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Research Article

Some New Results on the Number of Tagged Parts Over the Partitions with Designated Summands

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Abstract: The partitions with designated summands were introduced. In such partitions, among those parts of the same magnitude, one is tagged or designated. This work presents some new results on partitions with designated summands. The total number of partitions of n with designated summands is denoted by $PD(n)$. $PDO(n)$ indicates how many partitions of n there are with designated summands where every component is odd. The number of tagged parts across all partitions of n with specified summands is known as $PD_t(n)$, and the number of tagged parts across all partitions of n with designated summands in which all parts are odd is known as $PDO_t(n)$. In this study, we demonstrate a few new congruences modulo 2 and 4. We employed the dissection approach to access our recent discoveries.

Keywords: designated summands, partitions, dissections, congruence

MSC: 11P83, 05A17

1. Introduction

Andrews et al. [1] discovered and worked on a different group of partitions, partitions with designated summands. Take regular partitions and tag precisely one of each component size to create the partitions with designated summands. As an illustration, there are five partitions of 3 with designated summands, specifically,

$$3', 2' + 1', 1' + 1 + 1, 1 + 1' + 1, 1 + 1 + 1'.$$

$PD(n)$ displays the number of partitions having designated summands. Hence, $PD(3) = 5$. Andrews et al. [1] further researched $PDO(n)$ the number of partitions of n with designated summands, in which all components are odd. From the aforementioned illustration, $PDO(3) = 4$. Further research on $PD(n)$ and $PDO(n)$ were performed by Chen et al. [2], Baruah and Ojah [3], and Xia [4]. Two new partition functions, $PD_t(n)$ and $PDO_t(n)$ have been developed by Lin [5]. They count the number of tagged parts across all partitions of n with designated summands and, in turn, the number of tagged parts across all partitions of n with designated summands, in which all parts are odd. As demonstrated by the above partition of 3, $PD_t(3) = 6$ and $PDO_t(3) = 4$.

Moreover, Lin [5] has provided proofs for the following generating functions:

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$$\sum_{n=0}^{\infty} PD_t(n) q^n = \frac{1}{2}\left(\frac{f_3^5}{f_1^3 f_6^2} - \frac{f_6}{f_1 f_2 f_3} \right)$$

and

$$\sum_{n=0}^{\infty} PDO_t(n) q^n = q\frac{f_2 f_3^2 f_{12}^2}{f_1^2 f_6}.$$

Here and in the succeeding sections, we use

$$(a;q)_\infty = \prod_{n=1}^{\infty}\left(1-aq^{n-1}\right), |q|<1$$

and $f_k = (q^k;q^k)_\infty$, where a and q are two complex numbers and k is a positive integer.

Baruah and Kaur [6] and Vandna and Kaur [7] proved some new results on $PD_t(n)$ and $PDO_t(n)$. Baruah and Kaur [6] discovered the exact generating functions of $PDO_t(8n+6)$ and $PDO_t(8n+7)$ Moreover, they discovered several infinite families of congruences modulo 2 and 4 for $PD_t(n)$. In this paper, we use the various congruences found by Baruah and Kaur [6] to find infinite families of congruences and various novel congruences modulo 2 and 4.

2. Preliminaries

We now list several significant 2- and 3-dissections.

2.1 *Lemma 1*

2-dissections of q-products:

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2 f_8}, \tag{1}$$

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{2}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{3}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{4}$$

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}, \tag{5}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}. \tag{6}$$

$$\frac{f_1^3}{f_3}=\frac{f_4^3}{f_{12}}-3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2} \tag{7}$$

$$\frac{f_3}{f_1^3}=\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2}+3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7} \tag{8}$$

$$\frac{f_3^3}{f_1}=\frac{f_4^3 f_6^2}{f_2^2 f_{12}}+q\frac{f_{12}^3}{f_4} \tag{9}$$

$$\frac{f_1^2}{f_3^2}=\frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}}-2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4} \tag{10}$$

$$\frac{f_3^2}{f_1^2}=\frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}}+2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}} \tag{11}$$

2.2 *Lemma 2*

3-dissections of q-products:

$$\frac{f_1^2}{f_2}=\frac{f_9^2}{f_{18}}-2q\frac{f_3 f_{18}^2}{f_6 f_9} \tag{12}$$

$$\frac{f_2}{f_1^2}=\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3}+2q\frac{f_6^3 f_9^3}{f_3^7}+4q^2\frac{f_6^2 f_{18}^3}{f_3^6}, \tag{13}$$

$$f_1^3=f_3 a\left(q^3\right)-3qf_9^3, \tag{14}$$

$$\frac{1}{f_1^3}=a^2\left(q^3\right)\frac{f_9^3}{f_3^{10}}+3qa\left(q^3\right)\frac{f_9^6}{f_3^{11}}+9q^2\frac{f_9^9}{f_3^{12}} \tag{15}$$

$$\frac{1}{f_1 f_2}=a\left(q^6\right)\frac{f_9^3}{f_3^4 f_6^3}+qa\left(q^3\right)\frac{f_{18}^3}{f_3^3 f_6^4}+3q^2\frac{f_9^3 f_{18}^3}{f_3^4 f_6^4}, \tag{16}$$

where

$$a(q)=\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}=1+6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}}-\frac{q^{3n+2}}{1-q^{3n+2}}\right).$$

Here, we also use some of the following useful results:

$$a(q)=a(q^4)+6q\frac{f_4^2 f_{12}^2}{f_2 f_6},$$

$$a(q)+2a(q^2)=3\frac{f_2 f_3^6}{f_1^2 f_6^3},$$

$$a(q)+a(q^2)=2\frac{f_2^6 f_3}{f_1^3 f_6^2}.$$

In the sections that follow, we will regularly make use of the identities and congruences listed below, sometimes without explicitly mentioning them.

$$a(q) \equiv 1 \pmod 2,$$
$$a^2(q) \equiv 1 \pmod 4,$$
$$f_1^2 \equiv f_2 \pmod 2,$$
$$f_1^4 \equiv f_2^2 \pmod 4,$$
$$f_1^8 \equiv f_2^4 \pmod 8.$$

3. Results

Theorem 1. For $n > 0$, we have

$$\sum_{n=0}^{\infty} PD_t(432n+54)q^n \equiv f_1^3 \pmod 2, \tag{17}$$

$$PD_t(432n+198) \equiv 0 \pmod 2, \tag{18}$$

$$PD_t(432n+342) \equiv 0 \pmod 2, \tag{19}$$

$$\sum_{n=0}^{\infty} PD_t\left(216n+81\right)q^n \equiv f_3^3 \pmod 2, \tag{20}$$

$$\sum_{n=0}^{\infty} PD_t\left(216n+9\right)q^n \equiv f_1 \pmod 2, \tag{21}$$

$$\sum_{n=0}^{\infty} PD_t(36n+9)q^n \equiv \frac{f_2^4}{f_1^2} \pmod 4, \tag{22}$$

$$\sum_{n=0}^{\infty} PD_t(72n+9)q^n \equiv \frac{f_2^2}{f_1^2 f_4} \pmod 4, \tag{23}$$

$$\sum_{n=0}^{\infty} PD_t(72n+45)q^n \equiv 2\frac{f_4^5}{f_1 f_8^2} \pmod 4 \tag{24}$$

$$\sum_{n=0}^{\infty} PD_t(2592n+216)q^n \equiv f_1^2 a(q^2) \pmod 4, \tag{25}$$

$$\sum_{n=0}^{\infty} PD_t\left(2592n+1944\right)q^n \equiv f_6^3 \pmod 4, \tag{26}$$

$$PD_t(7776n+4536) \equiv 0 \pmod 4, \tag{27}$$

$$PD_t(7776n+7128) \equiv 0 \pmod 4. \tag{28}$$

4. Proof of the theorem

Using an equality in the proof of the Theorem 1 [6], we have

$$\sum_{n=0}^{\infty} PD_t\left(48n+6\right)q^n \equiv f_1^3 \equiv f_3 a\left(q^3\right) - 3qf_9^3 \pmod 2.$$

From which we extract

$$\sum_{n=0}^{\infty} PD_t\left(144n+54\right)q^n \equiv -3f_3^3 \pmod 2 \equiv f_3^3 \pmod 2.$$

From which when we extract the terms having q^{3n}, q^{3n+1}, q^{3n+2}, we obtain (17), (18), and (19), respectively.

Now, from [6, equation (5.4)], we have

$$\sum_{n=0}^{\infty} PD_t\left(72n+9\right)q^n \equiv f_1^3 \equiv f_3 a\left(q^3\right) - 3qf_9^3 \pmod 2. \tag{29}$$

Extracting the terms involving q^{3n+1} from both sides, we get

$$\sum_{n=0}^{\infty} PD_t\left(216n+81\right)q^n \equiv f_3^3 \pmod 2,$$

which is (20).

From (29), extracting the terms involving q^{3n} from both sides, we get

$$\sum_{n=0}^{\infty} PD_t\left(216n+9\right)q^n \equiv f_1 a\left(q\right) \pmod 2,$$

which is (21).

Now, from an equality in the proof of the Theorem 1 [6], we have

$$\sum_{n=0}^{\infty} PD_t\left(12n+9\right)q^n \equiv \frac{f_6^4}{f_3^2} \pmod 4, \tag{30}$$

from which we extract

$$\sum_{n=0}^{\infty} PD_t\left(36n+9\right)q^n \equiv \frac{f_2^4}{f_1^2} \pmod 4,$$

which is (22).

Also, from (6), we further get

$$\sum_{n=0}^{\infty} PD_t\left(36n+9\right)q^n \equiv f_2^4\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right) \equiv \left(\frac{f_8^5}{f_2 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2 f_8}\right) \pmod 4 \tag{31}$$

from which we extract

$$\sum_{n=0}^{\infty} PD_t\left(72n+9\right)q^n \equiv \frac{f_2^2}{f_1^2 f_4} \pmod 4,$$

which is (23).

Again, extracting the terms involving q^{2n+1} from (31), we have

$$\sum_{n=0}^{\infty} PD_t(72n+45)q^n \equiv 2\frac{f_4^5}{f_1^2 f_8^2} \pmod 4,$$

which is (24).

Now, from an equality in the proof of the Theorem 1 [6], we have

$$\sum_{n=0}^{\infty} PD_t(864n+216)q^n \equiv \frac{f_3^2}{f_6}\left(f_6 a\left(q^6\right) - 3q^2 f_{18}^3\right) \pmod 4$$

from which we extract the terms involving q^{3n} to get

$$\sum_{n=0}^{\infty} PD_t(2592n+216)q^n \equiv f_1^2 a\left(q^2\right) \pmod 4,$$

which is (25).

Again, on further extraction, we get

$$\sum_{n=0}^{\infty} PD_t(2592n+1944)q^n \equiv f_6^3 \pmod 4,$$

which is (26).

Also, extracting the terms q^{3n+1} and q^{3n+2} from (26), we arrive at (27) and (28), respectively.

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Conflict of interest

There is no conflict of interest for this study.

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