi'(r)(\psi(\varsigma) - \psi(r))^{\vartheta-1}$$

$$\times E_{\vartheta,\,\vartheta}(A(\psi(\varsigma) - \psi(r))^\vartheta) W(\theta,\, r) f(r, z(r)) \Big) dr - \eta \int_\tau^\varsigma \psi'(r)(\psi(\varsigma) - \psi(r))^{\vartheta-1} \tag{28}$$

$$\left. \times E_{\vartheta,\,\vartheta}(A(\psi(\varsigma) - \psi(r))^\vartheta) f(r, z(r)) dr\right].$$

Consequently, it is sufficient that $W^{-1}(\theta,\, \varsigma)$ and the solution of (28) must be exists and the solution of (28) must be unique, for the system (22) and (23) to be observable on $[\tau_0,\, \tau_1]$. If we assume that there exist solutions z_1, z_2 of (28), for a given w such that $z_1 \neq z_2$, we have

$$\|z_1(\tau)-z_2(\tau)\| \le k\eta \|(E_\vartheta(A(\psi(\varsigma)-\psi(r))^\vartheta)^{-1}\| \int_\theta^\varsigma \|\psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}\|$$

$$\|E_{\vartheta,\,\vartheta}(A(\psi(\varsigma)-\psi(r))^\vartheta)\|\|z_1-z_2\|dr + k\eta\|E_\vartheta(A(\psi(\varsigma)-\psi(r))^\vartheta)^{-1}\|$$

$$\|W^{-1}(\theta,\,\varsigma)\| \int_\theta^\varsigma \|\psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}\|\|E_{\vartheta,\,\vartheta}(A(\psi(\varsigma)-\psi(r))^\vartheta)\|\|z_1-z_2\|dr$$

$$\le \eta\ [k_1(\varsigma,\,\theta)+k_2(\varsigma,\,\theta)]\|z_1-z_2\|,$$

where

$$k_1(\varsigma,\,\theta) = \max_{\theta<\tau<s<t} \|E_\vartheta(A(\psi(\varsigma)-\psi(r))^\vartheta)^{-1}\psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}E_{\vartheta,\,\vartheta}(A(\psi(\varsigma)-\psi(r))^\vartheta)\|\|t-\theta\|k,$$

$$k_2(\varsigma,\,\theta) = \max_{\theta<\tau<s<t} \|E_\vartheta(A(\psi(\varsigma)-\psi(r))^\vartheta)^{-1}W^{-1}(\theta,\,\varsigma)\psi'(r)(\psi(\varsigma)-\psi(r))^{\vartheta-1}\|$$

$$\times \|E_{\vartheta,\,\vartheta}(A(\psi(\varsigma)-\psi(r))^\vartheta)\|\|t-\theta\|k.$$

From this, choose $k(\varsigma,\,\theta)$ such that

$$\|z_1(\tau)-z_2(\tau)\| \le \eta\ k(\varsigma,\,\theta)\|z_1-z_2\|,$$

where $k(\varsigma,\,\theta) = k_1(\varsigma,\,\theta)+k_2(\varsigma,\,\theta)$. If η satisfies the inequality

$$\eta < \frac{1}{k(\varsigma,\,\theta)}, \tag{29}$$

it follows that $z_1 = z_2$ on $[\theta,\,\varsigma]$. Clearly this is contradiction to $z_1 \neq z_2$ that directs to the sufficient condition for the nonlinear system (22) with linear observation (23) observable on $[\tau_0,\,\tau_1]$. Obviously, the condition (29) assures that the existence of solutions of (28).

5. Numerical examples

In this section, the first example, by taking $\psi(t) = t$ on the system (7), we provide and compare the numerical solution for the observability problems of ψ-Caputo fractional derivative with the observability problems of the well-known Caputo fractional derivative. In the last two examples, we present numerical results for the observability of fractional dynamical systems with respect to ψ-Caputo fractional derivative to demonstrate the texture's applicability and validate the numerical technique.

Example 1 Consider the linear system governed by ψ-Caputo fractional derivative

$$^{C}\mathbb{D}_{0+}^{0.5;\,\varsigma} z(\varsigma) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z(\varsigma), \quad \varsigma \in [0,\, 1], \tag{30}$$

with linear observation

$$w(\varsigma) = \begin{bmatrix} 1 & 0 \end{bmatrix} z(\varsigma), \quad \varsigma \in [0,\, 1]. \tag{31}$$

Comparing (30) with (7), we get $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\tau_0 = 0$, $\tau_1 = 1$, $\vartheta = 0.5$ and $\psi(\varsigma) = \varsigma$. Let $z(\varsigma) = \begin{bmatrix} z_1(\varsigma) \\ z_2(\varsigma) \end{bmatrix}$ and the linear observation for the system (32) is

$$w(\varsigma) = z_1(\varsigma) \;=\; \frac{1}{4}\big[N_1(1-i) + N_2(1+i)\big].$$

We utilize the $w(\varsigma)$ to identify $z_1(0)$ and $z_2(0)$. The Mittag-Leffler matrix function for the matrix A is

$$E_\vartheta(A(t-\tau_0)^\vartheta) = \frac{1}{2}\begin{bmatrix} N_1 + N_2 & -i(N_1 - N_2) \\ i(N_1 - N_2) & N_1 + N_2 \end{bmatrix},$$

where

$$N_1 = E_\vartheta(i(\varsigma - \tau_0)^\vartheta)$$

$$N_2 = E_\vartheta(-i(\varsigma - \tau_0)^\vartheta).$$

The observability Grammian matrix is

$$W[0,\, 1] = \int_0^1 E_\vartheta(A^*(\varsigma - \tau_0)^\vartheta) K^* K E_\vartheta(A(\varsigma - \tau_0)^\vartheta) d\varsigma \;=\; \begin{bmatrix} 0.4323 & 0.3097 \\ 0.3097 & 0.2864 \end{bmatrix}.$$

Noting that $W[0,\, 1]$ is symmetric, positive definite and its inverse is given by

$$W[0,\, 1]^{-1} = \begin{bmatrix} 10.2534 & -11.0856 \\ -11.0856 & 15.4767 \end{bmatrix}.$$

Let the reconstruction kernel is

$$Q(\varsigma) = W[0, 1]^{-1} E_{\vartheta}(A^*(\varsigma - \tau_0)^{\vartheta}) K^*$$

and clearly this $Q(\varsigma)$ satisfies

$$\int_0^1 Q(\varsigma) K E_{\vartheta}(A^*(\varsigma - \tau_0)^{\vartheta}) d\varsigma = I.$$

From the reconstruction formula

$$z(0) = \int_0^1 Q(\varsigma) w(\varsigma) d\varsigma = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

We get $z_1(0) = 0.5$ and $z_2(0) = 0.5$.

Example 2 Consider the linear system governed by ψ-Caputo fractional derivative

$$^{C}\mathbb{D}_{1+}^{0.75;\, \log\varsigma} z(\varsigma) = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{-1}{3} \end{bmatrix} z(\varsigma), \quad \varsigma \in [1, 2], \tag{32}$$

with linear observation

$$w(\varsigma) = \begin{bmatrix} 1 & 1 \end{bmatrix} z(\varsigma), \quad \varsigma \in [1, 2]. \tag{33}$$

Comparing(32) with (7), we get $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & \frac{-1}{3} \end{bmatrix}$, $K = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $\tau_0 = 1$, $\tau_1 = 2$, $\vartheta = 0.75$ and $\psi(\varsigma) = \log\varsigma$. Let $z(\varsigma) = \begin{bmatrix} z_1(\varsigma) \\ z_2(\varsigma) \end{bmatrix}$ and the linear observation for the system (32) is

$$\begin{aligned} w(\varsigma) &= z_1(\varsigma) + z_2(\varsigma) \\ &= \frac{11}{12} E_{\vartheta}\left(\frac{1}{2}(\log\varsigma - \log\tau_0)^{\vartheta}\right) - \frac{1}{12} E_{\vartheta}\left(\frac{-1}{3}(\log\varsigma - \log\tau_0)^{\vartheta}\right). \end{aligned}$$

We utilize the $w(\varsigma)$ to identify $z_1(1)$ and $z_2(1)$. The Mittag-Leffler matrix for the matrix A is

$$E_{\vartheta}(A(log\varsigma - \log\tau_0)^{\vartheta}) = \begin{bmatrix} P_1 & 0 \\ \frac{5}{6}(P_1 - P_2) & P_2 \end{bmatrix},$$

where

$$P_1 = E_\vartheta\left(\frac{1}{2}(\log\varsigma - \log\tau_0)^\vartheta\right)$$

$$P_2 = E_\vartheta\left(\frac{-1}{3}(\log\varsigma - \log\tau_0)^\vartheta\right).$$

The observability Grammian matrix is

$$W[1,\, 2] = \int_1^2 E_\vartheta(A^*(\log\varsigma - \log\tau_0)^\vartheta)K^*KE_\vartheta(A(\log\varsigma - \log\tau_0)^\vartheta)d\varsigma$$

$$= \begin{bmatrix} 6.6198 & 1.9667 \\ 1.9667 & 0.6003 \end{bmatrix}.$$

Noting that $W[1,\, 2]$ is symmetric, positive definite and its inverse is given by

$$W[1,\, 2]^{-1} = \begin{bmatrix} 5.6753 & -18.5942 \\ -18.5942 & 62.5875 \end{bmatrix}.$$

Let the reconstruction kernel is

$$Q(\varsigma) = W[1,\, 2]^{-1}E_\vartheta(A^*(\log\varsigma - \log\tau_0)^\vartheta)K^*$$

and clearly this $Q(\varsigma)$ satisfies

$$\int_1^2 Q(\varsigma)KE_\vartheta(A^*(\log\varsigma - \log\tau_0)^\vartheta)d\varsigma = I.$$

From the reconstruction formula

$$z(1) = \int_1^2 Q(\varsigma)w(\varsigma)d\varsigma \ = \begin{bmatrix} 0.5 \\ 0.3333 \end{bmatrix}.$$

We get $z_1(1) = 0.3333$ and $z_2(1) = 0.5$.

Example 3 Consider the nonlinear system governed by ψ-Caputo fractional derivative

$$
{}^{C}\mathbb{D}_{0+}^{0.2;\,\varsigma^2+\varsigma} z(\varsigma) = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} z(\varsigma) + \begin{bmatrix} 0 \\ \frac{1}{1+sin^2(z_1(\varsigma))} \end{bmatrix}, \quad \varsigma \in [0,\, 2], \tag{34}
$$

with linear observation

$$
w(\varsigma) = \begin{bmatrix} 0 & 1 \end{bmatrix} z(\varsigma), \quad \varsigma \in [0,\, 2]. \tag{35}
$$

Comparing (34) with (15), we get $A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$, $f(\varsigma,\, z(\varsigma)) = \begin{bmatrix} 0 \\ \frac{1}{1+sin^2(z_1(\varsigma))} \end{bmatrix}$, $\tau_0 = 0$, $\tau_1 = 2$, $K = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $\vartheta = 0.2$ and $\psi(\varsigma) = \varsigma^2 + \varsigma$. Let $z(\varsigma) = \begin{bmatrix} z_1(\varsigma) \\ z_2(\varsigma) \end{bmatrix}$ and the linear observation for the system (34) is

$$
w(\varsigma) = z_2(\varsigma) \;=\; \frac{1}{2} E_\vartheta(-(\varsigma^2 + \varsigma - {\tau_0}^2 - \tau_0)^\vartheta) - \frac{3}{2} E_\vartheta((\varsigma^2 + \varsigma - {\tau_0}^2 - \tau_0)^\vartheta).
$$

The Mittag-Leffler matrix for the matrix A is

$$
E_\vartheta(A(\varsigma^2 + \varsigma - {\tau_0}^2 - \tau_0)^\vartheta) = \begin{bmatrix} L_1 & 0 \\ \frac{L_1 - L_2}{2} & L_2 \end{bmatrix},
$$

where

$$
L_1 = E_\vartheta(-(\varsigma^2 + \varsigma - {\tau_0}^2 - \tau_0)^\vartheta)
$$

$$
L_2 = E_\vartheta((\varsigma^2 + \varsigma - {\tau_0}^2 - \tau_0)^\vartheta).
$$

The observability Grammian matrix for the system

$$
W[0,\, 2] = \int_0^2 E_\vartheta(A^*(\varsigma^2 + \varsigma - {\tau_0}^2 - \tau_0)^\vartheta) K^* K E_\vartheta(A(\varsigma^2 + \varsigma - {\tau_0}^2 - \tau_0)^\vartheta) d\varsigma = \begin{bmatrix} 25540.4 & -49420.6 \\ -49420.6 & 95635.3 \end{bmatrix}.
$$

Clearly $W[0,\, 2]$ is symmetric and positive definite. Then its inverse is

$$
W[0,\, 2]^{-1} = \begin{bmatrix} 0.5708 & 0.2950 \\ 0.2950 & 0.1524 \end{bmatrix}.
$$

Furthermore, for constant $k = 2$, the nonlinear function $f(\varsigma, z(\varsigma))$ fulfils condition (25). This system has a solution that is unique due to the use of Banach's fixed point theorem. The system (34) is (a) completely observable and globally (b) observable during the interval $[0, 2]$, according to Theorem (5).

6. Conclusion

In this study, we investigated the observability of linear and non-linear fractional dynamical systems in the sense of the ψ-Caputo fractional derivative. We have used the observability Grammian matrix for the linear case, which is described by the Mittag-Leffler function and Banach's fixed point theorem used to establish sufficient conditions for the observability of fractional dynamical systems for nonlinear case. Moreover, examples were provided to explain the theoretical results.

Author's contributions

A. Panneer Selvam: Conceptualization, Formal analysis, Investigation, Resources, Visualization, Software, Writing—original draft. V. Govindaraj: Investigation, Formal analysis, Writing- review and editing.

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Data availability statement

Data sharing does not apply to this article as no data sets were generated or analyzed during the current study.

Conflict of interest

The authors declare no competing financial interest.

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