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Research Article

A Semi-Analytical Method for Solving Nonlinear Fractional-Order Swift-Hohenberg Equations

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Abstract: In this study, we find approximate series solutions to fractional-order Swift-Hohenberg equations by using the hybrid method, i.e., accelerated homotopy perturbation transformation method (AHPTM). The accelerated homotopy perturbation method was merged with the Laplace transform to create the proposed method. We also compare the results of our proposed method with the exact solution and demonstrate that it is the useful tool for tackling nonlinear problems of fractional order. Results are presented through graphs using Mathematica software.

Keywords: fractional-order Swift-Hohenberg (S-H) equations, Liouville-Caputo fractional order derivative, Laplace transform, homotopy perturbation method

MSC: 35R11

1. Introduction

The study of derivatives and integrals of non-integer orders is known as fractional calculus. This field has gained much attention recently due to its numerous applications in various scientific and engineering disciplines. Fractional differential equations have been found to have many useful applications. The fractional derivative is defined by an integral and is a non-local operator with a singular kernel, providing a useful method for exploring the memory and hereditary qualities of a range of physical processes. The meaning of fractional derivative is not fixed, and various forms of fractional integration and differentiation exist, including those based on Riemann-Liouville, Liouville-Caputo, Riesz, and Weyl [1-3]. The two most widely used fractional calculus operations are the Riemann-Liouville fractional integral and the Caputo fractional derivative.

In 1695, Leibniz mentioned the use of fractional differential operators in a letter to L'Hôpital. Unlike traditional derivatives and integrals, fractional derivatives and integrals are not based on the immediate characteristics of a particular function. This is what makes the concept of fractional calculus so appealing [4]. Fractional calculus has been extensively applied in various fields by numerous authors. Examples included, but are not limited to, Baleanu et al. [5] employ fixed-point theorems to demonstrate the singularity and exclusivity of solutions to nonlinear fractional differential equations that have particular boundary conditions. Chen et al. [6] created a fractional variational optical flow model for motion estimation from video sequences, and their experiments demonstrate that this model could effectively generalize the derivative order.

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By utilizing the features of fractional derivatives, Li and Zhang [7] were able to convert a different rational-order system into a fractional system that had identical order. This simplification made it much easier to conduct stability analysis and numerical simulations. Klimek [8] examines the utilization of reflection symmetry in the Euler-Lagrange equations of fractional mechanics, demonstrating the concept with a practical example. Okur and Yigider [9] applied conformable fractional reduced differential transformation method (CFRDTM). To solve time fractional differential equations. Shah et al. [10] applied a new semi-analytical method called variational iteration transformation method for solving nonlinear homogeneous and nonhomogeneous fractional-order gas dynamics equations. A realistic method for resolving fractional hyperbolic telegraph equations (FHTEs) and other fractional differential equations is presented by Kapoor and Khosla [11]. Khater et al. [12] utilized five semi-analytical and numerical techniques, namely Adomain decomposition (AD), El Kalla (EK), cubic B-spline (CBS), extended cubic B-spline (ECBS), exponential cubic B-spline (ExCBS), to check the accuracy of analytical solutions of the fractional nonlinear space-time telegraph equation. Rezaei and Izadi [13] suggested a novel method for resolving a time-space fractional Black-Scholes equation that occurs in financial market. Singh and Singh [14] used the New Laplace variational iteration method (NLVIM) to solve the coupled Burgers' equation in three dimensions. The method is also applicable to system partial differential equations in three dimensions that arise in variety of scientific and engineering applications.

In this paper, we apply the accelerated homotopy perturbation transformation method (AHPTM) to the time-fractional Swift-Hohenberg (S-H) equation. The S-H equation, which was originally established and obtained from the equations of thermal convection by Swift and Hohenberg [15], rapidly became known as an approximate model for the creation of nonlinear patterns. It is well-known for its capacity to generate results that are remarkably similar to those of the Navier-Stokes equations, which can be difficult to solve with numerical methods [16].

The fractional order of S-H equation is [17]

$$\frac{\partial^\alpha \omega}{\partial t^\alpha} = \beta\omega - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 - N(\omega), \quad 0 < \alpha \le 1. \tag{1}$$

where ω is the scalar function, β the real constant, and the nonlinear term $N(\omega)$.

This equation has been utilised to make sense of a variety of pattern development wonders, including truing examples, hexagonal designs, and spirals. It is ordinarily utilised to model frameworks, for example, Rayleigh-Benard convection, laser elements, and synthetic responses. Various strategies and methodologies have been employed by researchers to study the S-H equation, such as the power series method [18], the q-homotopy analysis transform method [19], the iterative method [20], the homotopy analysis method [21], the homotopy perturbation method with the fractional complex transform [17], the fractional natural transform decomposition method (FNTDM) [22], the Laplace residual power series (LRPS) technique [23], etc.

In this paper, we present a variant of He's polynomial known as the accelerated He's polynomial [24], which increases the method's convergence. We then apply the AHPTM, a semi-analytical approach that does not require a problem critique or linearization of a nonlinear problem. Instead, the solution can be obtained after a few repetitions and with ease. With the help of Mathematica software, we are able to compute more terms with ease, which lowers the computational cost of handling such complicated problems. Finally, a comparison between the accelerated homotopy perturbation transformation technique's output and the exact solution is made.

2. Preliminaries

In this part, we will go over definitions that will be useful for this article, such as the Riemann-Liouville fractional integral, the Gamma function, the Mittag-Leffler function, the Liouville-Caputo derivative fractional, and the Laplace transform.

Definition 2.1. The Riemann-Liouville fractional integral $\mathbb{I}^\beta f(t)$ of order $\beta \in \mathbb{C}\ \Re(\beta) > 0$), for a function $f(t)$ is defined as [25]

$$(\mathbb{I}^{\beta} f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau) d(\tau), \quad (t > 0; \Re(\beta) > 0).$$

Definition 2.2. Liouville-Caputo fractional order derivative: Let $w(t) \in AC^n(a, b)$, the Liouville-Caputo [26] is presented as

$$D_t^{\beta} w(t) = \frac{1}{\Gamma(l-\beta)} \int_0^t \frac{w^{(l)}(z)}{(t-z)^{l-\beta+1}} dz,$$

for $l-1 < \beta < l,\ l \in \mathbb{N},\ t > 0.$

Definition 2.3. Laplace transform: The Liouville-Caputo derivative's Laplace transform is provided by Caputo [27].

$$\mathcal{L}[D_t^{\alpha} f(t)] = s^{\alpha} \mathcal{L}[f(t)] - \sum_{m=0}^{m-1} s^{\alpha-n-1} f^{(k)}(0), \quad m-1 < \alpha \le m.$$

Definition 2.4. Mittag-Leffler function: The Mittag-Leffler function [28] is defined as

$$E_{\alpha}(x) = \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(\alpha m+1)}, \quad \alpha > 0.$$

Definition 2.5. Gamma function: The gamma function was developed by Swiss mathematician, Leonhard Euler (1707- 1783), in an effort to expand the notion of the fractional to a non-integer number [29].

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt.$$

3. AHPTM

To demonstrate the basic concept of this technique, take the generic nonlinear nonhomogeneous fractional partial differential equation with an initial condition.

$$D_t^{\alpha} \omega(\varphi, t) + R(\varphi, t) + N\omega(\varphi, t) = g(\varphi, t), \text{ with condition } \omega(\varphi, 0) = k(\varphi). \tag{2}$$

where $D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the fractional Liouville-Caputo derivative of the function $\omega(\varphi, t)$, R and N are the linear and the nonlinear differential operator and $g(\varphi, t)$ is the source term.

Taking the Laplace transform on both sides of equation (2), we get

$$\mathcal{L}[D_t^{\alpha} \omega(\varphi, t)] + \mathcal{L}\omega[R(\varphi, t)] + \mathcal{L}[N\omega(\varphi, t)] = \mathcal{L}[g(\varphi, t)], \tag{3}$$

operating the properties of Laplace transformation to equation (3), we get

$$\mathcal{L}[\omega(\varphi, t)] = \frac{1}{s^{\alpha}} \omega(\varphi, 0) + \frac{1}{s^{\alpha}} \mathcal{L}\left(g(\varphi, t) - R\omega(\varphi, t) - N\omega(\varphi, t)\right), \tag{4}$$

operating the inverse of Laplace transform to both sides of equation (4), we get

$$\omega(\varphi,t)=(\varphi,0)+\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left(g(\varphi,t)-R\omega(\varphi,t)-N\omega(\varphi,t)\right)\right\}, \tag{5}$$

by using homotopy perturbation method (HPM), we get

$$0=(1-p)[\omega(\varphi,t)-\omega(\varphi,0)]+p\left[\omega(\varphi,t)-\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left(g(\varphi,t)-R\omega(\varphi,t)-N\omega(\varphi,t)\right)\right\}\right], \tag{6}$$

where $p \in [\,0, 1]$ is a parameter. Let

$$\omega(\varphi,t)=\sum_{n=0}^{\infty}p^{n}\omega_{n}(\varphi,t), \tag{7}$$

and the nonlinear term decompose as

$$N\omega(\varphi,t)=\sum_{n=0}^{\infty}p^{n}\tilde{H}_{n}, \tag{8}$$

where $\tilde{H}_n$ represents accelerated He's polynomial with

$$\tilde{H}_{n}(\omega_0,\omega_1,\omega_2\ldots\omega_n)=N(S_k)-\sum_{i=0}^{n-1}\tilde{H}_{n}, \tag{9}$$

where $\tilde{H}_0=N(\omega(\varphi_0))$ and $S_k=(\omega_0+\omega_1+\ldots+\omega_k)$.

Substituting equations (7) and (8) in equation (6), we get

$$\sum_{n=0}^{\infty}p^{n}\omega_{n}(\varphi,t)=\omega(\varphi,0)+p\left[\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left\{\sum_{n=0}^{\infty}p^{n}\omega_{n}(\varphi,t)-\sum_{n=0}^{\infty}p^{n}\tilde{H}_{n}\left(\omega(\varphi,t)\right)\right\}\right\}\right]. \tag{10}$$

When we compare similar powers of p, we get

$$\begin{aligned}
&p^{0}:\omega_0=\omega(\varphi,0),\\
&p^{1}:\omega_1=\omega_1(\varphi,t)=\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left[g(\varphi,t)-R\omega_0-\tilde{H}_0\omega)\right]\right\},\\
&p^{2}:\omega_2=\omega_2(\varphi,t)=-\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left[R\omega_1+\tilde{H}_1\omega)\right]\right\},\\
&p^{3}:\omega_3=\omega_3(\varphi,t)=-\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left[R\omega_2+\tilde{H}_2\omega)\right]\right\},\\
&\quad\vdots
\end{aligned}$$

Hence, when $p \to 1$, the approximate solution of equation (2) is obtained as

$$\omega(\varphi,t)=\omega_0+\omega_1+\omega_3\ldots \tag{11}$$

4. Convergence analysis

The condition of convergence of the above-mentioned approach is mentioned below.

Theorem 4.1. If there exists an η in the range $0 < \eta < 1$ for the ω and $\omega_n(\varphi, t)$ described in Banach space [30], then the equation defined series solution (10) converges to the solution defined by equation (2), if $\omega_{n+1} \| \leq \eta \| \omega_n \|$.

Proof. For convergence of sequence $\{S_n\}$ of the partial sums of the series (10), we prove that $\{S_n\}$ is a Cauchy sequence in $(C[0, K], \| . \|)$.

$$\| s_{n+1} - s_n \| = \| \omega_{n+1} \| \leq \eta \| \omega_n \| \leq \eta^2 \| \omega_{n-1} \| \leq \ldots \leq \eta^{n+1} \| \omega_0 \|.$$

Here,

$$\begin{aligned} \| s_n - s_m \| &= \left\| \sum_{i=m+1}^{n} \omega_i \right\| \leq \sum_{i=m+1}^{n} \|\omega_i\| \\ &\leq \eta^{m+1} \left(\sum_{i=0}^{n-m} \eta^i \right) \|\omega_0\| \\ &= \eta^{m+1} \frac{(1-\eta^{n-m})}{(1-\eta)} \|\omega_0\|, \quad n, m \in \mathbb{N}. \end{aligned}$$

Since $0 < \eta < 1$, hence

$$\| s_n - s_m \| \leq \frac{\eta^{n+1}}{1-\eta} \|\omega_0\|,$$

also, ω_0 is bounded, therefore $\| s_n - s_m \| \to 0$ as $m, n \to \infty$. So, $\{S_n\}$ is a Cauchy sequence in $C[0, K]$. Hence, $\sum_{n=0}^{\infty} \omega_n(\varphi, t)$ is convergent [30, 31].

5. Numerical examples

Example 5.1. Regarding the nonlinear time fractional S-H equation in the Liouville-Caputo sense [32].

$$\frac{\partial^\alpha \omega(\varphi,t)}{\partial t^\alpha} + \frac{\partial^4 \omega(\varphi,t)}{\partial \varphi^4} + (1-\beta)\omega(\varphi,t) + 2\frac{\partial^2 \omega(\varphi,t)}{\partial \varphi^2} - \omega^2(\varphi,t) + \left(\frac{\partial \omega(\varphi,t)}{\partial \varphi} \right)^2 = 0, \quad 0 < \alpha \leq 1. \tag{12}$$

With the initial condition $\omega(\varphi, 0) = e^{\varphi}$.

Apply AHPTM, we get

$$\sum_{n=0}^{\infty} p^n \omega_n = e^{\varphi} - p\left[\mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left[\left\{ \frac{\partial^4 \omega(\varphi,t)}{\partial \varphi^4} + (1-\beta)\omega(\varphi,t) + 2\frac{\partial^2 \omega(\varphi,t)}{\partial \varphi^2} \right\} - \left\{ \sum_{n=0}^{\infty} p^n \tilde{H}_n(\omega) \right\} \right] \right\} \right]. \tag{13}$$

The first few components of $\tilde{H}_n$ are given as

$$\begin{aligned} \tilde{H}_0 &= (\omega_0)^2 - (\omega_{0\varphi})^2, \\ \tilde{H}_1 &= 2\omega_0\omega_1 + \omega_1^2 - \omega_{1\varphi}(2\omega_{0\varphi} + \omega_{1\varphi}), \\ \tilde{H}_2 &= 2\omega_0\omega_2 + 2\omega_1\omega_2 + (\omega_2)^2 - 2\omega_{0\varphi}\omega_{2\varphi} - 2\omega_{1\varphi}\omega_{2\varphi} - (\omega_{2\varphi})^2, \\ &\vdots \end{aligned}$$

by equating the same power of p on both sides of equation (13), we obtain

$$\omega_0 = e^{\varphi},$$
$$\omega_1 = \frac{e^{\varphi} t^{\alpha} (\beta - 4)}{\Gamma(\alpha + 1)},$$
$$\omega_2 = \frac{e^{\varphi} t^{2\alpha} (\beta - 4)^2}{\Gamma(2\alpha + 1)},$$
$$\omega_3 = \frac{e^{\varphi} t^{3\alpha} (\beta - 4)^3}{\Gamma(3\alpha + 1)},$$
$$\vdots$$

The approximate series solution of equation (12) is obtained by AHPTM.

$$\omega(\varphi,t) = \sum_{m=0}^{\infty} \omega_m(\varphi,t) = \omega_0(\varphi,t) + \omega_1(\varphi,t) + \omega_2(\varphi,t) + \ldots$$
$$\omega(\varphi,t) = e^{\varphi} + \frac{e^{\varphi} t^{\alpha} (\beta - 4)}{\Gamma(\alpha + 1)} + \frac{e^{\varphi} t^{2\alpha} (\beta - 4)^2}{\Gamma(2\alpha + 1)} + \frac{e^{\varphi} t^{3\alpha} (\beta - 4)^3}{\Gamma(3\alpha + 1)} + \ldots \tag{14}$$

and the exact solution of equation (12) is

$$\omega(\varphi,t) = e^{\varphi} \mathrm{E}_{\alpha}\left((\beta - 4)t^{\alpha}\right).$$

Example 5.2. Consider the nonlinear time fractional S-H equation [32].

$$\frac{\partial^{\alpha} \omega(\varphi,t)}{\partial t^{\alpha}} + \frac{\partial^4 \omega(\varphi,t)}{\partial \varphi^4} + (1-\beta)\omega(\varphi,t) + 2\frac{\partial^2 \omega(\varphi,t)}{\partial \varphi^2} - \rho \frac{\partial^3 \omega(\varphi,t)}{\partial \varphi^3} - \omega^2(\varphi,t) + \left(\frac{\partial \omega(\varphi,t)}{\partial \varphi}\right)^2 = 0, \quad 0 < \alpha \le 1. \tag{15}$$

With the initial condition $\omega(\varphi, 0) = e^{\varphi}$.

Apply AHPTM, we get

$$\sum_{n=0}^{\infty} p^n \omega_n = e^{\varphi} - p\left[\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left[\left\{\frac{\partial^4 \omega(\varphi,t)}{\partial \varphi^4} + (1-\beta)\omega(\varphi,t) + 2\frac{\partial^2 \omega(\varphi,t)}{\partial \varphi^2} - \rho\frac{\partial^3 \omega(\varphi,t)}{\partial \varphi^3}\right\} - \left\{\sum_{n=0}^{\infty} p^n \tilde{H}_n(\omega)\right\}\right]\right\}\right]. \tag{16}$$

The first few components of $\tilde{H}_n$ are given as

$$\tilde{H}_0 = (\omega_0)^2 - (\omega_{0\varphi})^2,$$
$$\tilde{H}_1 = 2\omega_0\omega_1 + \omega_1^{\,2} - \omega_{1\varphi}(2\omega_{0\varphi} + \omega_{1\varphi}),$$
$$\tilde{H}_2 = 2\omega_0\omega_2 + 2\omega_1\omega_2 + \omega_2^{\,2} - 2\omega_{0\varphi}\omega_{2\varphi} - 2\omega_{1\varphi}\omega_{2\varphi} - \omega_{2\varphi}^{\,2},$$
$$\vdots$$

by equating the same power of p on both sides of equation (16), we obtain

$$\omega_0 = e^{\varphi},$$
$$\omega_1 = \frac{e^x t^{\alpha}(\rho+\beta-4)}{\Gamma(\alpha+1)},$$
$$\omega_2 = \frac{e^{\varphi} t^{2\alpha}(\rho+\beta-4)^2}{\Gamma(2\alpha+1)},$$
$$\omega_3 = \frac{e^{\varphi} t^{3\alpha}(\rho+\beta-4)^3}{\Gamma(3\alpha+1)},$$
$$\vdots$$

The approximate series solution of equation (15) is obtained by AHPTM.

$$\omega(\varphi,t) = \sum_{m=0}^{\infty} \omega_m(\varphi,t) = \omega_0(\varphi,t) + \omega_1(\varphi,t) + \omega_2(\varphi,t) + \ldots$$
$$\omega(\varphi,t) = e^{\varphi} + \frac{e^x t^{\alpha}(\rho+\beta-4)}{\Gamma(\alpha+1)} + \frac{e^{\varphi} t^{2\alpha}(\rho+\beta-4)^2}{\Gamma(2\alpha+1)} + \frac{e^{\varphi} t^{3\alpha}(\rho+\beta-4)^3}{\Gamma(3\alpha+1)} + \ldots \tag{17}$$

and the exact solution of equation (15) is

$$\omega(\varphi,t) = e^{\varphi} \mathrm{E}_{\alpha}\left((\rho+\beta-4)t^{\alpha}\right).$$

Example 5.3. Consider the non-linear time fractional S-H equation [27].

$$\frac{\partial^{\alpha}\omega(\varphi,t)}{\partial t^{\alpha}} + \frac{\partial^4\omega(\varphi,t)}{\partial\varphi^4} + 2\frac{\partial^2\omega(\varphi,t)}{\partial\varphi^2} - \rho\frac{\partial^3\omega(\varphi,t)}{\partial\varphi^3} + \beta\omega(\varphi,t) - 2\omega^2(\varphi,t) + \omega^3(\varphi,t) = 0, \quad 0 < \alpha \le 1. \tag{18}$$

With the initial condition $\omega(\varphi, 0) = \cos(x)$.

Apply AHPTM, we get

$$\sum_{n=0}^{\infty} p^n \omega_n = \cos(x) - p\left[\mathcal{L}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{L}\left[\left\{\frac{\partial^4\omega}{\partial\varphi^4} + 2\frac{\partial^2\omega(\varphi,t)}{\partial\varphi^2} - \rho\frac{\partial^3\omega(\varphi,t)}{\partial\varphi^3} + \beta\omega(\varphi,t)\right\} - \left\{\sum_{n=0}^{\infty} p^n \tilde{H}_n(\omega)\right\}\right]\right\}\right]. \tag{19}$$

The first few components of $\tilde{H}_n$ are given as

$$\tilde{H}_0 = 2(\omega_0)^2 - (\omega_0)^3,$$
$$\tilde{H}_1 = 2\omega_1^2 + 4\omega_0\omega_1 - 3\omega_0^2\omega_1 - 3\omega_0\omega_1^3$$
$$\tilde{H}_2 = 4\omega_0\omega_2 + 4\omega_1\omega_2 + 2\omega_2^2 - 3\omega_0^2\omega_2 - 6\omega_0\omega_1\omega_2 - 3\omega_2^2\omega_0 - 3\omega_1^2\omega_2 - 3\omega_2^2 - \omega_2^3,$$
$$\vdots$$

by equating the same power of p on both sides of equation (19), we obtain

$$\omega_0 = \cos(x),$$

$$\omega_1 = \left((1+\beta)\cos(x) + 2\cos(x)^2 - \cos(x)^3 + \rho\sin(x)\right)\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$\omega_2 = \frac{1}{64}[-(4t^\alpha(-2+3\cos(x)\Gamma(2\alpha+1)$$

$$(4+\cos(x)+4\beta\cos(x)+4\cos(2x)-\cos(3x)+4\rho\sin(x))^2)]\frac{t^{2\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}$$

$$-\Gamma(3\alpha+1)(4+\cos(x)+4\beta\cos(x)+4\cos(2x)-\cos(3x)+4\rho\sin(x))^3\frac{t^{2\alpha}}{\Gamma(\alpha+1)^3\Gamma(4\alpha+1)}$$

$$+\frac{1}{\Gamma(2\alpha+1)}64(-4(-5+\beta)\cos(x)^3 - 10\cos(x))^4 + 3\cos(x)^5$$

$$+\cos(x)^2(-4+6\beta-24\rho\sin(x)) + \cos(x)(-\rho^2+(1+\beta)^2+20k\sin(x)-48\sin(x)^2)$$

$$+2\sin(x)(\rho+\rho\beta+4\sin(x)+3\rho))],$$

$$\vdots$$

The approximate series solution of equation (18) is obtained by AHPTM.

$$\omega(\varphi,t) = \sum_{m=0}^{\infty}\omega_m(\varphi,t) = \omega_0(\varphi,t)+\omega_1(\varphi,t)+\omega_2(\varphi,t)+\ldots$$

$$\omega(\varphi,t) = \cos(x) + \left((1+\beta)\cos(x)+2\cos(x)^2-\cos(x)^3+\rho\sin(x)\right)\frac{t^\alpha}{\Gamma(\alpha+1)}+\ldots \tag{20}$$

6. Results and discussion

This study presents the AHPTM for solving time-fractional S-H equations. Graphical representations are used to showcase the results. Figures 1(a) and 1(b) show the 3D graph of the exact solution and AHPTM, and Figure 1(c) shows the combined exact and AHPTM 2D plot solution for $\alpha = 1$. The results in Figure 2 show that the solutions for varying fractional-orders become closer to the solution for an integer-order as the fractional-orders approach an integer-order. The graphs are drawn for $\beta = 5$ and $0 < \varphi \leq 1$. Figures 3 and 4 present the same graphical representation for $\beta = 5$, $\rho = 1$ and $0 \leq \varphi \leq 1$. Figure 5 represents the line graph of Example 5.3 at different fractional order $\alpha = 1, 0.95, 0.85, 0.75$, $\beta = 0.5$, $\rho = 0.3$, $0 < \varphi \leq 1$ and $t = 0.2$. Figure 5 shows the approximate solution of the fractional S-H equation up to three terms by using $\rho = 0.3$ and $\beta = 0.5$. Furthermore, we investigate the convergence analysis of this method and observed that $\|\omega_1\| > \|\omega_2\| > \|\omega_3\|$ (Tables 1 and 2).

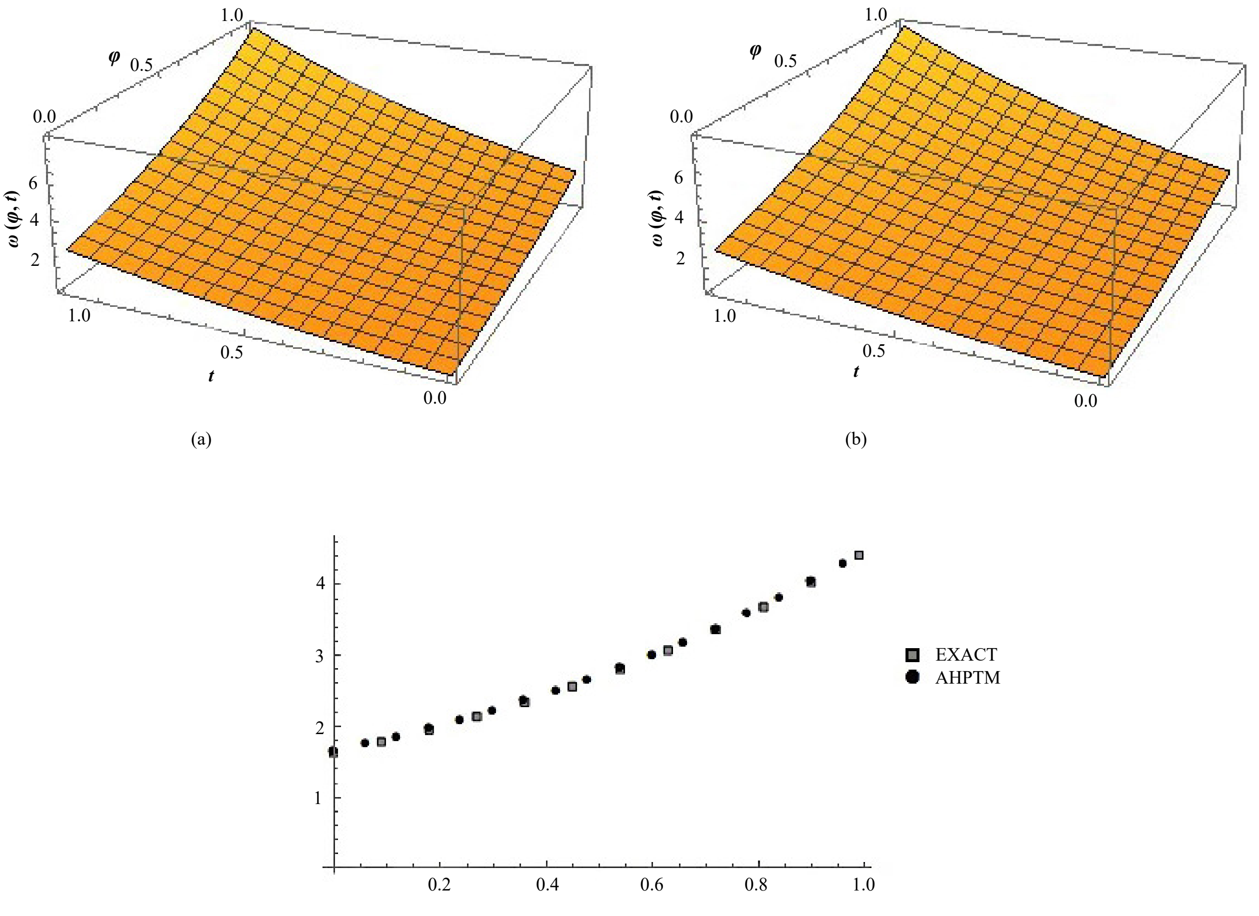


Figure 1. (a) Surface graph of exact solution at $\alpha = 1$; (b) Surface graph AHPTM at $\alpha = 1$; (c) Line graph of exact solution and AHPTM at $\alpha = 1$, $\beta = 5$, $0 < \varphi \leq 1$ and $t = 0.5$

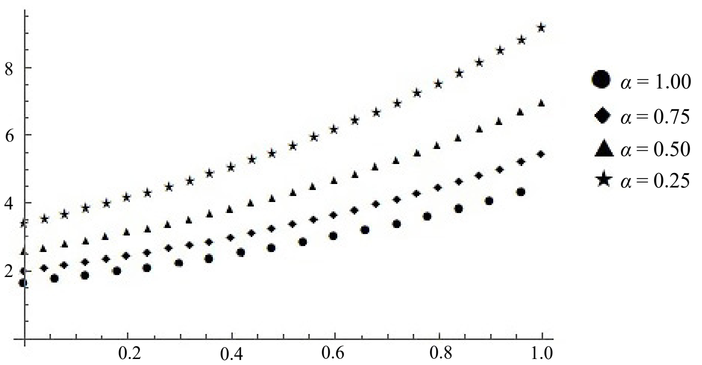
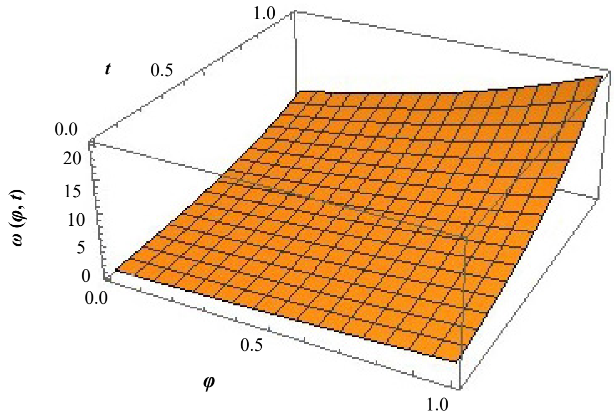
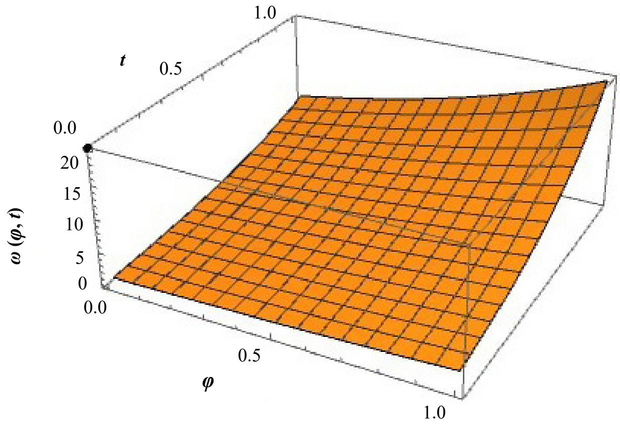


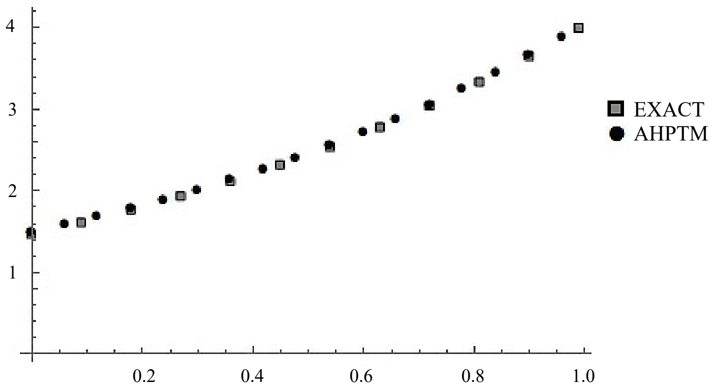
Figure 2. Solution of AHPTM at various fractional order $\alpha = 1, 0.75, 0.5, 0{,}25$, $\beta = 5$, $0 < \varphi \leq 1$ and $t = 0.5$



(a)



(b)



(c)

Figure 3. (a) Surface graph of exact solution at $\alpha = 1$; (b) Surface graph AHPTM at $\alpha = 1$; (c) Line graph of exact solution and AHPTM at $\alpha = 1$, $\beta = 5$, $\rho = 1$, $0 < \varphi \leq 1$ and $t = 0.2$

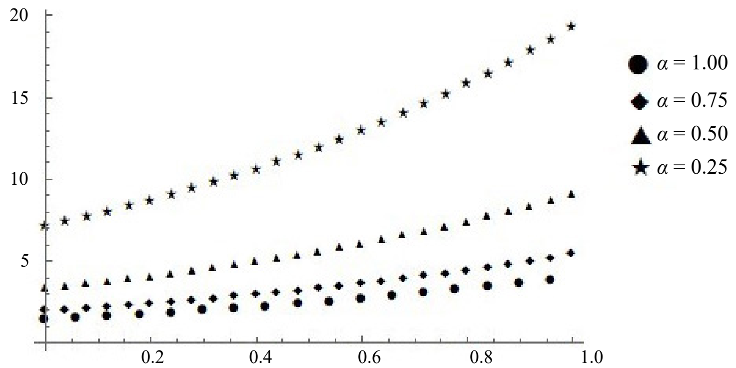


Figure 4. Solution of AHPTM at various fractional order $\alpha = 1, 0.75, 0.5, 0.25$, $\beta = 5$, $\rho = 1$, $0 < \varphi \leq 1$ and $t = 0.2$

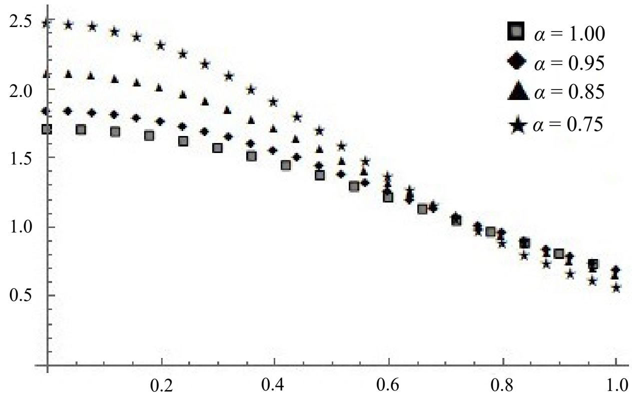


Figure 5. Solution of AHPTM at various fractional order $\alpha = 1, 0.75, 0.5, 0{,}25$, $\beta = 5$, $0 < \varphi \le 1$ and $t = 0.5$

Table 1. Error analysis of Example 5.1 at $\alpha = 1$ (up to the fourth order)

t	x	Exact	AHPTM	Absolute Error	$\lVert\omega_1\rVert$	$\lVert\omega_2\rVert$	$\lVert\omega_3\rVert$
0.1	0.1	1.2214027	1.2213981	4.6985E-06	0.11051709	0.0055259	0.00018419
	0.3	1.4918247	1.491819	5.7388E-06	0.13498588	0.0067493	0.00022497
	0.5	1.8221188	1.8221118	7.0093E-06	0.16487213	0.0082436	0.00024787
	0.7	2.2255409	2.2255324	8.56129E-06	0.20137527	0.0100688	0.00033563
	0.9	2.7182818	2.7182714	1.04568E-05	0.24596031	0.012298	0.00040993
0.3	0.1	1.4918247	1.4914282	0.00039654	0.33155128	0.0497327	0.00497327
	0.3	1.8221188	1.8216345	0.00048434	0.40495764	0.0607436	0.00607436
	0.5	2.2255409	2.2249494	0.00059157	0.49461638	0.0741925	0.00741925
	0.7	2.7182818	2.7175593	0.00072255	0.60412581	0.0906189	0.00906189
	0.9	3.3201169	3.3192344	0.00088252	0.73788093	0.1106821	0.01106821
0.5	0.1	1.8221188	1.8189271	0.00319166	0.55258546	0.1381464	0.0230244
	0.3	2.2255409	2.2216426	0.00389830	0.6749294	0.1687324	0.0281221
	0.5	2.7182818	2.7135204	0.00476140	0.82436064	0.2060902	0.0343484
	0.7	3.3201169	3.3143013	0.00581559	1.00687635	0.2517191	0.0419532
	0.9	4.0551999	4.0480968	0.00710318	1.22980156	0.3074504	0.0512417

Table 2. Error analysis Example 5.2 at $\alpha = 1$ (up to the fourth order)

t	x	Exact	AHPTM	Absolute Error	$\lVert\omega_1\rVert$	$\lVert\omega_2\rVert$	$\lVert\omega_3\rVert$
0.1	0.1	1.349858808	1.3497821	0.000076726	0.2210342	0.022103418	0.001473561
	0.3	1.648721271	1.6486276	9.37137E-05	0.269972	0.026997176	0.001799812
	0.5	2.013752707	2.0136382	0.000114462	0.3297443	0.032974425	0.002198295
	0.7	2.459603111	2.4594633	0.000139804	0.4027505	0.040275054	0.002685
	0.9	3.004166024	3.0039953	0.000170758	0.4919206	0.049192062	0.003279471
0.3	0.1	2.013752707	2.0069904	0.00676232	0.6631026	0.001473561	0.039786153
	0.3	2.459603111	2.4513436	0.008259517	0.8099153	0.001799812	0.048594917
	0.5	3.004166024	2.9940778	0.010088196	0.9892328	0.002198295	0.059353966
	0.7	3.669296668	3.6569749	0.012321751	1.2082516	0.002685	0.072495097
	0.9	4.48168907	4.4666392	0.01504982	1.4757619	0.003279471	0.088545712
0.5	0.1	3.004166023	2.9471224	0.057043576	1.1051709	0.552585459	0.184195153
	0.3	3.669296668	3.5996235	0.069673181	1.3498588	0.674929404	0.224976468
	0.5	4.48168907	4.3965901	0.085099015	1.6487213	0.824360635	0.274786878
	0.7	5.473947392	5.3700072	0.103940172	2.0137527	1.006876353	0.335625451
	0.9	6.685894442	6.5589416	0.126952813	2.4596031	1.229801556	0.409933852

7. Conclusion

The time-fractional S-H equations are solved using a semi-analytical approach. We show how successful this method is by applying it to the nonlinear S-H with an initial condition. The solutions are graphically shown for each example to illustrate the approach's simplicity, correctness, and ease of application. As a result, this method is an effective tool for solving other fractional-order differential equations, and we can also use this method for solving other types of nonlinear problems like Benjamin-Bona-Mahony equations, Boussinesq-type equations, etc.

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Conflict of interest

The authors declare that they have no conflict of interest.

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